

## Augmenting Edge-Connectivity and Vertex-Connectivity Simultaneously

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**Abstract** Given an undirected multigraph  $G = (V, E)$  and requirement functions  $\{r_\lambda(x, y) \in Z^+ \mid x, y \in V\}$  and  $\{r_\kappa(x, y) \in Z^+ \mid x, y \in V\}$  (where  $Z^+$  is the set of nonnegative integers), the edge and vertex-connectivities augmentation problem asks to augment  $G$  by adding the smallest number of new edges to  $G$  so that for every  $x, y \in V$ , the edge-connectivity and vertex-connectivity between  $x$  and  $y$  are at least  $r_\lambda(x, y)$  and  $r_\kappa(x, y)$ , respectively in the resulting graph  $G'$ . In this paper, we show that if  $r_\kappa(x, y) = 2$  holds for every  $x, y \in V$ , then the problem can be solved in polynomial time.

### 1 Introduction

Let  $G = (V, E)$  stand for an undirected multigraph with a set  $V$  of vertices and a set  $E$  of edges, where an edge with end vertices  $u$  and  $v$  is denoted by  $(u, v)$ . A singleton set  $\{x\}$  may be simply denoted by  $x$ . For two disjoint subsets of vertices  $X, Y \subset V$ , we denote by  $E_G(X, Y)$  the set of edges, one of whose end vertices is in  $X$  and the other is in  $Y$ , and also denote  $c_G(X, Y) = |E_G(X, Y)|$ . In particular,  $E_G(u, v)$  implies the set of edges with end vertices  $u$  and  $v$ . We denote  $n = |V|$  and  $e = |E|$ . For a subset  $V' \subseteq V$  in  $G$ ,  $G - V'$  denotes the subgraph induced by  $V - V'$ . A cut is defined as a subset  $X$  of  $V$  with  $\emptyset \neq X \neq V$ , and the size of a cut  $X$  is denoted by  $c_G(X, V - X)$ , which may also be written as  $c_G(X)$ . A cut with the minimum size is called a (global) minimum cut, and its size, denoted by  $\lambda(G)$ , is called the edge-connectivity of  $G$ . The local edge-connectivity  $\lambda_G(x, y)$  for two vertices  $x, y \in V$  is defined to be the minimum size of a cut in  $G$  that separates  $x$  and  $y$  (i.e.,  $x$  and  $y$  belong to different sides of  $X$  and  $V - X$ ), or equivalently the maximum number of edge-disjoint path between  $x$  and  $y$  by Menger's theorem [4].

For a subset  $X$  of  $V$ ,  $\{v \in V - X \mid (u, v) \in E \text{ for some } u \in X\}$  is called the neighbor set of  $X$ , denoted by  $\Gamma_G(X)$ . Let  $p(G)$  denote the number of components in  $G$ . A separator of  $G$  is defined as a cut  $S$  of  $V$  such that  $p(G - S) > p(G)$  holds and no  $S' \subset S$  has this property. A separator always exists, unless  $G$  contains the complete graph  $K_n$ . If  $G$  does not contain  $K_n$ , then a separator of the minimum size is called a (global) minimum separator, and its size, denoted by  $\kappa(G)$ , is called the vertex-connectivity of  $G$ . If  $G$  contains the complete graph  $K_n$ , we define  $\kappa(G) = n - 1$ . The local vertex-connectivity  $\kappa_G(x, y)$  for two vertices  $x, y \in V$  is defined to be the number of internally-disjoint paths between  $x$  and  $y$  in  $G$ .

For any separator  $S$ , there is the component  $X$  of  $G$  such that  $X \supseteq S$ , and we call the components in  $G[X] - S$  the  $S$ -components. Let

$$\beta(G) = \max\{p(G - S) \mid S \text{ is a minimum separator in } G\}. \quad (1.1)$$

A cut  $T \subset V$  is called tight if  $\Gamma_G(T)$  is a minimum separator in  $G$  and no  $T' \subset T$  has this property (hence,  $G[T]$  induces a connected graph). Let  $t(G)$  denotes the maximum number of pairwise disjoint tight sets in  $G$ .

In this paper, for a given function  $a: \binom{V}{2} \rightarrow R^+$  (resp.,  $b: \binom{V}{2} \rightarrow R^+$ ), where  $R^+$  denotes the set of nonnegative real numbers, we call  $G$   $a$ -edge-connected (resp.,  $b$ -vertex-connected) if  $\lambda_G(x, y) \geq a(x, y)$  (resp.,  $\kappa_G(x, y) \geq b(x, y)$ ) holds for every  $x, y \in V$ . Given a multigraph  $G = (V, E)$  and a requirement function  $r_\lambda: \binom{V}{2} \rightarrow Z^+$ , (resp., a requirement function  $r_\kappa: \binom{V}{2} \rightarrow Z^+$ ), where  $Z^+$  denotes the set of nonnegative integers, the edge-connectivity augmentation problem, (resp., the vertex-connectivity augmentation problem) asks to augment  $G$  by adding the smallest number of new edges so that the resulting graph  $G'$  becomes  $r_\lambda$ -edge-connected (resp.,  $r_\kappa$ -vertex-connected). When the requirement function  $r_\lambda$  (resp.,  $r_\kappa$ ) satisfies  $r_\lambda(x, y) = k \in Z^+$  for all  $x, y \in V$  (resp.,  $r_\kappa(x, y) = \ell \in Z^+$  for all  $x, y \in V$ ), this problem is called the global  $k$ -edge-connectivity problem (resp., the global  $\ell$ -vertex-connectivity problem).

Watanabe and Nakamura [16] first proved that the global  $k$ -edge-connectivity augmentation problem can be solved in polynomial time for any given integer  $k$ . Their algorithm increases edge-connectivity one by one, each time augmenting edges on the basis of structural information of the current  $G$ . Currently,  $O(e + k^2 n \log n)$  time algorithm due to Gabow [6] and  $\tilde{O}(n^3)$  time randomized algo-

rithm due to Benczúr [1], whose deterministic running time is  $O(n^4)$ , are the fastest among existing algorithms. Different from the approach by Watanabe and Nakamura, Cai and Sun [2] first pointed out that the augmentation problem for a given  $k$  can be directly solved by applying the Mader's edge-splitting theorem. Based on this, Frank [5] gave an  $O(n^5)$  time augmentation algorithm. Afterwards, Gabow [7] and Nagamochi and Ibaraki [14] improved it to  $O(mn^2 \log(n^2/m))$  and  $O(n^2(m + n \log n))$ , respectively. Recently, Nagamochi and Ibaraki [15] gave an  $O(n(m + n \log n) \log n)$  time algorithm. For a general requirement function  $r_\lambda$ , Frank [5] showed that the edge-connectivity augmentation problem can be solved in polynomial time by using Mader's edge-splitting theorem, and recently the time complexity was improved by Gabow [7] to  $O(n^3 m \log(n^2/m))$ .

