

# LEXICOGRAPHIC GRÖBNER BASES OF TORIC IDEALS ARISING FROM ROOT SYSTEMS

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ABSTRACT. The present paper is a brief draft based on a joint work with Takayuki Hibi. Gröbner bases of toric ideals arising from root systems are studied.

## INTRODUCTION

Let  $\mathcal{A} \subset \mathbb{Z}^n$  be a finite set and let  $K[t, t^{-1}, s] = K[t_1, t_1^{-1}, \dots, t_n, t_n^{-1}, s]$  denote the Laurent polynomial ring over a field  $K$ . We associate each  $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n$  with the monomial  $t^\alpha s = t_1^{\alpha_1} \dots t_n^{\alpha_n} s \in K[t, t^{-1}, s]$  and write  $\mathcal{R}_K[\mathcal{A}]$  for the subalgebra of  $K[t, t^{-1}, s]$  generated by all monomials  $t^\alpha s$  with  $\alpha \in \mathcal{A}$ . Let  $K[\mathbf{x}] = K[\{x_\alpha; \alpha \in \mathcal{A}\}]$  denote the polynomial ring in  $\#(\mathcal{A})$  variables over  $K$  and  $I_{\mathcal{A}} \subset K[\mathbf{x}]$  the kernel of the surjective homomorphism  $\pi : K[\mathbf{x}] \rightarrow \mathcal{R}_K[\mathcal{A}]$  defined by setting  $\pi(x_\alpha) = t^\alpha s$  for all  $\alpha \in \mathcal{A}$ . The ideal  $I_{\mathcal{A}}$  is called the *toric ideal* of the configuration  $\mathcal{A}$ . It is known [9] that if  $I_{\mathcal{A}}$  possesses a squarefree initial ideal, then the convex hull of  $\mathcal{A}$  possesses a unimodular triangulation.

Fix  $n \geq 2$ . Let  $\mathbf{e}_i$  denote the  $i$ -th unit coordinate vector of  $\mathbb{R}^n$ . We write  $\mathbf{A}_{n-1}^+$ ,  $\mathbf{B}_n^+$ ,  $\mathbf{C}_n^+$ ,  $\mathbf{D}_n^+$  and  $\mathbf{BC}_n^+$  for the set of positive roots of root systems  $\mathbf{A}_{n-1}$ ,  $\mathbf{B}_n$ ,  $\mathbf{C}_n$ ,  $\mathbf{D}_n$  and  $\mathbf{BC}_n$ , respectively ([3, pp. 64 – 65]):

$$\begin{aligned}\mathbf{A}_{n-1}^+ &= \{\mathbf{e}_i - \mathbf{e}_j; 1 \leq i < j \leq n\}; \\ \mathbf{B}_n^+ &= \{\mathbf{e}_i; 1 \leq i \leq n\} \cup \{\mathbf{e}_i + \mathbf{e}_j; 1 \leq i < j \leq n\} \cup \{\mathbf{e}_i - \mathbf{e}_j; 1 \leq i < j \leq n\}; \\ \mathbf{C}_n^+ &= \{2\mathbf{e}_i; 1 \leq i \leq n\} \cup \{\mathbf{e}_i + \mathbf{e}_j; 1 \leq i < j \leq n\} \cup \{\mathbf{e}_i - \mathbf{e}_j; 1 \leq i < j \leq n\}; \\ \mathbf{D}_n^+ &= \{\mathbf{e}_i + \mathbf{e}_j; 1 \leq i < j \leq n\} \cup \{\mathbf{e}_i - \mathbf{e}_j; 1 \leq i < j \leq n\}; \\ \mathbf{BC}_n^+ &= \mathbf{B}_n^+ \cup \mathbf{C}_n^+.\end{aligned}$$

Let, in addition,  $\tilde{\Phi}^+ = \Phi^+ \cup \{(0, 0, \dots, 0)\}$ , where  $\Phi = \mathbf{A}_{n-1}, \mathbf{B}_n, \mathbf{C}_n, \mathbf{D}_n$  or  $\mathbf{BC}_n$  and where  $(0, 0, \dots, 0)$  is the origin of  $\mathbb{R}^n$ .

