

# ON A PRINCIPLE OF OGUS: THE HASSE INVARIANT'S ORDER OF VANISHING AND “FROBENIUS AND THE HODGE FILTRATION”

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## 1. INTRODUCTION

This work is inspired by the following result by Ogus in [Ogu01b] and [Ogu01a]. Let  $k$  be an algebraically closed field of characteristic  $p > 0$ . Let  $X \rightarrow S$  be a family of Calabi-Yau<sup>1</sup> varieties over  $k$ . Under some technical conditions on  $X \rightarrow S$ , including degeneration of the Hodge spectral sequence at  $E_1$ , Ogus proves that the order of vanishing of the Hasse invariant on  $S$  is equal to the “conjugate line position”, i.e. the largest piece of the Hodge filtration containing the line of the conjugate filtration. We call this equality *Ogus’ principle*.

**1.1. Main results.** We introduce a group-theoretical Ogus’ principle associated to triples  $(G, \mu, r)$ , where  $G$  is a connected, reductive  $\mathbf{F}_p$ -group,  $\mu: \mathbf{G}_m \rightarrow G_k$  is a cocharacter, and  $r$  is a representation of  $G$ . To such a triple we associate a global section  $\text{Ha}(G, \mu, r)$  of a line bundle on  $G\text{-Zip}^\mu$ , the stack of  $G$ -zips of type  $\mu$ , which generalizes the classical Hasse invariant defined via de Rham cohomology. We say that Ogus’ principle is satisfied if the vanishing order of  $\text{Ha}(G, \mu, r)$  equals the conjugate line position of the pushforward along  $r$  of the universal  $G$ -zip of type  $\mu$  over  $G\text{-Zip}^\mu$ . We prove Ogus’ principle for several triples  $(G, \mu, r)$ .

**Theorem 1.1.1.** *Ogus’ principle holds for the following triples  $(G, \mu, r)$ :*

- (a)  $\text{type}(G) = \mathbf{A}_1^d$ ,  $\mu$  is regular and  $r$  is faithful of dimension  $2d$
- (b)  $G = \text{GL}(n)$ ,  $\text{type}(L) = \mathbf{A}_{n-2}$  and  $r = \wedge^n(\text{Std} \oplus \text{Std}^\vee)$ ,
- (c)  $G = \text{GSp}(2n)$ ,  $\text{type}(L) = \mathbf{A}_{n-1}$  and  $r = \wedge^n \text{Std}^\vee$ ,
- (d)  $G = \text{SO}(2n+1)$ ,  $\text{type}(L) = \mathbf{B}_{n-1}$  and  $r = \text{Std}$
- (e)  $G = \text{SO}(2n)$ ,  $\text{type}(L) = \mathbf{D}_{n-1}$  and  $r = \text{Std}$
- (f)  $G = \text{GL}(4)$ ,  $\text{type}(L) = \mathbf{A}_1 \times \mathbf{A}_1$  and  $r = \wedge^2(\text{Std})$ .

*Remark 1.1.2.* We also give an explicit value of the vanishing order of the Hasse invariant  $\text{Ha}(G, \mu, r)$  on each Ekedahl-Oort and Bruhat stratum of  $G\text{-Zip}^\mu$ . This is also done for  $(G, \mu, r)$  where  $G$  is a spin similitude group of type  $\mathbf{B}_m$  (resp.  $\mathbf{D}_m$ ),  $\text{type}(L) = \mathbf{B}_{m-1}$  (resp.  $\mathbf{D}_{m-1}$ ) and  $r$  is the spin representation.