As to the vertex-connectivity augmentation problem, the problem of adding the minimum number of new edges to a  $k$ -vertex-connected graph to make it  $(k+1)$ -vertex-connected has been studied by several researchers. It is easy to see that  $M(G) = \max\{\beta(G) - 1, \lceil t(G)/2 \rceil\}$  provides a lower bound on the optimal value to this problem. Eswaran and Tarjan [3] proved that the vertex-connectivity augmentation problem can be solved by adding  $M(G)$  edges to  $G$  for  $k = 1$ . Watanabe and Nakamura [17] stated the same result for  $k = 2$ . However,  $M(G)$  can be smaller than the optimal value for general  $k \geq 3$ . Recently Jordán presented an  $O(n^5)$  time approximation algorithm for this problem [11, 12]. The difference between the number of new edges added by his algorithm and the optimal value is at most  $(k-2)/2$ .

It is known that if the requirement function  $r_\kappa$  satisfies  $r_\kappa(x, y) = k$  for all  $x, y \in V$ , where  $k \in \{2, 3, 4\}$ , then the global  $k$ -vertex-connectivity augmentation problem can be solved in polynomial time due to [3, 9], [17, 8], [10], where an input graph  $G$  may not be  $(k-1)$ -vertex-connected. However, whether there is an polynomial time algorithm for the global vertex-connectivity augmentation problem for an arbitrary  $k$  is an open question (even if  $G$  is  $(k-1)$ -vertex-connected).

In this paper, we consider the problem of augmenting the edge-connectivity and the vertex-connectivity of a given graph  $G$  simultaneously by adding the smallest number of new edges. For two given functions  $a : \binom{V}{2} \rightarrow R^+$  and  $b : \binom{V}{2} \rightarrow R^+$ , we say that  $G$  is  $(a, b)$ -connected if  $G$  is  $a$ -edge-connected and  $b$ -vertex-connected.

Given a multigraph  $G = (V, E)$ , and two requirement functions  $r_\lambda : \binom{V}{2} \rightarrow Z^+$  and  $r_\kappa : \binom{V}{2} \rightarrow Z^+$ , the *edge-and-vertex-connectivity augmentation problem*, denoted by  $\text{EVAP}(r_\lambda, r_\kappa)$ , asks to augment  $G$  by adding the smallest number of new edges to  $G$  so that the resulting graph  $G'$  becomes  $(r_\lambda, r_\kappa)$ -connected. Without loss of generality,  $r_\lambda(x, y) \geq r_\kappa(x, y)$  is assumed for all  $x, y \in V$ , since if a graph is  $r_\kappa$ -vertex-connected then it is  $r_\kappa$ -edge-connected. Clearly,

$\text{EVAP}(r_\lambda, r_\kappa)$  contains the edge-connectivity augmentation problem and the vertex-connectivity augmentation problem as its special cases.

When the requirement function  $r_\kappa$  satisfies  $r_\kappa(x, y) = \ell \in Z^+$  for all  $x, y \in V$ , this problem is denoted by  $\text{EVAP}(r_\lambda, \ell)$ , if no confusion arises. In this paper, we consider this problem in case  $r_\kappa(x, y) = 2$  holds for every  $x, y \in V$  (but  $r_\lambda(x, y)$  are arbitrary). We first present a lower bound on the number of edges that is necessary to make a given graph  $G$   $(r_\lambda, 2)$ -connected. We then show that this problem can be solved in polynomial time, by actually presenting a polynomial time algorithm that adds a new edge set whose size is equal to this lower bound.

In Section 2, after introducing basic definitions and the concept of edge-splitting, we derive a lower bound on the number of edges that are necessary to make a given graph  $G$   $(r_\lambda, r_\kappa)$ -connected. In Section 3, we outline our algorithm for making a given graph  $G$   $(r_\lambda, 2)$ -connected by adding a new edge set whose size is equal to the above lower bound. In Sections 4 – 7, we prove the correctness of each step in our algorithm.

## 2 Preliminaries

### 2.1 Definitions

For a multigraph  $G = (V, E)$ , its vertex set  $V$  and edge set  $E$  may be denoted by  $V[G]$  and  $E[G]$ , respectively. For a subset  $V' \subseteq V$  (resp.,  $E' \subseteq E$ ) in  $G$ ,  $G[V']$  (resp.,  $G[E']$ ) denotes the subgraph induced by  $V'$  (resp.,  $E'$ ). For  $V' \subseteq V$  (resp.,  $E' \subseteq E$ ) in  $G$ , we denote  $G[V - V']$  (resp.,  $G[E - E']$ ) simply by  $G - V'$  (resp.,  $G - E'$ ). For an edge set  $F$  with  $F \cap E = \emptyset$ , we denote  $G = (V, E \cup F)$  by  $G + F$ . A *partition*  $X_1, \dots, X_t$  of vertex set  $V$  means a family of nonempty disjoint subsets of  $V$  whose union is  $V$ , and a *subpartition* of  $V$  means a partition of a subset of  $V$ .

We say that a cut  $X$  *separates* two disjoint subsets  $Y$  and  $Y'$  of  $V$  if  $Y \subseteq X$  and  $Y' \subseteq V - X$  (or  $Y \subseteq V - X$  and  $Y' \subseteq X$ ) hold. In particular, a cut  $X$  separates  $x$  and  $y$  if  $x \in X$  and  $y \in V - X$  (or  $x \in V - X$  and  $y \in X$ ) hold. A cut  $X$  *crosses* another cut  $Y$  if none of subsets  $X \cap Y$ ,  $X - Y$ ,  $Y - X$  and  $V - (X \cup Y)$  is empty. We say that a separator  $S \subset V$  separates two disjoint subsets  $Y$  and  $Y'$  of  $V - S$  if no two vertices  $x \in Y$  and  $y \in Y'$  are connected in  $G - S$ . In particular, a separator  $S$  separates vertices  $x$  and  $y$  in  $V - S$  if  $x$  and  $y$  are contained in different components of  $G - S$ .

### 2.2 Edge-Splitting

In this section, we introduce an operation of transforming a graph, called *edge-splitting*, which is helpful to solve the edge-connectivity augmentation problem.

