

A generalization of a non-symmetric numerical semigroup generated by three elements

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§1. Non-symmetric numerical semigroups generated by three elements.

Let $\mathbb{N}$ be the additive semigroup of non-negative integers. Let H be a *numerical semigroup of genus g* , i.e., a subsemigroup of $\mathbb{N}$ whose complement $\mathbb{N} \setminus H$ consists of g elements. We denote by $g(H)$ the genus of H . We set

$$c(H) = \text{Min}\{c \in \mathbb{N} | c + \mathbb{N} \subset H\},$$

which is called the *conductor* of H . Then $c(H) \leq 2g(H)$. A numerical semigroup H is said to be *symmetric* if $c(H) = 2g(H)$. Let $M(H) = \{a_1, a_2, \dots, a_n\}$ be the minimal set of generators for H . We set

$$\alpha_i = \text{Min}\{\alpha | \alpha a_i \in \langle a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n \rangle\}$$

where for any non-negative integers $b_1, \dots, b_m$ the set $\langle b_1, \dots, b_m \rangle$ means the semigroup generated by $b_1, \dots, b_m$.

Example 1. i) Let $H = \langle 4, 5, 6 \rangle$. Then $g(H) = 4$ and $c(H) = 8$. Hence H is symmetric. If we set $a_1 = 4$, $a_2 = 5$ and $a_3 = 6$, then $\alpha_1 = 3$, $\alpha_2 = 2$ and $\alpha_3 = 2$.
ii) Let $H = \langle 4, 5, 7 \rangle$. Then $g(H) = 4$ and $c(H) = 7$. Hence H is non-symmetric. If we set $a_1 = 4$, $a_2 = 5$ and $a_3 = 7$, then $\alpha_1 = 3$, $\alpha_2 = 3$ and $\alpha_3 = 2$.

Remark 2 (Herzog [1]). Let H be a non-symmetric numerical semigroup with $M(H) = \{a_1, a_2, a_3\}$. Then

$$\alpha_1 a_1 = \alpha_{12} a_2 + \alpha_{13} a_3, \alpha_2 a_2 = \alpha_{21} a_1 + \alpha_{23} a_3 \text{ and } \alpha_3 a_3 = \alpha_{31} a_1 + \alpha_{32} a_2$$

where $\alpha_1 = \alpha_{21} + \alpha_{31}$, $\alpha_2 = \alpha_{12} + \alpha_{32}$, $\alpha_3 = \alpha_{13} + \alpha_{23}$ and $0 < \alpha_{ij} < \alpha_j$, all i, j . In this case α_{ij} 's are uniquely determined.

Proposition 3. Let the notation be as in Remark 2. Then we have
$$\begin{vmatrix} \alpha_1 & -\alpha_{12} \\ -\alpha_{21} & \alpha_2 \end{vmatrix} = a_3.$$

Example 4. Let $H = \langle a_1 = 4, a_2 = 5, a_3 = 7 \rangle$. Then

$$3a_1 = a_2 + a_3, 3a_2 = 2a_1 + a_3, 2a_3 = a_1 + 2a_2$$

and

$$\begin{vmatrix} \alpha_1 & -\alpha_{12} \\ -\alpha_{21} & \alpha_2 \end{vmatrix} = \begin{vmatrix} 3 & -1 \\ -2 & 3 \end{vmatrix} = 7 = a_3.$$

§2. Numerical semigroups of quasi-toric type.

In this section we introduce the notion of a numerical semigroup of quasi-toric type and give several examples.

Definition 5. i) Let H be a numerical semigroup with $M(H) = \{a_1, \dots, a_n\}$. A system of relations

[illegible]

satisfying

$$\alpha_j = \alpha_{1j} + \cdots + \alpha_{j-1j} + \alpha_{j+1j} + \cdots + \alpha_{nj}$$

for any j and $0 \leq \alpha_{ij} < \alpha_j$ for all i, j is said to be *neat*.

ii) A numerical semigroup H is said to be *neat* if it has a neat system of relations.

