

Periodicity of solutions to some parabolic-elliptic
variational inequalities

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§1. Results

In this paper, we report the results of Kenmochi-Kubo [2] and give the outline of proofs. Let $\Omega \subset \mathbb{R}^N$ ($N \geq 1$) be an open bounded set with smooth boundary Γ . We are interested in periodic behavior of solutions to parabolic-elliptic problems with mixed-type boundary conditions prescribed on time-dependent parts of the boundary. Assume that Γ admits the decomposition: $\Gamma = \Gamma_D(t) \cup \Gamma_N(t) \cup \Gamma_U(t)$, for each $t \in \mathbb{R}$, where $\Gamma_i(t)$ ($i=D, N, U$) are mutually disjoint measurable subsets of Γ . Let $\rho: \mathbb{R} \rightarrow \mathbb{R}$ be a non-decreasing Lipschitz-continuous function. The following system is studied:

$$\rho(v)' - \Delta v = f \quad \text{in } (0, \infty) \times \Omega,$$

$$\rho(v(0, \cdot)) = u_0 \quad \text{in } \Omega,$$

$$v = 0 \quad \text{on } \bigcup_{t>0} \{t\} \times \Gamma_D(t),$$

$$\partial_\nu v = 0 \quad \text{on } \bigcup_{t>0} \{t\} \times \Gamma_N(t),$$

$$v \leq 0, \quad \partial_\nu v \leq 0, \quad v \cdot \partial_\nu v = 0 \quad \text{on } \bigcup_{t>0} \{t\} \times \Gamma_U(t).$$

Here $\rho(v)' = \frac{\partial}{\partial t} \rho(v)$ and ∂_v is the outward normal derivative on Γ . These kinds of problems arise from the free boundary problems for saturated-unsaturated flows in porous media. We refer to [3, 4] and their references for related topics. In order to give a notion of weak solutions in variational sense, let us introduce the convex sets

$$K(t) = \{z \in H^1(\Omega); z=0 \text{ a.e. on } \Gamma_D(t), z \leq 0 \text{ a.e. on } \Gamma_U(t)\}, \quad \text{for } t \in \mathbb{R}.$$

Definition. Let $J = \mathbb{R}$ or $\mathbb{R}_+$. Let $f \in L^2_{loc}(J; L^2(\Omega))$. Then a function $v \in L^2_{loc}(J; H^1(\Omega))$ is called a weak solution to $E(K(t), \rho, f)$ on J , if $v(t) \in K(t)$ for a.e. $t \in J$, $\rho(v) \in W^{1,2}_{loc}(J; L^2(\Omega))$ and v satisfies the following variational inequality for a.e. $t \in J$:

$$\int_{\Omega} (\rho(v)'(t) - f(t))(v(t) - z) dx + \int_{\Omega} \nabla v(t) \cdot \nabla (v(t) - z) dx \leq 0,$$

for all $z \in K(t)$.

Let us assume the following geometric condition.

(A.1) For each $t \in \mathbb{R}_+$ there is a C^1 -diffeomorphism $\theta(t, \cdot): \bar{\Omega} \rightarrow \bar{\Omega}$ such that

- (i) $\theta(0, \cdot) = \text{Id}$;
- (ii) $\Gamma_i(t) = \theta(t, \Gamma_i(0))$, $i = D, N, U$, for all $t \in \mathbb{R}_+$;
- (iii) $\frac{\partial}{\partial x_j} \theta, \frac{\partial}{\partial t} \theta, \frac{\partial^2}{\partial x_j \partial t} \theta \in C^0(\mathbb{R}_+ \times \bar{\Omega})$;
- (iv) $\text{meas}_{\Gamma} \bigcap_{t \geq 0} \Gamma_D(t) > 0$ (meas_{Γ} denotes the surface measure on Γ).

Lemma 1 (cf. Kenmochi-Pawlow [3]). Assume (A.1) holds as well as

$$(A.2) \quad f \in W_{loc}^{1,1}(R_+; L^2(\Omega)).$$

Let u_0 be such that there is $v_0 \in K(0)$ with $u_0 = \rho(v_0)$. Then there is a unique weak solution v to $E(K(t), \rho, f)$ on R_+ satisfying $\rho(v)|_{t=0} = u_0$.

Also the existence of a periodic solution is known.