Given a multigraph  $G = (V, E)$ , a designated vertex  $s \in V$ , vertices  $u, v \in \Gamma_G(s)$  (possibly  $u = v$ ) and a nonnegative integer  $\delta \leq \min\{c_G(s, u), c_G(s, v)\}$ , we construct graph  $G' = (V, E')$  from  $G$  by deleting  $\delta$  edges from  $E_G(s, u)$  and  $E_G(s, v)$ , respectively, and adding new  $\delta$  edges to  $E_G(u, v)$ :

$$\begin{aligned} c_{G'}(s, u) &:= c_G(s, u) - \delta, \\ c_{G'}(s, v) &:= c_G(s, v) - \delta, \\ c_{G'}(u, v) &:= c_G(u, v) + \delta, \\ c_{G'}(x, y) &:= c_G(x, y) \text{ for all other pairs } x, y \in V. \end{aligned}$$

In case of  $u = v$ , we interpret that  $c_{G'}(s, u) := c_G(s, u) - 2\delta$ ,  $c_{G'}(u, u) := c_G(u, u) + 2\delta$ , and  $c_{G'}(x, y) := c_G(x, y)$  for all other pairs  $x, y \in V$ , where an integer  $\delta$  is chosen so as to satisfy  $0 \leq \delta \leq \frac{1}{2}c_G(s, u)$ . We say that  $G'$  is obtained from  $G$  by *splitting*  $\delta$  pair of edges  $(s, u)$  and  $(s, v)$  (or by splitting  $(s, u)$  and  $(s, v)$  by size  $\delta$ ), and denote the resulting graph  $G'$  by  $G/(u, v; \delta)$ . A sequence of splittings is *complete* if the resulting graph  $G'$  does not have any neighbor of  $s$ .

The following theorem is proven by Mader [13].

**Theorem 2.1** [13] *Let  $G = (V, E)$  be a multigraph with a designated vertex  $s \in V$  with  $c_G(s) \neq 1, 3$  and  $\lambda_G(x, y) \geq 2$  for all pairs  $x, y \in V$ . Then for any edge  $(s, u) \in E$  there is an edge  $(s, v) \in E$  such that  $\lambda_{G/(u, v; 1)}(x, y) = \lambda_G(x, y)$  holds for all pairs  $x, y \in V - s$ .*  $\square$

This says that if  $c_G(s)$  is even, there always exists a complete splitting at  $s$  such that the resulting graph  $G'$  satisfies  $\lambda_{G'-s}(x, y) = \lambda_G(x, y)$  for every pair of  $x, y \in V - s$ .

### 2.3 Lower Bound

In this section, we consider problem  $\text{EVAP}(r_\lambda, r_\kappa)$ , and give a lower bound on the number of edges that is necessary to make a graph  $G$   $(r_\lambda, r_\kappa)$ -connected, where  $r_\lambda$  and  $r_\kappa$  are given requirement functions. Define

$r_\lambda(X) \equiv \max\{r_\lambda(u, v) \mid u \in X, v \in V - X\}$   
for each cut  $X$ ,  
 $r_\kappa(X) \equiv \max\{r_\kappa(u, v) \mid u \in X, v \in V - X - \Gamma_G(X)\}$   
for each cut  $X$  with  $V - X - \Gamma_G(X) \neq \emptyset$ , where see Section 1 for the definition of  $\Gamma_G(X)$ . To make a graph  $G$   $r_\lambda$ -edge-connected, it is necessary to add

- (1) at least  $r_\lambda(X) - c_G(X)$  edges between  $X$  and  $V - X$  for each cut  $X$ .

Also, to make a graph  $G$   $r_\kappa$ -vertex-connected, it is necessary to add

- (2) at least  $r_\kappa(X) - |\Gamma_G(X)|$  edges between  $X$  and  $V - X - \Gamma_G(X)$  for each cut  $X$  with  $V - X - \Gamma_G(X) \neq \emptyset$ .

For a separator  $S$  of  $G$ , let  $T_1, \dots, T_q$  denote all components of  $G - S$ . Now we consider a graph  $H_S = (\{T_1, \dots, T_q\}, \mathcal{E})$  in which we regard each  $T_i$  as one vertex of  $H_S$  and the edge set  $\mathcal{E}$  is defined as

follows:

$$(T_i, T_j) \in \mathcal{E} \longleftrightarrow \begin{aligned} &\text{There is a pair of vertices} \\ &x \in T_i \text{ and } y \in T_j \text{ with} \\ &r_\kappa(x, y) \geq |S| + 1. \end{aligned}$$

In a  $r_\kappa$ -vertex-connected graph, any pair of vertices  $x, y \in V$  with  $r_\kappa(x, y) \geq |S| + 1$  cannot be separated by such separator  $S$ . Hence if there is a pair of vertices  $x \in T_i$  and  $y \in T_j$  with  $r_\kappa(x, y) \geq |S| + 1$ , then we must add at least one edge between  $T_i$  and  $T_j$  (i.e., the number of  $S$ -components must become at most  $p(H_S)$ ), in order to make  $G$   $r_\kappa$ -vertex-connected. Therefore in this case, it is necessary to add

- (3) at least  $p(G - S) - p(H_S)$  edges to connect components of  $G - S$  for a separator  $S$ .

(See Section 1 for the definition of  $p(G - S)$ .) Now define  $\delta(G) = \max\{p(G - S) - p(H_S) \mid S \text{ is a separator in } G\}$ .

Given a subpartition  $\{X_1, \dots, X_p, X_{p+1}, \dots, X_q\}$  of  $V$  such that  $q \geq p \geq 0$  and  $V - X_i - \Gamma_G(X_i) \neq \emptyset$  ( $i = p+1, \dots, q$ ), we need to add  $\max\{r_\lambda(X_i) - c_G(X_i), 0\}$  edges for each  $X_i$ ,  $i = 1, \dots, p$ , and to add  $\max\{r_\kappa(X_i) - |\Gamma_G(X_i)|, 0\}$  edges for each  $X_i$ ,  $i = p+1, \dots, q$ , based on observations (1) and (2). Now note that adding one edge to  $G$  can contribute to the requirements of at most two  $X_i$ . Therefore, we need to add  $\lceil \alpha(G)/2 \rceil$  new edges to make  $G$   $(r_\lambda, r_\kappa)$ -edge-connected, where

$$\alpha(G) = \max \left\{ \sum_{i=1}^p (r_\lambda(X_i) - c_G(X_i)) + \sum_{i=p+1}^q (r_\kappa(X_i) - |\Gamma_G(X_i)|) \right\}, \quad (2.1)$$

and the max is taken over all subpartitions  $\{X_1, \dots, X_p, X_{p+1}, \dots, X_q\}$  of  $V$  such that  $q \geq p \geq 0$  and  $V - X_i - \Gamma_G(X_i) \neq \emptyset$ ,  $i = p+1, \dots, q$ . On the other hand, from observation (3), to make  $G$   $r_\kappa$ -vertex-connected, at least  $\max\{p(G - S) - p(H_S) \mid S \text{ is a separator in } G\}$  new edges are necessarily added to  $G$ . Consequently, we have the next lemma.

**Lemma 2.1 (Lower Bound)** *To make a given graph  $G$   $(r_\lambda, r_\kappa)$ -connected, at least*

$$\gamma(G) \equiv \max\{\lceil \alpha(G)/2 \rceil, \delta(G)\}$$

*new edges must be added.*  $\square$

Now we specialize this lower bound to problem  $\text{EVAP}(r_\lambda, 2)$  based on which we give a polynomial time algorithm for solving  $\text{EVAP}(r_\lambda, 2)$  in the next section.

