

On weak notion of $\mathfrak{p}$ -dividing

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Abstract

I considered the restricted notions of weak dividing. In this note, I try to define a weak notion of $\mathfrak{p}$ -dividing (thorn-dividing).

1. Preliminaries

We recall some definitions.

Definition 1 Let $\varphi(x_0, x_1, \dots, x_{n-1})$ be a formula and $p(x)$ be a type. We denote the type $\{\varphi(x_0, x_1, \dots, x_{n-1})\} \cup p(x_0) \cup p(x_1) \cup \dots \cup p(x_{n-1})$ by $[p]^\varphi$.

Let $A \subset B$ and $p(x) \in S(B)$.

$p(x)$ *divides over* A if there is a formula $\varphi(x, b) \in p(x)$ and an infinite sequence $\{b_i : i < \omega\}$ with $b \equiv b_i(A)$ such that $\{\varphi(x, b_i) : i < \omega\}$ is k -inconsistent for some $k < \omega$.

$p(x)$ *weakly divides over* A if there is a formula $\varphi(\bar{x}) \in L_n(A)$ such that $[p \upharpoonright A]^\varphi$ is consistent, while $[p]^\varphi$ is inconsistent.

We can define weak dividing for formulas.

Let $b \notin A$.

$\psi(x, b)$ *weakly divides over* A if there is a formula $\varphi(\bar{x}) \in L_n(A)$ and a realization a of $\psi(x, b)$ such that $[\text{tp}(a/A)]^\varphi$ is consistent, while $[\psi(x, b)]^\varphi$ is inconsistent.

And we can consider weak forking.

$p(x)$ *weakly forks over* A if there is a $q(x, y) \in S(A)$ such that $p(x) \cup q(x, y)$ is consistent, and any completion $r(x, y) \in S(B)$ of $p(x) \cup q(x, y)$ weakly divides over A .

If we exchange the role between variables and parameters in the definition of weak dividing, we could define weak forking naturally.

In this note, we call such formula " $\varphi(\bar{x})$ " in the definition above the *witness formula* of weak dividing for the sake of convenience.

I introduce an example from [3].

Example 2 Let T be the theory of an equivalence relation with two infinite classes of the language $L = \{ \text{a binary relation } E(x, y) \}$. And let $\models \neg E(a, b)$. Then the type $\text{tp}(a/b)$ does not divide over $\emptyset$, while $\text{tp}(a/b)$ weakly divides over $\emptyset$ by the formula $\neg E(x, y)$.

I tried to divide witness formulas into some classes according to their properties ago. And I told about the next characterization at the RIMS meeting last year.

Definition 3 Let $A \subset B$ and $p(x) \in S(B)$.

$p(x)$ $\mathcal{M}$ -weakly divides over A if there is a formula $\varphi(\bar{x}) \in L_n(A)$ and a Morley sequence $I = \{a_i : i < n+1\}$ of $p \upharpoonright A$ such that $\models \varphi(a_0, a_1, \dots, a_{n-1})$, while the type $[p]^\varphi$ is inconsistent.

Theorem 4 Let T be simple.

Then T is stable if and only if $\mathcal{M}$ -weak dividing over models is symmetric.

2. Weak notion of p -dividing

In recent years another variant of dividing, "thorn"-dividing has been characterized in rosy theory (see e.g. [4]). I tried to define weak notion of p -dividing (thorn-dividing). We recall some definitions first.

Definition 5 Let $A \subset B$ and $p(x) \in S(B)$.

$p(x)$ strongly divides over A if there is a formula $\varphi(x, b) \in p(x)$ such that $b \notin \text{acl}(A)$ and $\{\varphi(x, b_i) : b_i \models \text{tp}(b/A)\}$ is k -inconsistent for some $k < \omega$.

$p(x)$ p -divides over A if $p(x)$ strongly divides over Ac for some parameter c .

