

Strongly closed subgraphs in a distance-regular graph

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1 Introduction

First we recall the notation and terminologies. All graphs we considered are undirected finite graphs without loops or multiple edges. Let Γ be a connected graph with usual shortest path distance ∂_Γ . We identify Γ with the set of vertices. We denote by

$$d_\Gamma := \max\{\partial_\Gamma(x, y) \mid x, y \in \Gamma\}$$

which is called the *diameter* of Γ . Let

$$\Gamma_j(u) := \{x \in \Gamma \mid \partial_\Gamma(u, x) = j\} \quad \text{and} \quad k_\Gamma(u) := |\Gamma_1(u)|.$$

Γ is called a *regular graph of valency k* if $k_\Gamma(u) = k$ for all vertices $u \in \Gamma$.

For two vertices u and x in Γ with $\partial_\Gamma(u, x) = j$, let

$$C(u, x) = C_j(u, x) := \Gamma_{j-1}(u) \cap \Gamma_1(x),$$

$$A(u, x) = A_j(u, x) := \Gamma_j(u) \cap \Gamma_1(x)$$

$$\text{and} \quad B(u, x) = B_j(u, x) := \Gamma_{j+1}(u) \cap \Gamma_1(x).$$

A connected graph Γ is said to be *distance-regular* if

$$c_j := |C_j(u, x)|, \quad a_j := |A_j(u, x)| \quad \text{and} \quad b_j := |B_j(u, x)|$$

depend only on $j = \partial_\Gamma(u, x)$ rather than individual vertices. The numbers c_j , a_j and b_j are called the *intersection numbers* of Γ . It is clear that Γ is a regular graph of valency $k_\Gamma = b_0$ if Γ is distance-regular.

The reader is referred to [1],[2] for a general theory of distance-regular graphs.

Let $\Delta \subseteq \Gamma$. We identify Δ with the induced subgraph on it.

A subgraph Δ is called *strongly closed* if $S(x, y) \subseteq \Delta$ for any $x, y \in \Delta$, where

$$S(x, y) := \{y\} \cup C(x, y) \cup A(x, y).$$

It is known that a strongly closed subgraph is distance-regular if it is regular.

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Let q be a positive integer. A quadruple (w, x, y, z) of vertices is called a *root of size q* if $\partial_\Gamma(w, x) = \partial_\Gamma(y, z) = q$, $y \in S(x, w)$ and $z \in S(w, x)$. (See Figure 1.)

A triple (x, y, z) of vertices with $\partial_\Gamma(x, z) = \partial_\Gamma(y, z) = q$ is called a *conron of size q* if there exist three sequences of vertices

$$(x_0, x_1, \dots, x_m = x), (y_0, y_1, \dots, y_m = y) \quad \text{and} \quad (z_0, z_1, \dots, z_m = z)$$

such that $\partial_\Gamma(x_0, y_0) \leq 1$, $(x_{i-1}, z_{i-1}, x_i, z_i)$ and $(y_{i-1}, z_{i-1}, y_i, z_i)$ are roots of size q for all $1 \leq i \leq m$. (See Figure 2.)

Figure 1.

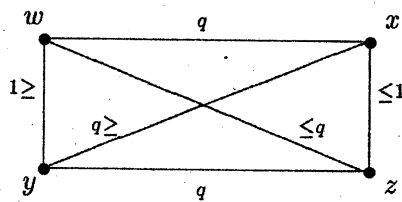
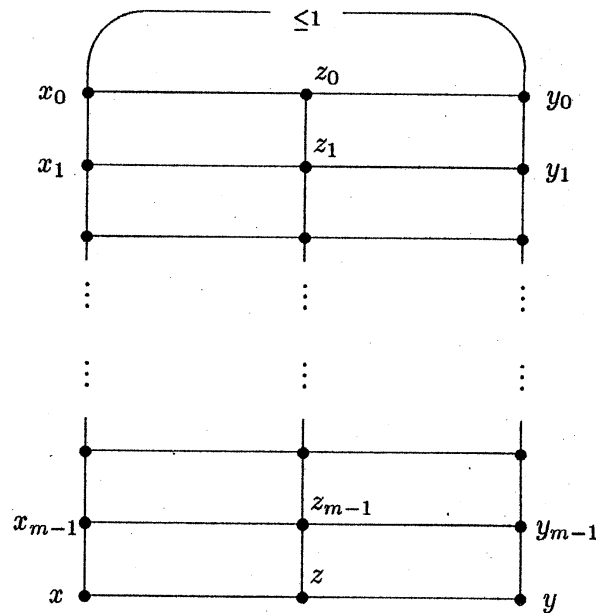
a root (w, x, y, z) of size q

Figure 2.

a conron (x, y, z) of size q

The conditions $(SS)_q$, $(CR)_q$ and $(SC)_q$ are defined as follow:

$(SS)_q$: $S(x, z) = S(y, z)$ for any triple of vertices (x, y, z) with $\partial_\Gamma(x, z) = \partial_\Gamma(y, z) = q$ and $\partial_\Gamma(x, y) \leq 1$.

$(CR)_q$: $S(x, z) = S(y, z)$ for any conron (x, y, z) of size q .

$(SC)_q$: For any given pair of vertices at distance q , there exists a strongly closed subgraph of the diameter q containing them.