In problem  $\text{EVAP}(r_\lambda, 2)$ , we can assume  $r_\lambda(x, y) \geq r_\kappa(x, y) = 2$  for all  $x, y \in V$ . Now the  $\alpha(G)$  in (2.1) can be simplified to

$$\alpha(G) = \max \left\{ \sum_{i=1}^p (r_\lambda(X_i) - c_G(X_i)) + \sum_{i=p+1}^q (2 - |\Gamma_G(X_i)|) \right\}, \quad (2.2)$$

where the maximization is taken over all subpartitions  $\{X_1, \dots, X_p, X_{p+1}, \dots, X_q\}$  of  $V$  such that  $q \geq p \geq 0$  and  $V - X_i - \Gamma_G(X_i) \neq \emptyset$  for  $i = p+1, \dots, q$ .

Also we specialize the second lower bound  $\delta(G)$ . Now, to derive  $\delta(G)$ , the maximization is taken over all separators  $S$  that satisfy  $|S| \leq 1$ , since each pair of vertices  $x, y \in V$  satisfy  $r_\kappa(x, y) = 2$ . Note that  $p(H_S) = 1$  holds for any separator  $S$  with  $|S| \leq 1$ , since any pair of  $S$ -components  $T_i$  and  $T_j$  has a pair of vertices  $x \in T_i$  and  $y \in T_j$  where  $r_\kappa(x, y) = 2 > |S|$ . Hence this lower bound can be rewritten by

$$\max\{p(G - S) - 1 \mid S \text{ is a separator with } |S| \leq 1\}. \quad (2.3)$$

A vertex  $v$  is called a *cut vertex* in  $G = (V, E)$  if  $S = \{v\}$  is a minimum separator in  $G$ . If  $G$  has a cut vertex  $v \in V$ , then  $p(G - v) > p(G)$  holds from the definition of a separator; otherwise  $p(G - v) = p(G)$  holds for all  $v \in V$ . Hence the lower bound in (2.3) can be simplified to

$$\max_{v \in V} \{p(G - v) - 1\}.$$

Also note that if  $\kappa(G) \leq 1$  holds, then (1.1) in Section 1 satisfies  $\beta(G) = \max_{v \in V} \{p(G - v)\}$  and the lower bound in (2.3) can be simplified to  $\beta(G) - 1$ . In case of  $\kappa(G) \geq 2$ , the lower bound in (2.3) is not defined but  $\max_{v \in V} \{p(G - v) - 1\} = 0$  holds. Therefore, in Problem EVAP( $r_\lambda, 2$ ), we can define the lower bound in (2.3) by  $\max_{v \in V} \{p(G - v) - 1\}$  without confusion. This means that we can define

$$\beta(G) = \max_{v \in V} \{p(G - v)\}. \quad (2.4)$$

and the lower bound in (2.3) becomes

$$\beta(G) - 1.$$

Now define  $\gamma(G) = \max\{\lceil \alpha(G)/2 \rceil, \beta(G) - 1\}$ . From the above discussion, a set of new edges gives an optimal solution to EVAP( $r_\lambda, 2$ ) if its size is equal to  $\gamma(G)$  and the graph obtained by adding  $\gamma(G)$  edges to  $G$  is  $(r_\lambda, 2)$ -connected. We now show that this is always possible, by presenting a polynomial time algorithm in the next section for making  $G$   $(r_\lambda, 2)$ -connected by adding  $\gamma(G)$  new edges.

**Lemma 2.2** *If  $\kappa(G) = 1$  (i.e.,  $G$  is connected and has a cut vertex), then any two tight sets  $X$  and  $Y$  in  $G$  are disjoint.*  $\square$

### 3 A Polynomial Time Algorithm for EVAP( $r_\lambda, 2$ )

We now present a polynomial time algorithm, based on the argument in the previous section. Call an edge  $e = (u, u')$  *admissible* with respect to a vertex  $v$ , if  $v$  is a cut vertex such that  $v \neq u, u'$  and  $p(G - v) = p((G - e) - v)$ . For a subset  $F$  of edges in a graph  $G$ , we say that two edge  $e_1 = (u_1, w_1)$  and  $e_2 = (u_2, w_2)$  are *switched* in  $F$  if we delete  $e_1$  and  $e_2$  from  $F$ , and add edges  $(u_1, u_2)$  and  $(w_1, w_2)$  to  $F$ . Our algorithm for solving the EVAP( $r_\lambda, 2$ ), denoted by Algorithm EVA( $r_\lambda, 2$ ), consists of the following four major steps.

#### Algorithm EVA( $r_\lambda, 2$ )

**Input:** An undirected multigraph  $G = (V, E)$ , and a requirement function  $\{r_\lambda(x, y) \in \mathbb{Z}^+ \mid x, y \in V\}$ .

**Output:** An undirected multigraph  $G^* = G + F$  with  $\lambda_{G^*}(x, y) \geq r_\lambda(x, y)$  for every  $x, y \in V$  and  $\kappa(G^*) \geq 2$  where the size of new edge set  $F$  is the minimum.

#### Step I. (Addition of vertex $s$ and associated edges):

After adding a new vertex  $s$ , add a set  $F'$  of a sufficiently large number of edges between  $s$  and  $V$  so that the resulting graph  $G' = (V \cup \{s\}, E \cup F')$  satisfies

$$c_{G'}(X) \geq r_\lambda(X) \quad (3.1)$$

for all  $X$  with  $\emptyset \neq X \subset V$ ,

$$|\Gamma_{G'}(X \cup s)| \geq 2 \quad (3.2)$$

for all  $X$  with  $\emptyset \neq X \subset V$  and  $V - X - \Gamma_{G'}(X) \neq \emptyset$ . (This can be done for example by adding  $\max\{r_\lambda(x, y) \mid x, y \in V\}$  edges between  $s$  and each vertex  $v \in V$ .)

Next, to make  $F'$  minimal we discard new edges in  $F'$ , one by one, as long as (3.1) and (3.2) remain valid. Denote the resulting set of new edges by  $F_1$  and the resulting graph by  $G_1 = (V \cup \{s\}, E \cup F_1)$ , where  $F_1 = E_{G_1}(s, V)$ .

Clearly, these operations can be performed in polynomial time. We claim the next.

**Remark:** Note that if the original graph  $G$  is not connected, then  $\kappa_{G_1}(x, y) \geq 2$  cannot be attained for some  $x, y \in V$ , since a subset  $X \subset V$  which induces a component  $G[X]$  of  $G$  satisfies  $\Gamma_{G_1}(X) = \emptyset$  or  $\{s\}$ , and hence  $\kappa_{G_1}(x, y) \leq 1$  for  $x \in X$  and  $y \in V - X$ .