Example 6. i) Any non-symmetric numerical semigroup H with $\sharp M(H) = 3$ is neat by Remark 2.

ii) Let $H = \langle a_1 = 20, a_2 = 24, a_3 = 25, a_4 = 31 \rangle$. We have a unique neat system of relations

$$4a_1 = a_2 + a_3 + a_4, 4a_2 = 2a_1 + a_3 + a_4, 3a_3 = a_1 + a_2 + a_4 \text{ and } 3a_4 = a_1 + 2a_2 + a_3,$$

which implies that H is neat.

Definition 7. Let H be a numerical semigroup with $M(H) = \{a_1, \dots, a_n\}$. The $\mathbb{Z}$ -module

$$R = \{(r_1, r_2, \dots, r_n) \in \mathbb{Z}^n \mid \sum_{i=1}^m r_i a_i = 0\}$$

is called a *relation module* for H .

Lemma 8. *Let the notation be as in Definition 7. Then a relation module for H is a free $\mathbb{Z}$ -module of rank $n - 1$.*

Definition 9. Let S be a subsemigroup of $\mathbb{Z}^n$. S is said to be *saturated* if the condition $nr \in S$, where n is a positive integer and r an element of $\mathbb{Z}^n$, implies that $r \in S$.

Definition 10. Consider an order on the set

$$I = \{(i, j) | 1 \leq i \leq n, 1 \leq j \leq n, i \neq j\},$$

which is fixed. Let H be a neat numerical semigroup with $M(H) = \{a_1, \dots, a_n\}$.

We define the order on the set

$$\{(i, j) | i, j = 1, 2, 3 \text{ and } i \neq j\}$$

as follows: $(i, j) \leq (i', j')$ if " $j < j'$ " or " $j = j', i \leq i'$ ". The associated subsemigroup S of $\mathbb{Z}^4$ is generated by $b_1 = e_1, b_2 = e_2, b_3 = e_3, b_4 = e_4, b_5 = (1, 1, -1, 0)$ and $b_6 = (-1, 0, 1, 1)$. Then S is saturated, which implies that H is of quasi-toric type.

Proposition 13. *Let H be a neat numerical semigroup with $M(H) = \{a_1, \dots, a_n\}$ such that it has a neat system of relations*

[illegible]

with

$$\begin{vmatrix} \alpha_1 & -\alpha_{12} & \cdots & -\alpha_{1n-1} \\ -\alpha_{21} & \alpha_2 & \cdots & -\alpha_{2n-1} \\ \vdots & \vdots & & \vdots \\ -\alpha_{n-11} & -\alpha_{n-12} & \cdots & \alpha_{n-1} \end{vmatrix} \neq 0.$$

- i) If there is some j_0 with $1 \leq j_0 \leq n$ such that $\alpha_{ij_0} > 0$ for all i , then H is of quasi-toric type.
- ii) If there is some i_0 with $1 \leq i_0 \leq n$ such that $\alpha_{i_0j} > 0$ for all j , then H is of quasi-toric type.

Example 14. Let $H = \langle a_1 = 20, a_2 = 24, a_3 = 25, a_4 = 31 \rangle$. We have a neat system of relations

$$4a_1 = a_2 + a_3 + a_4, 4a_2 = 2a_1 + a_3 + a_4,$$

$$3a_3 = a_1 + a_2 + a_4 \text{ and } 3a_4 = a_1 + 2a_2 + a_3.$$

Since we have

$$\begin{vmatrix} 4 & -1 & -1 \\ -2 & 4 & -1 \\ -1 & -1 & 3 \end{vmatrix} = 31 \neq 0,$$

by Proposition 13 H is of quasi-toric type (Cf. Example 6 ii)).