Lemma 2 (cf. Kenmochi-Kubo [1]). In addition to (A.1) and (A.2) assume that there is a constant $T > 0$ such that

$$(A.3) \quad f(t+T) = f(t) \quad \text{and} \quad \Gamma_i(t+T, \cdot) = \Gamma_i(t) \quad (i=D, N, U), \quad \text{for all } t \in R_+.$$

Then there is a weak solution ω to $E(K(t), \rho, f)$ on R_+ such that

$$\omega(t+T) = \omega(t), \quad \text{for a.e. } t \in R_+.$$

Such a solution ω is called a T-periodic solution. Any T-periodic solution can be extended as a solution on the whole of R by using T-periodicity, provided that we extend the function f and $\Gamma_i(t)$ ($i=D, N, U$) periodically on R . The main result is stated as follows.

Theorem. Under conditions (A.1), (A.2) and (A.3), T-periodic solution ω to $E(K(t), \rho, f)$ is unique and asymptotically stable in the sense that for any weak solution v to $E(K(t), \rho, f)$ on R_+

$$\rho(v)(t) - \rho(\omega)(t) \rightarrow 0 \quad \text{in } L^2(\Omega) \text{ and weakly in } H^1(\Omega) \text{ as } t \rightarrow \infty.$$

Moreover the T-periodic solution ω is the only one weak solution on R such that the trajectory $\{\omega(t); t \in R\}$ is bounded in $L^2(\Omega)$.

We shall give the outline of the proof of this theorem in the next section. For the detailed proof, see [2].

Remark (cf. [1, 3, 4]). As far as Lemmas 1 and 2 are concerned, condition (iv) of (A.1) can be replaced by weaker one:

$$(iv)' \quad \text{meas}_{\Gamma} \Gamma_D(t) > 0, \quad \text{for all } t \in R_+.$$

§2. Outline of Proof

The proof of Theorem is based on the following two lemmas.

Lemma 3. Assume (A.1), (A.2) and (A.3) hold. Let ω and v be weak solutions to $E(K(t), \rho, f)$ on R_+ . Suppose that ω is T-periodic and that $\omega \leq v$ (or $\omega \geq v$) a.e. in $R_+ \times \Omega$. Then we have

$$(1) \quad \rho(v)(t+nT) \rightarrow \rho(\omega)(t) \quad \text{in } L^2(\Omega) \text{ and weakly in } H^1(\Omega) \text{ as } n \rightarrow \infty$$

for all $t \in R_+$.

Lemma 4. Let v and $\hat{v}$ be weak solutions such that $v \leq \hat{v}$ a.e in $R_+ \times \Omega$. Then

$$(2) \quad \partial_v v(t) \geq \partial_v \hat{v}(t) \quad \text{in the sense of } H^{-1/2}(\Gamma) \quad \text{for a.e. } t \in R_+,$$

that is $\langle \partial_v v(t), z \rangle \geq \langle \partial_v \hat{v}(t), z \rangle$ for all $z \in H^{1/2}(\Gamma)$ with $z \geq 0$. Here $\langle \cdot, \cdot \rangle$ denotes the duality between $H^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$.

Proof of Lemma 4. Fix $t \in R_+$. For each $\lambda > 0$ and $\mu > 0$ let $v_{\lambda, \mu}(t) \in H^1(\Omega)$ be the solution to

$$\begin{cases} v_{\lambda, \mu}(t) - \lambda \Delta v_{\lambda, \mu}(t) = v(t) & \text{in } \Omega, \\ -\partial_v v_{\lambda, \mu}(t) = \frac{1}{\mu} \chi_{\Gamma_D}(t) \cdot v_{\lambda, \mu}(t) + \frac{1}{\mu} \chi_{\Gamma_U}(t) \cdot [v_{\lambda, \mu}(t)]^+ & \text{on } \Gamma, \end{cases}$$

where $\chi_{\Gamma_D}(t)$ and $\chi_{\Gamma_U}(t)$ are the characteristic functions of the sets $\Gamma_D(t)$ and $\Gamma_U(t)$, respectively. And let $\hat{v}_{\lambda, \mu}(t)$ be similarly defined. The boundary conditions imply that $\partial_v v_{\lambda, \mu}(t), \partial_v \hat{v}_{\lambda, \mu}(t) \in L^2(\Gamma)$. Also it follows from $v(t) \leq \hat{v}(t)$ that $v_{\lambda, \mu}(t) \leq \hat{v}_{\lambda, \mu}(t)$. Consequently $-\partial_v v_{\lambda, \mu}(t) \leq -\partial_v \hat{v}_{\lambda, \mu}(t)$ on Γ . Since $\partial_v v_{\lambda, \mu}(t)$ and $\partial_v \hat{v}_{\lambda, \mu}(t)$ converge to $\partial_v v(t)$ and $\partial_v \hat{v}(t)$, respectively in $H^{-1/2}(\Gamma)$ as $\mu \rightarrow 0$ and $\lambda \rightarrow 0$, we have (2). See [2; Proposition 4.1] for the detail.

q.e.d.