**Property 3.1** *In the above step, it is possible to choose a subset  $F_1$  for which  $|F_1| = \alpha(G)$  holds.*  $\square$

**Step II. (Edge-splitting):** If  $c_{G_1}(s)$  is odd, then we add one edge  $(s, w)$  to  $G$  by choosing vertex an arbitrary  $w \in V$  which is not a cut vertex in  $G$ .

Next we find a complete edge-splitting at  $s$  in  $G_1 = (V \cup \{s\}, E \cup F_1)$  which preserves condition (3.1) (i.e., the  $r_\lambda$ -edge-connectivity). By Mader's theorem, there always exists such a complete edge-splitting at  $s$ , and it can be computed in polynomial time. Let  $G_2 = (V, E \cup F_2)$  denote the graph obtained by such a complete edge-splitting, ignoring the isolated vertex  $s$ . The next is immediate from Mader's theorem.

**Property 3.2** *There is a complete edge-splitting at  $s$  of  $G_1$ , so that the resulting graph  $G_2$  is  $r_\lambda$ -edge-connected.*  $\square$

If  $G_2$  is also 2-vertex-connected, then we are done because  $|F_2| = |F_1|/2 = \lceil \alpha(G)/2 \rceil$  implies that  $G_2$  is optimally augmented by lower bound  $\lceil \alpha(G)/2 \rceil$ . Otherwise, go to Step III.

**Step III. (Switching edges):** Now  $G_2$  has cut vertices. Then, by property (3.2) for  $G_1$ ,  $G_2$  satisfies

$$G_2[X \cup \{v\}] \text{ contains at least one edge in } F_2 \text{ for any cut vertex } v \text{ and its } v\text{-component } X. \quad (3.3)$$

**Property 3.3** *Assume that  $G_2$  has an admissible edge  $e_1 \in F_2$  with respect to a cut vertex  $v$ . Let  $X$  be a  $v$ -component with  $e_1 \notin E[G_2[X \cup \{v\}]]$ , and  $e_2$  be chosen arbitrarily from  $F_2 \cap E[G_2[X \cup \{v\}]]$ . Then switching  $e_1$  and  $e_2$  decreases the number of  $v$ -components in  $G_2$  at least by one while preserving the  $r_\lambda$ -edge-connectivity. Moreover, the resulting graph  $G'_2$  from switching  $e_1$  and  $e_2$  still satisfies (3.3), and  $\kappa_{G'_2}(x, y) \geq 2$  holds for any pair of vertices  $x$  and  $y$  with  $\kappa_{G_2}(x, y) \geq 2$ .*  $\square$

**Property 3.4** *If  $G_2$  has two cut vertices  $v_1$  and  $v_2$ , then there are  $v_1$ -component  $X_1$  and  $v_2$ -component  $X_2$  such that  $X_1 \cap X_2 = \emptyset$ . Let edge  $e_1$  be arbitrarily chosen from  $F_2 \cap E[G_2[X_1 \cup \{v_1\}]]$ . Then  $e_1$  is admissible with respect to  $v_2$ .*  $\square$

Based on Property 3.3, Step III repeats switching pairs of edges in  $F_2$  until the resulting graph has no admissible edge in  $F_2$ .

Let  $G_3 = (V, E \cup F_3)$  be the resulting graph obtained by such a sequence of switching edges in  $F_2$ , where  $F_3$  denotes the final  $F_2$ . Then Property 3.4 implies that, if there are at least two cut vertices, then  $G_3$  has an admissible edge in  $F_3$ , which is a contradiction. Hence  $G_3$  has the following property.

**Property 3.5**  $G_3$  has at most one cut vertex.  $\square$

If  $G_3$  has no cut vertex, then we are done, since  $|F_3| = \lceil \alpha(G)/2 \rceil$  implies that  $G_3$  is optimally augmented. Otherwise, go to Step IV.

**Step IV. (Edge augmentation):** Now  $G_3$  has exactly one cut vertex  $v$ . Then  $G_3$  and  $v$  satisfy the following property.

**Property 3.6** *For the graph  $G_3$  and its cut vertex  $v$ , it holds  $p(G_3 - v) = p(G - v) - \lceil \alpha(G)/2 \rceil$ .*  $\square$

Now let  $T_1, \dots, T_q$  be all  $v$ -components in  $G_3$ , where  $q = p(G_3 - v)$ . We can make  $G_3$  2-vertex-connected by adding one edge between  $T_i$  and  $T_{i+1}$  for each  $i = 1, \dots, q-1$  (i.e.,  $p(G_3 - v) - 1$  edges in total). Let  $F_4$  denote a set of these  $p(G_3 - v) - 1$  edges added. Note that  $p(G_3 - v) = p(G - v) - \lceil \alpha(G)/2 \rceil \leq \beta(G) - \lceil \alpha(G)/2 \rceil$  holds from Property 3.6 and  $\beta(G) \geq p(G - v)$  (see (2.4)). Also note that  $|F_4| + |F_3| = p(G_3 - v) - 1 + \lceil \alpha(G)/2 \rceil \geq \beta(G) - 1$  holds since  $\beta(G) - 1$  is a lower bound on the number of edges that must be added to make  $G$   $(r_\lambda, 2)$ -connected. These imply  $|F_4| = \beta(G) - 1 - \lceil \alpha(G)/2 \rceil$ . Therefore we have the following property.

**Property 3.7** *There is a set of  $\beta(G) - 1 - \lceil \alpha(G)/2 \rceil$  new edges  $F_4$  obtained for  $G_3$  such that the resulting graph  $G_4 = (V, E \cup F_3 \cup F_4)$  is 2-vertex-connected.*

Finally, we are done since  $|F_3| + |F_4| = \beta(G) - 1$  implies that  $G_4$  is optimally augmented by lower bound  $\beta(G) - 1$ .

We shall explain in the subsequent sections that the required properties (summarized as Properties 3.1 – 3.7) always hold. Together with these proofs, this algorithm establishes the next theorem, which is the main goal of this thesis.

**Theorem 3.1** *Given a requirement function  $\{r_\lambda(x, y) \in \mathbb{Z}^+ \mid x, y \in V\}$ , a multigraph  $G$  can be made  $(r_\lambda, 2)$ -connected by adding  $\gamma(G) = \max \{\lceil \alpha(G)/2 \rceil, \beta(G) - 1\}$  new edges in  $O(n^3 m \log \frac{n^2}{m})$  time.*  $\square$

## 4 Correctness of Step I

We give a proof of Property 3.1 in order to prove the correctness of Step I.

**Proof of Property 3.1:** It is clear that  $\lambda_{G_1}(x, y) \geq r_\lambda(x, y) \geq 2$  holds for all  $x, y \in V$  by (3.1).