Proposition 15. *Let H be a neat numerical semigroup with $M(H) = \{a_1, \dots, a_n\}$ such that it has a neat system of relations*

[illegible]

with

$$\begin{vmatrix} \alpha_1 & -\alpha_{12} & \cdots & -\alpha_{1n-1} \\ -\alpha_{21} & \alpha_2 & \cdots & -\alpha_{2n-1} \\ \vdots & \vdots & & \vdots \\ -\alpha_{n-11} & -\alpha_{n-12} & \cdots & \alpha_{n-1} \end{vmatrix} \neq 0$$

where we set $\alpha_{ij} = 0$ if $\alpha_{ij}a_j$ does not appear in the system of relations. Then H is of quasi-toric type.

Example 16 (Komeda [3]). For any $n \geq 5$, let H_n be the numerical semigroup with

$$M(H_n) = \{a_1 = n, a_2 = n + 1, a_3 = 2n + 3, a_4 = 2n + 4, \dots, a_{n-1} = 2n + n - 1\}.$$

Then we have a neat system of relations

$$\alpha_1 a_1 = 4a_1 = a_2 + a_{n-1}, \alpha_2 a_2 = 3a_2 = a_1 + a_3, \alpha_3 a_3 = 2a_3 = 2a_2 + a_4,$$

$$\alpha_i a_i = 2a_i = a_{i-1} + a_{i+1} \quad (4 \leq i \leq n-2), \alpha_{n-1} a_{n-1} = 2a_{n-1} = 3a_1 + a_{n-2}.$$

By Proposition 15, H_n is a neat numerical semigroup of quasi-toric type.

Theorem 17. Let H be a neat numerical semigroup with $M(H) = \{a_1, a_2, a_3, a_4\}$. Then H is of quasi-toric type.

Proof. Let

$$\begin{cases} \alpha_1 a_1 = \alpha_{12} a_2 + \alpha_{13} a_3 + \alpha_{14} a_4 \\ \alpha_2 a_2 = \alpha_{21} a_1 + \alpha_{23} a_3 + \alpha_{24} a_4 \\ \alpha_3 a_3 = \alpha_{31} a_1 + \alpha_{32} a_2 + \alpha_{34} a_4 \\ \alpha_4 a_4 = \alpha_{41} a_1 + \alpha_{42} a_2 + \alpha_{43} a_3 \end{cases}$$

be a unique neat system of relations for H . We note that

$$D = \begin{vmatrix} \alpha_1 & -\alpha_{12} & -\alpha_{13} \\ -\alpha_{21} & \alpha_2 & -\alpha_{23} \\ -\alpha_{31} & -\alpha_{32} & \alpha_3 \end{vmatrix} > 0.$$

By Proposition 13 first we may assume that $\alpha_1 a_1 = \alpha_{12} a_2 + \alpha_{14} a_4$, which implies that $\alpha_3 = \alpha_{23} + \alpha_{43}$. Moreover, we have

$$"\alpha_2 a_2 = \alpha_{21} a_1 + \alpha_{23} a_3 \text{ OR } \alpha_{23} a_3 + \alpha_{24} a_4"$$

and

$$"\alpha_4 a_4 = \alpha_{41} a_1 + \alpha_{43} a_3 \text{ OR } \alpha_{42} a_2 + \alpha_{43} a_3".$$

i) $\alpha_2 a_2 = \alpha_{21} a_1 + \alpha_{23} a_3$, $\alpha_4 a_4 = \alpha_{41} a_1 + \alpha_{43} a_3$. Then $\alpha_3 a_3 = \alpha_{32} a_2 + \alpha_{34} a_4$. This case is reduced to Proposition 15.

$i = 1, \dots, n-1$. Assume that

$$\begin{vmatrix} r_{11} & r_{12} & \cdots & r_{1n-1} \\ r_{21} & r_{22} & \cdots & r_{2n-1} \\ \vdots & \vdots & & \vdots \\ r_{n-11} & r_{n-12} & \cdots & r_{n-1n-1} \end{vmatrix} = \pm a_n.$$

Then $r_1, \dots, r_{n-1}$ form a basis for the $\mathbb{Z}$ -module R .