In their combinatorial study of hypergeometric functions associated with root systems, Gelfand, Graev and Postnikov [2, Theorem 6.3] discovered a squarefree quadratic initial ideal of the toric ideal  $I_{\tilde{\mathbf{A}}_{n-1}^+}$  of  $\tilde{\mathbf{A}}_{n-1}^+$ . Moreover, for *any* subconfiguration  $\mathcal{A}$  of  $\mathbf{A}_{n-1}^+$ , the configuration  $\tilde{\mathcal{A}} = \mathcal{A} \cup (0, 0, \dots, 0)$  possesses a regular unimodular triangulation ([7, Example 2.4 (a)]). Stanley [8, Exercise 6.31 (b), p. 234] computed the Ehrhart polynomial of the convex polytope  $\text{conv}(\tilde{\mathbf{A}}_{n-1}^+)$ . Fong [1] constructed certain triangulations of the configurations  $\tilde{\mathbf{B}}_n^+ (= \text{conv}(\tilde{\mathbf{D}}_n^+) \cap \mathbb{Z}^n)$

and  $\text{conv}(\tilde{C}_n^+) \cap \mathbb{Z}^n (= \widetilde{BC}_n^+)$ , and computes the Ehrhart polynomials of  $\text{conv}(\tilde{B}_n^+)$  and  $\text{conv}(\tilde{C}_n^+)$ . The triangulations studied in [1] are, however, non-unimodular. Motivated by their results, Ohsugi–Hibi [6] showed that

**Proposition 0.1.** *Let  $\Phi \subset \mathbb{Z}^n$  be one of the root systems  $A_{n-1}$ ,  $B_n$ ,  $C_n$ ,  $D_n$  and  $BC_n$ . Then, there exists a reverse lexicographic order such that the initial ideal of  $I_{\Phi^+}$  is generated by squarefree quadratic monomials.*

Moreover, Ohsugi–Hibi [5] discussed subconfigurations  $\tilde{\mathcal{A}} = \mathcal{A} \cup \{(0, 0, \dots, 0)\}$  of  $\tilde{B}_n^+ \cup \tilde{C}_n^+$  which possesses a (regular) unimodular triangulation (i.e.,  $I_{\tilde{\mathcal{A}}}$  which possesses a squarefree initial ideal).

Hence, it is natural to study the same problem as above for  $I_{\Phi^+}$  where  $\Phi \subset \mathbb{Z}^n$  is one of the root systems  $A_{n-1}$ ,  $B_n$ ,  $C_n$ ,  $D_n$  and  $BC_n$ . (Then,  $I_{\Phi^+}$  is not generated by quadratic binomials if  $n \geq 6$ .)

### 1. SQUAREFREE LEXICOGRAPHIC INITIAL IDEALS

Let  $\Phi^+ \subset \mathbb{Z}^n$  denote one of the configurations  $A_{n-1}^+$ ,  $B_n^+$ ,  $C_n^+$ ,  $D_n^+$  and  $BC_n^+$ . Let  $K[A_{n-1}^+]$ ,  $K[B_n^+]$ ,  $K[C_n^+]$ ,  $K[D_n^+]$  and  $K[BC_n^+]$  denote the polynomial rings

$$\begin{aligned} K[A_{n-1}^+] &= K[\{f_{i,j}\}_{1 \leq i < j \leq n}], \\ K[B_n^+] &= K[\{y_i\}_{1 \leq i \leq n} \cup \{e_{i,j}\}_{1 \leq i < j \leq n} \cup \{f_{i,j}\}_{1 \leq i < j \leq n}], \\ K[C_n^+] &= K[\{a_i\}_{1 \leq i \leq n} \cup \{e_{i,j}\}_{1 \leq i < j \leq n} \cup \{f_{i,j}\}_{1 \leq i < j \leq n}], \\ K[D_n^+] &= K[\{e_{i,j}\}_{1 \leq i < j \leq n} \cup \{f_{i,j}\}_{1 \leq i < j \leq n}], \\ K[BC_n^+] &= K[\{a_i\}_{1 \leq i \leq n} \cup \{y_i\}_{1 \leq i \leq n} \cup \{e_{i,j}\}_{1 \leq i < j \leq n} \cup \{f_{i,j}\}_{1 \leq i < j \leq n}] \end{aligned}$$

over  $K$ . Write  $\pi : K[\Phi^+] \rightarrow K[t, t^{-1}, s]$  for the homomorphism defined by setting

$$\pi(a_i) = t_i^2 s, \quad \pi(y_i) = t_i s, \quad \pi(e_{i,j}) = t_i t_j s, \quad \pi(f_{i,j}) = t_i t_j^{-1} s.$$

Thus the kernel of  $\pi$  is the toric ideal  $I_{\Phi^+}$ .

First, an explicit initial ideals of  $I_{A_{n-1}^+}$  generated by squarefree monomials of degree  $\leq 3$  will be constructed. Let  $<_{lex}$  be the lexicographic order induced by the ordering of variables

$$f_{n-1,n} > f_{n-2,n-1} > f_{n-2,n} > \cdots > f_{1,2} > f_{1,3} > \cdots > f_{1,n},$$

and let  $<_{rev}$  be the reverse lexicographic order induced by the ordering of variables

$$f_{n-1,n} > f_{n-2,n} > f_{n-2,n-1} > \cdots > f_{2,3} > f_{1,n} > \cdots > f_{1,3} > f_{1,2}.$$

Then, the reduced Gröbner basis with respect to  $<_{lex}$  (and  $<_{rev}$ ) is as follows.