First, we show  $|F_1| \geq \alpha(G)$ . Let  $\mathcal{F}^* = \{X_1^*, \dots, X_p^*, X_{p+1}^*, \dots, X_q^*\}$  be a subpartition of  $V$

with  $V - X_i^* - \Gamma_{G_1}(X_i^*) \neq \emptyset$  for  $i = p+1, \dots, q$  that attains the maximum of (2.2); i.e.,  $\alpha(G) = \sum_{i=1}^p (r_\lambda(X_i^*) - c_G(X_i^*)) + \sum_{i=p+1}^q (2 - |\Gamma_G(X_i^*)|)$ . If  $|F_1| < \alpha(G)$  holds, then there must be at least one cut  $X_i^* \in \mathcal{F}^*$  that violates (3.1) or (3.2), contradicting construction of  $G_1$ .

Now we prove the converse,  $|F_1| \leq \alpha(G)$ , through five claims.

A cut  $X \subset V$  is called *critical* in  $G_1$  if  $s \in \Gamma_{G_1}(X)$  holds and the removal of any edge  $e \in E_{G_1}(s, X)$  violates (3.1) or (3.2). Clearly, a subset  $X \subset V$  with  $s \in \Gamma_{G_1}(X)$  is critical if and only if  $X$  satisfies at least one of the following conditions:

- (1)  $c_{G_1}(X) = r_\lambda(X)$ .
- (2)  $c_{G_1}(s, X) = 1$ ,  $|\Gamma_{G_1}(X) - s| = 1$ , and  $V - X - \Gamma_{G_1}(X) \neq \emptyset$ .
- (3)  $\Gamma_{G_1}(X) = \{s\}$ ,  $|\Gamma_{G_1}(s) \cap X| = 2$ , and there is a vertex  $v \in \Gamma_{G_1}(s) \cap X$  with  $c_{G_1}(s, v) = 1$ .

We call a critical cut  $X$  *v-minimal* if  $v \in \Gamma_{G_1}(s) \cap X$  and there is no critical cut  $X'$  with  $\{v\} \subseteq X' \subset X$ . A subset  $X$  is called *critical of type (1)* (resp., (2), (3)) if it satisfies (1) (resp., (2), (3)).

We will prove that  $G_1$  has a set of critical cuts  $X_1, \dots, X_q$  only of type (1) and (2) such that

$$\begin{aligned} X_i \cap X_j &= \emptyset, \quad 1 \leq i < j \leq q \text{ and} \\ \Gamma_{G_1}(s) &\subseteq X_1 \cup \dots \cup X_q, \end{aligned} \quad (4.1)$$

This implies that

$$|F_1| = \left\{ \sum_{i=1}^p (r_\lambda(X_i) - c_G(X_i)) + \sum_{i=p+1}^q (2 - |\Gamma_G(X_i)|) \right\}$$

where  $X_i, i = 1, \dots, p$  is of type (1) and  $X_i, i = p+1, \dots, q$  is of type (2), from which  $|F_1| \leq \alpha(G)$  by definition of  $\alpha(G)$ .

**Claim 4.1** Any critical cut  $X$  of type (3) is also critical of type (1).  $\square$

By this claim, we can regard critical cuts of type (3) as those of type (1). The next property is known in [5].

**Claim 4.2** Let  $X$  and  $Y$  be critical cuts of type (1) in  $G_1$ . Then at least one of the following statements holds.

- (i) Both  $X \cap Y$  and  $X \cup Y$  are critical.
- (ii) Both  $X - Y$  and  $Y - X$  are critical, and  $c_{G_1}(X \cap Y, (V \cup \{s\}) - (X \cup Y)) = 0$ .  $\square$

An analogous property holds for type (2) critical cuts.

**Claim 4.3** Let  $X$  and  $Y$  be critical cuts of type (2). If  $Y$  is *v-minimal* for some  $v \in V - X$ , then they do not cross each other.  $\square$

**Claim 4.4** Let  $X$  be a critical cut of type (1), and  $Y$  be a critical cut of type (2) such that  $\Gamma_{G_1}(s) \cap (Y - X) \neq \emptyset$ . If  $X$  and  $Y$  cross each other, then  $c_{G_1}(X \cap Y, s) = 0$  holds and cut  $Y - X$  is critical of type (1).  $\square$

Now we are ready to prove that  $G_1$  has a set of critical cuts  $X_1, \dots, X_q$  that satisfies (4.1). Let  $N_1 \subseteq \Gamma_{G_1}(s)$  be the set of neighbors  $u$  of  $s$  such that there is a critical cut  $X$  of type (1) with  $u \in X$ . Let us choose a critical cut  $X_u$  of type (1) with  $u \in X_u$  for each  $u \in N_1$  so that  $\sum_{X \in \{X_u | u \in N_1\}} |X|$  is minimized. Denote such a set  $\{X_u | u \in N_1\}$  by  $\mathcal{F}_1$ . For  $N_2 = \Gamma_{G_1}(s) - N_1$ , we choose a  $u$ -minimal critical cut  $X_u$  for each  $u \in N_2$ , and let  $\mathcal{F}_2 = \{X_u | u \in N_2\}$ . Then we claim the next.

**Claim 4.5**  $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$  consists of disjoint critical cuts whose union contains  $\Gamma_{G_1}(s)$ .

**Proof.** Let  $\mathcal{F}_1 = \{X_1, \dots, X_p\}$  and  $\mathcal{F}_2 = \{X_{p+1}, \dots, X_q\}$  with each  $\emptyset \neq X_i \subset V$ . Clearly,  $\Gamma_{G_1}(s) \subseteq \bigcup_{X_i \in \mathcal{F}} X_i$  holds from construction of  $\mathcal{F}$ .

We show that  $X_i$  and  $X_j$  are pairwise disjoint for each  $X_i, X_j \in \mathcal{F}_1$ . Assume that  $\mathcal{F}_1$  contains  $X_i$  and  $X_j$  which are not pairwise disjoint. Note that  $X_i \subset X_j$  does not hold from construction of  $\mathcal{F}_1$ . If  $X_i$  and  $X_j$  cross each other, then Claim 4.2 implies that at least one of the following statements holds:

- (i) Both  $X_i \cap X_j$  and  $X_i \cup X_j$  are critical.
- (ii) Both  $X_i - X_j$  and  $X_j - X_i$  are critical, and  $c_{G_1}(X \cap Y, (V \cup \{s\}) - (X \cup Y)) = 0$ .

If the statement (i) holds, then  $\mathcal{F}'_1 = (\mathcal{F}_1 - X_i - X_j) \cup \{X_i \cup X_j\}$  would satisfy  $N_1 \subseteq \mathcal{F}'_1$  and  $\sum_{X \in \mathcal{F}'_1} |X| < \sum_{X \in \mathcal{F}_1} |X|$ , contradicting the minimality of  $\sum_{X \in \mathcal{F}_1} |X|$ . If the statement (ii) holds, then  $\mathcal{F}'_1 = (\mathcal{F}_1 - X_i - X_j) \cup \{X_i - X_j, X_j - X_i\}$  satisfies  $\sum_{X \in \mathcal{F}'_1} |X| < \sum_{X \in \mathcal{F}_1} |X|$  and  $N_1 \subseteq \mathcal{F}'_1$  (by  $c_{G_1}(X \cap Y, (V \cup \{s\}) - (X \cup Y)) = 0$ ). This again contradicts the minimality of  $\sum_{X \in \mathcal{F}_1} |X|$ . Therefore  $X_i$  and  $X_j$  are pairwise disjoint for each  $X_i, X_j \in \mathcal{F}_1$ .