Let H be a neat numerical semigroup with $M(H) = \{a_1, \dots, a_n\}$ with its fixed neat system of relations. We set $N = \#\{(i, j) | \alpha_{ij} \neq 0\} - (n-1)$. Let $S = \langle b_1, \dots, b_{N+n-1} \rangle$ of $\mathbb{Z}^N$ be the associated subsemigroup. Let k be a field. Let $\varphi_H : k[X] = k[X_1, \dots, X_n] \rightarrow k[H] = k[t^h]_{h \in H}$ be a k -algebra homomorphism sending X_i to t^{a_i} , $\pi : k[Y] = k[Y_1, \dots, Y_{N+n-1}] \rightarrow k[S] = k[T^b]_{b \in S}$ a k -algebra homomorphism sending Y_i to T^{b_i} , $\eta : k[Y] \rightarrow k[X]$ a k -algebra homomorphism sending Y_i to $g_i = X_j^{\alpha_{ij}}$ if b_i corresponds to $\alpha_{ij}a_j$ and $\zeta : k[\mathbb{N}^N] = k[t_1, \dots, t_N] \rightarrow k[H]$ a k -algebra homomorphism sending t_i to $t^{w(g_i)}$ where the weight w on $k[X]$ is defined by $w(X_i) = a_i$ and $w(c) = 0$ for $c \in k^\times$. By the definition of b_i 's, ζ extends to $\zeta' : k[S] \rightarrow k[H]$. Then we get $\varphi_H \circ \eta = \zeta' \circ \pi$, which implies that $\text{Ker } \varphi_H \supseteq \eta(\text{Ker } \pi)$.

Definition 22. A neat numerical semigroup H is said to be of *toric type* if it is of quasi-toric type and we have an isomorphism $k[H] \cong k[S] \otimes_{k[Y]} k[X]$, that is to say, $\text{Ker } \varphi_H = (\eta(\text{Ker } \pi))$.

Remark 23 (Komeda [2]). A numerical semigroup of toric type is Weierstrass, where a numerical semigroup H is said to be *Weierstrass* if there is a pointed non-singular complete curve (C, P) over an algebraically closed field such that

$$H = \{n \in \mathbb{N} | \text{there is a rational function } f \text{ on } C \text{ with } (f)_\infty = nP\}.$$

Example 24. Any non-symmetric numerical semigroup with $M(H) = \{a_1, a_2, a_3\}$ is of toric type, because we know that the ideal $\text{Ker } \varphi_H$ is generated by

$$X_1^{\alpha_1} - X_2^{\alpha_{12}} X_3^{\alpha_{13}}, X_2^{\alpha_2} - X_1^{\alpha_{21}} X_3^{\alpha_{23}} \text{ and } X_3^{\alpha_3} - X_1^{\alpha_{31}} X_2^{\alpha_{32}} \text{ (Herzog [1])}.$$

We note that H is 1-neat (Cf. Example 20).

Example 25. For any integer $n \geq 5$, let H_n be a numerical semigroup with

$$M(H_n) = \{a_1 = n, a_2 = n+1, a_3 = 2n+3, a_4 = 2n+4, \dots, a_{n-1} = 2n+n-1\}$$

(Cf. Example 16). Then the ideal $\text{Ker } \varphi_{H_n}$ is generated by

$$X_2^3 - X_1 X_3, X_2 X_j - X_1 X_{j+1} (3 \leq j \leq n-2), X_2 X_{n-1} - X_1^4,$$

$$X_3X_j - X_2^2X_{j+1} (3 \leq j \leq n-2), X_3X_{n-1} - X_2^2X_1^3, \\ X_iX_j - X_{i-1}X_{j+1} (4 \leq i \leq n-2, i \leq j \leq n-2), X_iX_{n-1} - X_{i-1}X_1^3 (4 \leq i \leq n-1).$$

It is proved that H_n is of toric type. In this case H_n is also 1-neat.

Theorem 26. *A 1-neat numerical semigroup with $M(H) = \{a_1, a_2, a_3, a_4\}$ is of toric type.*

Problem 27. Let $n \geq 5$. If H is a 1-neat numerical semigroup with $\#M(H) = n$, is it of toric type?

References

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