**Theorem 1.1** ([4]). *The set of the binomials*

$$\begin{aligned} f_{i,\ell} f_{j,k} - f_{i,k} f_{j,\ell}, & \quad i < j < k < \ell, \\ f_{i,j} f_{j,k} - f_{i,i+1} f_{i+1,k}, & \quad i+1 < j < k, \\ f_{i,j} f_{k,k+1} f_{k+1,\ell} - f_{i,i+1} f_{i+1,j} f_{k,\ell}, & \quad i+1 < j < k < \ell-1, \end{aligned}$$

is the reduced Gröbner basis of the toric ideal  $I_{A_{n-1}^+}$  with respect to both  $<_{lex}$  and  $<_{rev}$ , where the initial monomial of each binomial is the first monomial.

Then, we can associate the initial ideal of  $I_{\mathbf{A}_{n-1}^+}$  with respect to  $<_{lex}$  with the regular unimodular triangulation  $\Delta_{<_{lex}}$ . A graph-theoretical characterization of the maximal faces of the triangulation  $\Delta_{<_{lex}}$  is given in [4].

Second, we discuss the existence of squarefree initial ideals of the toric ideal  $I_{\Phi^+}$  where  $\Phi \subset \mathbb{Z}^n$  is one of the root systems  $\mathbf{B}_n$ ,  $\mathbf{C}_n$ ,  $\mathbf{D}_n$  and  $\mathbf{BC}_n$ . The similar argument as in [5] plays an important role in the proof of Theorems 1.2 and 1.4.

Let  $<_{lex}^c$  be the lexicographic order induced by the ordering of variables

$$\begin{aligned} a_1 &> a_2 > \cdots > a_n \\ &> f_{n-1,n} > f_{n-2,n-1} > f_{n-2,n} > \cdots > f_{1,2} > f_{1,3} > \cdots > f_{1,n} \\ &> e_{n-1,n} > e_{n-2,n-1} > e_{n-2,n} > \cdots > e_{1,2} > e_{1,3} > \cdots > e_{1,n}. \end{aligned}$$

**Theorem 1.2.** *The initial ideal of the toric ideal  $I_{\mathbf{C}_n^+}$  with respect to  $<_{lex}^c$  is generated by squarefree monomials.*

Let  $<_{lex}^d$  denote the lexicographic order obtained by restricting  $<_{lex}^c$  to  $K[\mathbf{D}_n^+]$ . By the elimination property of the lexicographic order  $<_{lex}^c$ , we have the following corollary from Theorem 1.2.

**Corollary 1.3.** *The initial ideal of the toric ideal  $I_{\mathbf{D}_n^+}$  with respect to  $<_{lex}^d$  is generated by squarefree monomials.*

We now consider the root systems  $\mathbf{B}_n$  and  $\mathbf{BC}_n$ . Let  $<_{lex}^{bc}$  be the lexicographic order induced by the ordering of variables

$$\begin{aligned} a_1 &> a_2 > \cdots > a_n \\ &> e_{n-1,n} > e_{n-2,n-1} > e_{n-2,n} > \cdots > e_{1,2} > e_{1,3} > \cdots > e_{1,n} \\ &> y_1 > y_2 > \cdots > y_n \\ &> f_{n-1,n} > f_{n-2,n-1} > f_{n-2,n} > \cdots > f_{1,2} > f_{1,3} > \cdots > f_{1,n}. \end{aligned}$$

**Theorem 1.4.** *The initial ideal of the toric ideal  $I_{\mathbf{BC}_n^+}$  with respect to  $<_{lex}^{bc}$  is generated by squarefree monomials.*

Let  $<_{lex}^b$  denote the lexicographic order obtained by restricting  $<_{lex}^{bc}$  to  $K[\mathbf{B}_n^+]$ . By the elimination property of the lexicographic order  $<_{lex}^{bc}$ , we have the following corollary from Theorem 1.4.

**Corollary 1.5.** *The initial ideal of the toric ideal  $I_{\mathbf{B}_n^+}$  with respect to  $<_{lex}^b$  is generated by squarefree monomials.*

**Remark 1.6.** Let  $n \geq 6$  and let  $\Phi^+$  denote one of the configurations  $\mathbf{A}_{n-1}^+$ ,  $\mathbf{B}_n^+$ ,  $\mathbf{C}_n^+$ ,  $\mathbf{D}_n^+$  and  $\mathbf{BC}_n^+$ . Then  $I_{\Phi^+}$  is not generated by quadratic binomials. Hence, in particular,  $I_{\Phi^+}$  does not possess a quadratic Gröbner basis.

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