Claim 4.3 implies that  $X_i$  and  $X_j$  are pairwise disjoint for each  $X_i, X_j \in \mathcal{F}_2$ .

Finally, we show that  $X_i$  and  $X_j$  are pairwise disjoint for each  $X_i \in \mathcal{F}_1$  and  $X_j \in \mathcal{F}_2$ . Note that  $\Gamma_{G_1}(s) \cap (X_j - X_i) \neq \emptyset$  holds from definition of  $N_1$ . Then  $X_j \subset X_i$  does not hold. Also note that  $X_i \subset X_j$  does not hold, otherwise  $\Gamma_{G_1}(s) \cap X_i \neq \emptyset$  and  $\Gamma_{G_1}(s) \cap (X_j - X_i) \neq \emptyset$  imply  $c_{G_1}(X_j, s) \geq c_{G_1}(X_i, s) + 1 \geq 2$ , contradicting that  $X_j$  is of type (2). Assume that  $X_i$  and  $X_j$  cross each other. Now  $\Gamma_{G_1}(s) \cap (X_j - X_i) \neq \emptyset$  holds. Therefore Claim 4.4 implies that  $c_{G_1}(s, X_i \cap X_j) = 0$  holds and  $X_j - X_i$  is a critical cut of type (1). This implies that any vertex in  $X_j$  cannot belong to  $N_2$ , contradicting  $X_j \in \mathcal{F}_2$ .  $\square$

Clearly  $\mathcal{F}$  is a subpartition of  $V$  by Claim 4.5. Since  $\Gamma_{G_1}(s) \subseteq X_1 \cup \dots \cup X_q$  with  $X_i \in \mathcal{F}$  holds, it

holds

$$|F_1| = \sum_{i=1}^p (r_\lambda(X_i) - c_G(X_i)) + \sum_{i=p+1}^q (2 - |\Gamma_G(X_i)|),$$

for  $\mathcal{F}_1 = \{X_1, \dots, X_p\}$  and  $\mathcal{F}_2 = \{X_{p+1}, \dots, X_q\}$ . From definition of  $\alpha(G)$ , we have  $|F_1| \leq \alpha(G)$ .  $\square$

## 5 Correctness of Step II

Let  $G_1 = (V \cup \{s\}, E \cup F_1)$  be the graph obtained from a given graph  $G = (V, E)$  after Step I. In this section, we describe about the correctness of Property 3.2 and the purpose of operations in case where  $c_{G_1}(s)$  is odd.

In Step II, a graph  $G_2 = (V, E \cup F_2)$  is constructed from  $G_1$  by a complete edge-splitting at  $s$ . Then the correctness of Property 3.2 is immediate from Mader's theorem (see Theorem 2.1).

In this step, a non cut vertex  $w$  is chosen when we add an extra edge  $(s, w)$  to  $G_1$  if  $c_{G_1}(s)$  is odd. Such choice of  $w$  will be used for the correctness of Step IV in Section 7 (i.e., by this choice of  $w$ , we will be able to make  $G$   $(r_\lambda, 2)$ -connected by adding  $\beta(G) - 1$  new edges in case of  $\beta(G) - 1 > \lceil \alpha(G)/2 \rceil$ ).

## 6 Correctness of Step III

Let  $G_2 = (V, E \cup F_2)$  be the graph obtained in Step II. Now  $G_2$  is 2-edge-connected but has cut vertices.

In order to justify Step III, we now prove Property 3.3 in Step III.

**Proof of Property 3.3:** We prove Property 3.3 via two claims.

**Claim 6.1** *Let  $v \in V$  denote a cut vertex in  $G_2$ . Assume that a  $v$ -component  $T$  contains an admissible edge  $e = (u, u')$  with respect to  $v$ . Then  $G_2[T] - e$  contains a path  $P$  between  $u$  and  $u'$ .*  $\square$

**Claim 6.2** *Let  $e_1 = (u_1, w_1)$  and  $e_2 = (u_2, w_2)$  be the edges in the statement of Property 3.3. Then the graph  $G'_2 = (V, E \cup F'_2)$  obtained by switching  $e_1$  and  $e_2$ , where  $F'_2 = F_2 \cup \{(u_1, u_2), (w_1, w_2)\} - \{e_1, e_2\}$ , satisfies followings:*

- (i)  $\lambda_{G'_2}(x, y) \geq r_\lambda(x, y)$  for every  $x, y \in V$ .
  - (ii)  $p(G'_2 - v) < p(G_2 - v)$ .
  - (iii)  $\kappa_{G'_2}(x, y) \geq 2$  holds for every pair of vertices  $x$  and  $y$  that satisfies  $\kappa_{G_2}(x, y) \geq 2$ .
- (The statements (ii) and (iii) and Lemma 2.2 imply that switching  $e_1$  and  $e_2$  decreases the number  $t(G_2)$  of tight sets in  $G_2$  by at least one if  $e_1$  or  $e_2$  is contained in a tight set in  $G_2$ .)

**Proof.** (i) We assume that there is a cut  $X$  such that  $c_{G'_2}(X) \leq r_\lambda(X) - 1$  holds. Note that  $c_{G_2}(X) \leq c_{G'_2}(X)$  holds if cut  $X$  does not separate  $\{u_1, u_2\}$

and  $\{w_1, w_2\}$  in  $G'_2$ . Since  $c_{G_2}(X) \geq r_\lambda(X)$  originally holds, cut  $X$  separates  $\{u_1, u_2\}$  and  $\{w_1, w_2\}$  and hence  $c_{G'_2}(X) = c_{G_2}(X) - 2$  holds. Since the cut  $X$  crosses both  $v$ -components  $T_1$  and  $T_2$  in  $G_2$ , either  $G_2[X]$  or  $G_2[V - X]$  consists of at least two components. Without loss of generality, assume that  $G_2[X]$  consists of at least two components. There are vertices  $x^* \in X$  and  $y^* \in V - X$  such that  $r_\lambda(x^*, y^*) = r_\lambda(X) \geq c_{G'_2}(X) + 1$ . Without loss of generality, assume that  $x^* \in X \cap T_1$ . Note that  $c_{G_2}(X \cap T_2) \geq r_\lambda(X \cap T_2) \geq 2$  and  $c_{G_2}(X \cap T_1) \geq r_\lambda(X \cap T_1) \geq r_\lambda(x^*, y^*) \geq c_{G'_2}(X) + 1$  hold. This implies  $c_{G_2}(X) = c_{G_2}(X \cap T_1) + c_{G_2}(X \cap T_2) \geq (c_{G'_2}(X) + 1) + 2$ , contradicting  $c_{G'_2}(X) = c_{G_2}(X) - 2$ .