Proof of Lemma 3. We shall prove in the case $\omega \leq v$. The case $\omega \geq v$ is similarly proved.

Since $t \mapsto \|\rho(v)(t) - \rho(\omega)(t)\|_{L^1(\Omega)}^+$ is non-increasing (cf. [3, 4]), we have by $v \geq \omega$

$$t \mapsto \int_{\Omega} \{\rho(v)(t) - \rho(\omega)(t)\} dx \quad \text{is non-increasing.}$$

In particular, since ω is T -periodic,

$$\int_{\Omega} \rho(v)(mT) dx \leq \int_{\Omega} \rho(v)(nT) dx \quad \text{for all } n \leq m \quad (n, m \in \mathbb{N}).$$

Therefore

$$(3) \quad \lim_{n \rightarrow \infty} \int_{\Omega} \rho(v)(nT) dx \quad \text{exists.}$$

Next by virtue of [1; Theorem 1], $\{\rho(v)(t); t \in \mathbb{R}_+\}$ is bounded in $H^1(\Omega)$. Hence on account of the convergence result [4; Theorem 1.4], there are a subsequence $\{n_k\}$ of $\{n\}$ and a weak solution v^* to $E(K(t), \rho, f)$ on $\mathbb{R}_+$ such that

$$(4) \quad \rho(v)(t + n_k T) \rightarrow \rho(v^*)(t) \quad \text{in } L^2(\Omega) \text{ and weakly in } H^1(\Omega) \text{ as } k \rightarrow \infty$$

for all $t \in \mathbb{R}_+$.

We are going to show that $v^* \equiv \omega$. Then the entire sequence $\rho(v)(t + nT)$ converges to $\rho(\omega)(t)$ and we have (1). First by (3) and

(4) we see that

$$\begin{aligned}
 (5) \quad \int_{\Omega} \rho(v^*)(nT) dx &= \lim_{k \rightarrow \infty} \int_{\Omega} \rho(v)(nT + n_k T) dx \\
 &= \lim_{m \rightarrow \infty} \int_{\Omega} \rho(v)(mT) dx \quad (\text{put } m = n + n_k) \\
 &= \lim_{k \rightarrow \infty} \int_{\Omega} \rho(v)(n_k T) dx \\
 &= \int_{\Omega} \rho(v^*)(0) dx, \quad \text{for all } n \in \mathbb{N}.
 \end{aligned}$$

Therefore from the equations for v^* and ω it follows that

$$\begin{aligned}
 0 &= \int_0^{nT} dt \frac{d}{dt} \int_{\Omega} \{\rho(v^*)(t) - \rho(\omega)(t)\} dx \\
 &= \int_0^{nT} dt \int_{\Omega} \Delta(v^*(t) - \omega(t)) dx \\
 &= \int_0^{nT} \langle \partial_v(v(t) - \omega(t)), 1 \rangle dt, \quad \text{for all } n \in \mathbb{N}.
 \end{aligned}$$

On the other hand, it is evident that $\omega \leq v^*$. Therefore by (2)

$$\partial_v \omega(t) \geq \partial_v v^*(t) \quad \text{in the sense of } H^{-1/2}(\Gamma) \quad \text{for a.e. } t \in \mathbb{R}_+.$$

Hence we have

$$\langle \partial_v(v(t) - \omega(t)), 1 \rangle = 0, \quad \text{for a.e. } t \in \mathbb{R}_+.$$

From this we can conclude that

$$(6) \quad \partial_\nu \omega(t) = \partial_\nu v^*(t) \quad \text{in } H^{-1/2}(\Gamma) \quad \text{for a.e. } t \in \mathbb{R}_+.$$

Next put $\Gamma_0 = \bigcap_{t \geq 0} \Gamma_D(t)$ and $V \equiv \{z \in H^1(\Omega); z=0 \text{ on } \Gamma_0\}$. Since $\text{meas}_\Gamma \Gamma_0 > 0$ by assumption, for each $t \in \mathbb{R}_+$ there is a unique solution $u(t) \in V$ of the following variational problem:

$$(7) \quad \int_{\Omega} \nabla u(t) \cdot \nabla z \, dx = \int_{\Omega} \{\rho(v^*)(t) - \rho(\omega)(t)\} z \, dx \quad \text{for all } z \in V.$$

It is seen from Poincaré's inequality that there exists a constant $C_1 > 0$ such that

$$(8) \quad \|\nabla u(t)\|_{L^2(\Omega)} \leq C_1 \|\rho(v^*)(t) - \rho(\omega)(t)\|_{L^2(\Omega)} \quad \text{for all } t \in \mathbb{R}_+.$$