$p(x)$ p -forks over A if there is a formula $\varphi(x, b) \in p(x)$ such that $\varphi(x, b)$ implies a finite disjunction of formulas which p -divides over A .

Given a formula φ , a set Δ of formulas in variables x, y , a set of formulas Π in variables y, z , and a number k , we define $p(\varphi, \Delta, \Pi, k)$ (thorn-rank) inductively as follows :

- (1) $p(\varphi, \Delta, \Pi, k) \geq 0, \infty, \lambda$ for limit ordinal λ is defined as usual.
- (2) $p(\varphi, \Delta, \Pi, k) \geq \alpha + 1$ if and only if there is a $\delta \in \Delta$, some $\pi(y, z) \in \Pi$ and parameters c such that

(a) $p(\varphi \wedge \delta(x, a), \Delta, \Pi, k) \geq \alpha$ for infinitely many $a \models \pi(y, c)$

(b) $\{\delta(x, a)\}_{a \models \pi(y, c)}$ is k -inconsistent.

For a type p , we define $p(p, \Delta, \Pi, k) = \min\{p(\varphi, \Delta, \Pi, k) \mid \varphi \in p\}$.

A theory T is rosy if for any type $p(x)$, any finite sets of formulas Δ and Π , and any finite k , $p(\varphi, \Delta, \Pi, k)$ is finite.

Remark 6 (1) In rosy theories, p -forking satisfies the independence axioms.

(2) If $a \models \varphi(x, b)$ and $\varphi(x, b)$ $\mathfrak{p}$ -divides over C by the set $\{b_i \models \theta(y, d)\}$, then $b \in \text{acl}(Cda) - \text{acl}(Cd)$.

Weak notions of $\mathfrak{p}$ -dividing could be defined in many ways. By the definition, $\mathfrak{p}$ -dividing implies dividing. So we expect that weak $\mathfrak{p}$ -dividing implies weak dividing.

Definition 7 Let $b \notin A$.

$\psi(x, b)$ *weakly $\mathfrak{p}$ -divides over A* if there is a formula $\varphi(\bar{x}) = \exists y \bigwedge_{i < n} \theta(x_i, y)$ $\in L_n(A)$ and a realization a of $\psi(x, b)$ such that $[\text{tp}(a/A)]^\varphi$ is consistent, while $[\psi(x, b)]^\varphi$ is inconsistent.

We define weak $\mathfrak{p}$ -dividing($\mathfrak{p}$ -forking) for types just like weak dividing(forking).

We can check the next fact easily.

Fact 8 *Let T be rosy. Then $\mathfrak{p}$ -forking implies weak $\mathfrak{p}$ -forking.*

3. Weak $\mathfrak{p}$ -dividing and NIP theories

Definition 9 A formula $\varphi(x, y)$ has the *independence property* if for every $n < \omega$, there are sequences a_l ($l < n$) such that for every $w \subset n$, $\models (\exists x) \left[\bigwedge_{l < n} \varphi(x, a_l)^{\text{if } (l \in w)} \right]$.

A theory T is *NIP* if no formula $\varphi(x, y)$ has the independence property.

Weak $\mathfrak{p}$ -dividing is a kind of algebraic extension.

Lemma 10 (T is any theory.) $A \subset B$.

Then $\text{tp}(a/B)$ does not weakly $\mathfrak{p}$ -divide over A if and only if

for any $n < \omega$, any C and any extension $q(x, C, A)$ of $\text{tp}(a/A)$ over AC , if $\bigcup_{i < n} q(x_i, C, A)$ is consistent, then $\bigcup_{i < n} q(x_i, Z, A) \cup \bigcup_{i < n} r(x_i, Y, A)$ is consistent where $\text{tp}(a/B) := r(x, B, A)$.

By the lemma above, we can prove the next fact.

Proposition 11 *Let T be NIP and unstable.*

Then weak $\mathfrak{p}$ -dividing is not symmetric.

References

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