The following are our main results.

Theorem 1 ([4, 6]) *Let Γ be a distance-regular graph with $a_1 > 0$. Let q be an integer with $b_{q-1} > b_q$. The following two conditions hold if and only if $(SC)_q$ holds.*

- (i) $(SS)_i$ holds for all $1 \leq i < q$,
- (ii) $(CR)_q$ holds.

Theorem 2 ([3]) *Let Γ be a distance-regular graph with $r = r(\Gamma) := \max\{i \mid (c_i, b_i) = (c_1, b_1)\}$. Then $(CR)_{r+1}$ holds if and only if $(SC)_{r+1}$ holds.*

2 The Proof of Main Result

First we show the following relations among the conditions $(SS)_q$, $(CR)_q$ and $(SC)_q$.

Proposition 3 (1) *If $(SC)_q$ holds, then $(CR)_q$ holds.*
 (2) *If $(SC)_q$ holds, then $(SS)_h$ holds for all $h \leq q$.*

Proof. (1) Let (x, y, z) be a conron with sequences

$$(x_0, x_1, \dots, x_m = x), (y_0, y_1, \dots, y_m = y) \quad \text{and} \quad (z_0, z_1, \dots, z_m = z)$$

as in Figure 2. Let Δ be a strongly closed subgraph of the diameter q containing x and z . Then $z_{m-1}, x_{m-1} \in S(x, z) \cup S(z, x) \subseteq \Delta$. Inductively, we have

$$z_{m-i}, x_{m-i} \in S(x_{m-i+1}, z_{m-i+1}) \cup S(z_{m-i+1}, x_{m-i+1}) \subseteq \Delta \quad \text{for all } 1 \leq i \leq m.$$

Whence $y_0 \in S(z_0, x_0) \subseteq \Delta$ and $y_i \in S(z_{i-1}, y_{i-1}) \subseteq \Delta$ for all $1 \leq i \leq m$. Therefore we have $y \in \Delta$ and

$$S(x, z) = \{z\} \cup \Delta_1(z) = S(y, z).$$

This proves our assertion.

(2) Let (x, y, z) be any triple of vertices in Γ with $\partial_\Gamma(x, z) = \partial_\Gamma(y, z) = h \leq q$ and $\partial_\Gamma(x, y) \leq 1$. Suppose there exists $w \in S(y, z) - S(x, z)$. Then $\partial_\Gamma(x, y) = 1, \partial_\Gamma(x, w) = h + 1$ and hence $\partial_\Gamma(y, w) = h$. Let $w_h := w$ and take $w_{i+1} \in B(x, w_i) \subseteq B(y, w_i)$ for $i = h, h + 1, \dots, q$. Then $\partial_\Gamma(x, w_q) = q + 1$ and $\partial_\Gamma(y, w_q) = q$. Since $(SC)_q$ holds, there exists Δ a strongly closed subgraph of diameter q containing y and w_q . Then $w_{j-1} \in S(y, w_j) \subseteq \Delta$ for all $j = q, \dots, h$. Thus $z \in S(y, w_h) \subseteq \Delta$ and $x \in S(z, y) \subseteq \Delta$. This contradicts $q = d_\Delta \geq \partial_\Gamma(x, w_q) = q + 1$. The desired result is proved. $\square$

Let Γ be a distance-regular graph with $b_{q-1} > b_q$. Assume that the conditions (i)(ii) in Theorem 1 holds. We sketch the construction of strongly closed subgraphs.

Fix a pair of vertices (u, v) in Γ at distance q . For any $x, y \in \Gamma_q(u)$, we define the relation $x \approx y$ iff (x, y, u) is a conron. Then this is an equivalence relation on $\Gamma_q(u)$. Set Ψ be the equivalence class containing v under this equivalence relation $\approx$. Define $\Delta(u, v) := P(u, \Psi)$ the subgraph induced on all vertices lying on shortest paths between u and vertices in Ψ .

Then $\Delta(u, v)$ becomes a strongly closed subgraph of diameter q . In particular, it is distance-regular. (See [4, 6]). $\square$

3 Applications

Theorem 4 ([6]) *Let Γ be a regular thick near polygon with $r = r(\Gamma)$. If $2r + 1 \leq d_\Gamma$, then $b_{q-1} > b_q$ and $(SC)_q$ holds for all q with $r + 1 \leq q \leq d - r$. In particular, $r \in \{1, 2, 3, 5\}$.*

Theorem 5 ([5]) *Let Γ be a distance-regular graph with $(c_1, b_1) = \dots = (c_r, b_r) \neq (c_{r+1}, b_{r+1}) = \dots = (c_{2r}, b_{2r})$ where $c_{r+1} \geq 2$. Then one of the following holds:*

- (i) $r \leq 2$,
- (ii) $a_1 = a_{r+1} = 0$, $c_{r+1} = 2$ and $r \equiv 0 \pmod{2}$.

Sketch of the Proof of Theorem 5. Assume $r \geq 3$.

- (1) Show $(CR)_{r+1}$ holds and r is even.
- (2) $(SC)_{r+1}$ holds from Theorem 1. A strongly closed subgraph Δ is a genelarized polygon.
- (3) Δ is an ordinary polygon. Hence (ii) holds. $\square$

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