From (6) and (7) we observe that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\nabla u(t)\|_{L^2(\Omega)}^2 &= \int_{\Omega} \nabla u'(t) \cdot \nabla u(t) \, dx \\ &= \int_{\Omega} \{\rho(v^*)(t) - \rho(\omega)(t)\}' u(t) \, dx \\ &= \int_{\Omega} \Delta(v^*(t) - \omega(t)) u(t) \, dx \\ &= - \int_{\Omega} \nabla(v^*(t) - \omega(t)) \cdot \nabla u(t) \, dx \\ &= - \int_{\Omega} (v^*(t) - \omega(t)) \{\rho(v^*)(t) - \rho(\omega)(t)\}' \, dx. \end{aligned}$$

Hence by (8) and the Lipschitz continuity of ρ ,

$$\begin{aligned}
 & \frac{1}{2} \frac{d}{dt} |\nabla u(t)|_{L^2(\Omega)}^2 + c_2 |\nabla u(t)|_{L^2(\Omega)}^2 \\
 & \leq \frac{1}{2} \frac{d}{dt} |\nabla u(t)|_{L^2(\Omega)}^2 + c_3 |\rho(v^*)(t) - \rho(\omega)(t)|_{L^2(\Omega)}^2 \\
 & \leq \frac{1}{2} \frac{d}{dt} |\nabla u(t)|_{L^2(\Omega)}^2 + \int_{\Omega} (v^*(t) - \omega(t)) \{ \rho(v^*)(t) - \rho(\omega)(t) \} dx \\
 & \leq 0, \quad \text{for a.e. } t \in \mathbb{R}_+.
 \end{aligned}$$

From this inequality we can conclude that

$$\frac{d}{dt} |\nabla u(t)|_{L^2(\Omega)}^2 \leq 0 \quad \text{and} \quad \int_0^\infty |\nabla u(t)|_{L^2(\Omega)}^2 dt < \infty.$$

Consequently

$$|\nabla u(t)|_{L^2(\Omega)} \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

Combining this with (7) we obtain

$$(9) \quad \int_{\Omega} \{ \rho(v^*)(t) - \rho(\omega)(t) \} z dx \rightarrow 0 \quad \text{as } t \rightarrow \infty \quad \text{for all } z \in V.$$

Since $\{ \rho(v^*)(t) - \rho(\omega)(t); t \in \mathbb{R}_+ \}$ is bounded in $L^2(\Omega)$ ([1; Theorem 1]) and V is dense in $L^2(\Omega)$, the convergence (9) holds for all $z \in L^2(\Omega)$.

In particular ($z=1$)

$$\int_{\Omega} \{\rho(v^*)(nT) - \rho(\omega)(nT)\} dx \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

On the other hand, the T-periodicity of ω and (5) imply that

$$\int_{\Omega} \{\rho(v^*)(nT) - \rho(\omega)(nT)\} dx = \int_{\Omega} \{\rho(v^*)(0) - \rho(\omega)(0)\} dx \quad \text{for all } n \in \mathbb{N}.$$

Hence

$$\int_{\Omega} \{\rho(v^*)(0) - \rho(\omega)(0)\} dx = 0.$$

Since $\rho(v^*)(0) \geq \rho(\omega)(0)$, we have $\rho(v^*)(0) = \rho(\omega)(0)$. This implies $v^* \equiv \omega$. Thus we have proved Lemma 3. q.e.d.

Proof of Theorem. First we shall show the uniqueness of $L^2(\Omega)$ -bounded solution on $\mathbb{R}$. Uniqueness of T-periodic solution follows from this. Let ω be a T-periodic solution and let v be a weak solution on $\mathbb{R}$ such that $\{v(t); t \in \mathbb{R}\}$ is bounded in $L^2(\Omega)$. We first assume that $\omega \leq v$ a.e. in $\mathbb{R} \times \Omega$. Since $L^2(\Omega)$ -boundedness implies $H^1(\Omega)$ -boundedness (cf. [1, 3, 4]), there is a subsequence $\{n_k\}$ of $\{n\}$ and a weak solution v^* on $\mathbb{R}$ such that

$$v(t - n_k T) \rightarrow v^*(t) \quad \text{in } L^2(\Omega) \text{ and weakly in } H^1(\Omega) \quad \text{as } k \rightarrow \infty.$$