(ii) It is sufficient to show that  $G'_2[T_1 \cup T_2]$  is connected. Since the removal of the admissible edge  $e_1$  does not increase the number of  $v$ -components,  $T_1$  remains a  $v$ -component in  $G_2 - e_1$ . If  $T_2$  remains a  $v$ -component in  $G_2 - e_2$ , then  $G[T_1]$  and  $G[T_2]$  are joined by the edges  $(u_1, u_2)$  and  $(w_1, w_2)$  obtained by switching  $e_1$  and  $e_2$  in  $G'_2$ . If  $T_2$  consists of two components  $T_2^1$  and  $T_2^2$  in  $G_2 - e_2$ , then  $u_2 \neq v \neq w_2$  holds and  $u_2$  and  $w_2$  are separated by  $T_2^1$ . Assume  $u_2 \in T_2^1$  and  $w_2 \in T_2^1$  without loss of generality. Now  $T_2^1$  (resp.,  $T_2^2$ ) and  $T_1$  are joined by the edges  $(u_1, u_2)$  (resp.,  $(w_1, w_2)$ ). This implies that  $G'_2[T_1 \cup T_2]$  is a component since  $T_1$  remains a  $v$ -component in  $G_2 - e_1$ . Therefore if  $v$  remains a cut vertex in  $G'_2$ , then  $T_1 \cup T_2$  is a  $v$ -component (otherwise, clearly,  $p(G_2 - v) = 1$ ).

(iii) Assume that there are vertices  $x, y \in V$  such that  $\kappa_{G_2}(x, y) = 2$  but  $\kappa_{G'_2}(x, y) = 1$ . Let  $v' \in V$  denote a cut vertex in  $G'_2$  that separates  $x$  and  $y$ . Clearly,  $v' \neq v$  (because  $v = v'$  would imply  $\kappa_{G_2}(x, y) = 1$ ). Let  $W_1, W_2, \dots, W_q$  ( $q \geq 2$ ) be the  $v'$ -components of  $G'_2$ , where  $x \in W_1$  and  $y \in W_2$ . Since a cut vertex  $v'$  does not separate  $x$  and  $y$  in  $G_2$ ,  $e_1 \in E_{G_2}(W_1, W_2)$  or  $e_2 \in E_{G_2}(W_1, W_2)$  holds. Also note that no edge other than  $e_1$  and  $e_2$  can not belong to  $E_{G_2}(W_1, W_2)$ . We can easily see that  $G_2[W_1 \cup W_2 \cup \{v'\}]$  contains  $u_1, w_1, u_2$ , and  $w_2$ . Then note that  $u_i, w_i \in W_j$  cannot hold for any  $i, j$  with  $1 \leq i \leq j \leq 2$ . Otherwise (assume  $u_1, w_1 \in W_1$  without loss of generality) then  $e_2 \in E_{G_2}(W_1, W_2)$  holds (assume  $u_2 \in W_1$  and  $w_2 \in W_2$  without loss of generality). Now  $(w_1, w_2) \in E_{G'_2}(W_1, W_2)$  holds and  $G'_2[W_1]$  and  $G'_2[W_2]$  are both connected from definition of  $W_1$  and  $W_2$ , contradicting that cut vertex  $v'$  separates  $x$  and  $y$  in  $G'_2$ . Therefore, for each  $i = 1, 2$ , we have now  $e_i = (u_i, w_i) \in E_{G_2}(W_1, W_2)$  or  $u_i = v'$  or  $w_i = v'$ .

We first consider the case of  $e_1 \in E_{G_2}(W_1, W_2)$ . Then  $v' \in T_1$  holds since  $G_2[T_1] - e_1$  is connected by Claim 6.1. Hence  $e_2 \in E_{G_2}(W_1, W_2)$  holds since  $v' \in T_1$  implies  $u_2 \neq v' \neq w_2$ . Let  $v \notin W_2$  and  $u_1, u_2 \in W_1$  without loss of generality. Now  $\Gamma_{G'_2}(T_2 \cap W_2) \cap (T_2 - W_2) = \emptyset$  holds since  $v'$  is a cut vertex of  $G'_2$  and  $v' \notin T_2$  hold. Note that  $E_{G'_2}(T_2 \cap W_2, V -$

$(T_2 \cap W_2) = \{(w_1, w_2)\}$  since  $T_2$  is a  $v$ -component of  $G_2$  and  $u_2 \in W_1$  holds. This implies  $\Gamma_{G_2}(T_2 \cap W_2) = \{u_2\}$  holds and hence  $e_2$  is a bridge of  $G_2$  from  $E_{G_2}(W_1, W_2) = \{e_1, e_2\}$ , which contradicts  $\lambda(G_2) \geq 2$ .

We then consider the case of  $e_1 \notin E_{G_2}(W_1, W_2)$  holds, i.e.,  $v' = u_1 \in T_1$  or  $v' = w_1 \in T_1$  holds. This implies that  $e_2 \in E_{G_2}(W_1, W_2)$  holds and  $v' \notin T_2$ . Therefore, this clearly leads to a contradiction, in a similar way to above case of  $e_1 \in E_{G_2}(W_1, W_2)$ .  $\square$

From the above claim, Property 3.3 is proved.

## 7 Correctness of Step IV

Let  $G_3 = (V, E \cup F_3)$  be obtained from  $G_2$  after Step III. Now clearly  $|F_3| = \lceil \alpha(G)/2 \rceil$ . This  $G_3$  has exactly one cut vertex  $v$ .

The correctness of Step IV clearly follows if we prove Property 3.6. The proof is now given below via two claims.

**Claim 7.1**  $G_3$  has no edge in  $F_3$  incident to the cut vertex  $v$ .  $\square$

**Claim 7.2**  $p(G-v) = p(G_3-v) + |F_3|$  holds. That is, deleting any edge  $e \in F_3$  increases the number of  $v$ -components in  $G_3$ .

**Proof.** If  $p(G-v) < p(G_3-v) + |F_3|$  holds, then there is at least one edge  $e \in F_3$  with  $p((G_3 - e) - v) = p(G_3 - v)$ . Then  $e$  is admissible with respect to  $v$  since Claim 7.1 implies that any edge in  $F_3$  is not incident to  $v$ , contradicting construction of  $G_3$ .  $\square$

This claim implies that since  $G_3$  has no edge in  $F_3$  incident to the cut vertex  $v$ , a graph  $H = (W, F_3)$  is a forest, where a vertex set  $W$  of  $H$  is obtained by removing the cut vertex  $v$  and contracting each component of  $G-v$  to one vertex. Now Claim 7.2 implies Property 3.6 since  $|F_3| = \lceil \alpha(G)/2 \rceil$  holds from construction.

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