On the other hand it follows from (2) and $\omega \leq v$ that

$$\begin{aligned}
 (10) \quad \frac{d}{dt} \int_{\Omega} \{\rho(v)(t) - \rho(\omega)(t)\} dx &= \int_{\Omega} \Delta(v(t) - \omega(t)) dx \\
 &= \langle \partial_v(v(t) - \omega(t)), 1 \rangle \\
 &\leq 0.
 \end{aligned}$$

Hence

$$(11) \quad \lim_{t \rightarrow -\infty} \int_{\Omega} \{\rho(v)(t) - \rho(\omega)(t)\} dx \equiv d \quad \text{exists.}$$

Therefore for all $t \in \mathbb{R}$

$$d = \lim_{k \rightarrow \infty} \int_{\Omega} \{\rho(v)(t - n_k T) - \rho(\omega)(t - n_k T)\} dx = \int_{\Omega} \{\rho(v^*)(t) - \rho(\omega)(t)\} dx,$$

By the way, since $\omega \leq v$, it follows from Lemma 3 that

$$v^*(t + nT) - \omega(t + nT) \rightarrow 0 \quad \text{in } L^2(\Omega) \text{ and weakly in } H^1(\Omega) \quad \text{as } n \rightarrow \infty.$$

Consequently

$$0 = \lim_{n \rightarrow \infty} \int_{\Omega} \{\rho(v^*)(t + nT) - \rho(\omega)(t + nT)\} dx = d.$$

Therefore it follows from (10) and (11) that $\int_{\Omega} \{\rho(v)(t) - \rho(\omega)(t)\} dx$

is non-negative and non-decreasingly converges to $d = 0$ as $t \rightarrow -\infty$.

Hence $\int_{\Omega} \{\rho(v)(t) - \rho(\omega)(t)\} dx \equiv 0$ so that $\rho(v) \equiv \rho(\omega)$ by $v \geq \omega$.

Therefore $v \equiv \omega$. Similarly we can show that $v \equiv \omega$ in the case $\omega \geq v$.

Now let v be an arbitrary $L^2(\Omega)$ -bounded solution on R . For each $n \in N$, put $u_{0,n} = \rho(v)(-nT) \vee \rho(\omega)(-nT)$ and let v_n be the weak solution to $E(K(t), \rho, f)$ on $[-nT, \infty)$ satisfying $\rho(v_n)|_{t=-nT} = u_{0,n}$. Comparison result implies that $v_n \geq v \vee \omega$ on $[-nT, \infty)$. Also $L^2(\Omega)$ -boundedness on v implies the uniform $L^2(\Omega)$ -boundedness of $\{v_n\}_{n \in N}$. Therefore there is a subsequence $\{n_k\}$ of $\{n\}$ and an $L^2(\Omega)$ -bounded solution v^* on R such that

$$v_{n_k}(t) \rightarrow v^*(t) \quad \text{in } L^2(\Omega) \text{ and weakly in } H^1(\Omega) \quad \text{as } k \rightarrow \infty$$

for all $t \in R$.

Clearly $v^* \geq v \vee \omega$ on R . Therefore from the argument before we have $v^* \equiv \omega$. Similarly there is an $L^2(\Omega)$ -bounded solution v_* on R such that $v_* \leq v \wedge \omega$. And $v_* \equiv \omega$. Hence we have $v \equiv \omega$.

Next we shall show the asymptotic stability of the T -periodic solution ω . Let v be any weak solution on R_+ . Then as before there are weak solutions $\bar{v}$ and $\underline{v}$ such that $\underline{v} \leq v \wedge \omega \leq v \vee \omega \leq \bar{v}$. By Lemma 3, $\rho(\bar{v})(t+nT) - \rho(\omega)(t+nT) \rightarrow 0$ ($n \rightarrow \infty$). On the other hand (cf. [3; Lemma 5.4]), $|\rho(\bar{v})(t) - \rho(\omega)(t)|_{L^1(\Omega)} \leq |\rho(\bar{v})(s) - \rho(\omega)(s)|_{L^1(\Omega)}$ for all $0 \leq s \leq t < \infty$. So we have $\rho(\bar{v})(t) - \rho(\omega)(t) \rightarrow 0$ ($t \rightarrow \infty$): Similarly $\rho(\underline{v})(t) - \rho(\omega)(t) \rightarrow 0$ ($t \rightarrow \infty$). Since $\rho(\underline{v})(t) - \rho(\omega)(t) \leq \rho(v)(t) - \rho(\omega)(t) \leq \rho(\bar{v})(t) - \rho(\omega)(t)$, we obtain $\rho(v)(t) - \rho(\omega)(t) \rightarrow 0$ ($t \rightarrow \infty$).

q.e.d.

References

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