Let  $(\mathbf{G}, \mathbf{X})$  be an abelian-type Shimura datum. Assume that  $\mathbf{G}$  is unramified at  $p$ . Let  $K_p \subset \mathbf{G}(\mathbf{Q}_p)$  be a hyperspecial maximal compact subgroup. By the work of Kisin [Kis10] and Vasiu [Vas99], as  $K^p$  ranges over open, compact subgroups of  $\mathbf{G}(\mathbf{A}_f^p)$ , the associated projective system of Shimura varieties admits an integral canonical model  $(\mathcal{S}_{K_p K^p}(\mathbf{G}, \mathbf{X}))_{K^p}$  in the sense of Milne [Mil92]. Set  $K := K_p K^p$  and let  $S_K$  be the special  $k$ -fiber of  $\mathcal{S}_{K_p K^p}(\mathbf{G}, \mathbf{X})$ . Let  $G$  be the reductive  $\mathbf{F}_p$ -group deduced from the  $\mathbf{Q}$ -group  $\mathbf{G}$  and let  $\mu \in X_*(G)$  be a representative of the conjugacy class of cocharacters deduced from the Hermitian symmetric space  $\mathbf{X}$ . By Zhang [Zha18] there is a smooth morphism  $\zeta_K: S_K \rightarrow G\text{-Zip}^\mu$ . We deduce the following.

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<sup>1</sup>An  $n$ -dimensional variety  $X$  is Calabi-Yau if  $\dim H^0(X, \mathcal{O}_X) = \dim H^n(X, \mathcal{O}_X) = 1$  and  $H^i(X, \mathcal{O}_X) = 0$  for all  $i \neq 0, n$ . The vanishing of  $H^1(X, \mathcal{O}_X)$  excludes abelian varieties.

**Corollary 1.1.3.** Ogus’ principle holds for  $S_K$  if the  $\mathbf{Q}$ -group  $\mathbf{G}$  is either of the following:

- (a) A restriction of scalars  $\mathrm{Res}_{F/\mathbf{Q}}(\mathrm{GL}(2))$  for a totally real extension  $F/\mathbf{Q}$ .
- (b) A unitary group associated to an imaginary quadratic field  $K/\mathbf{Q}$  such that the base change  $\mathbf{G}_{\mathbf{R}}$  to  $\mathbf{R}$  is a unitary similitude group of signature either  $(n-1, 1)$  or  $(2, 2)$  and  $p$  splits in  $K$ .
- (c) The  $\mathbf{Q}$ -split symplectic group (the Siegel case).
- (d) A  $\mathbf{Q}$ -form of the orthogonal group  $\mathrm{SO}(m)$  such that  $\mathbf{G}_{\mathbf{R}} \cong \mathrm{SO}(m-2, 2)$ . If  $m$  is even, assume that  $G$  is  $\mathbf{F}_p$ -split.

*Remark 1.1.4.* In upcoming joint work with Jean-Stefan Koskivirta we extend (b) to arbitrary signatures at inert primes.

We make the following conjecture.

**Conjecture 1.1.5.** *Suppose that  $(\mathbf{G}, \mathbf{X})$  is of Hodge-type. Let  $Y/S_K$  be the abelian variety obtained from a symplectic embedding of  $(\mathbf{G}, \mathbf{X})$ . Let  $n := \dim(Y/S_K)$ . Assume that the conjugacy class of  $\mu$  is defined over  $\mathbf{F}_p$ . For every  $k$ -point  $x$  of  $S_K$ ,*

$$\mathrm{ord}_x \mathrm{Ha}(H_{\mathrm{dR}}^n(Y/S_K)) = \mathrm{cl}_p H_{\mathrm{dR}}^n(Y/S_K).$$

Finally, using Madapusi-Pera’s extension of the Kuga-Satake construction to mixed characteristic [Mad15] we also recover Ogus’ result for K3-surfaces.

**Corollary 1.1.6.** Let  $M_{2d,K}^\circ$  be the moduli space of K3 surfaces with level  $K$  and a polarization of degree  $2d$ . Then Ogus’ principle holds for the universal family  $Y/M_{2d,K}^\circ$ .

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#### 2. OGUS PRINCIPLE FOR FAMILIES MOD $p$

Let  $f: X \rightarrow S$  be a smooth proper morphism over  $k$  of dimension  $n$ . Let  $F_S: S \rightarrow S$  denote the absolute Frobenius morphism, let  $X^{(p)} := X \times_{S, F_S} S$  with structure morphism  $f^{(p)}: X^{(p)} \rightarrow S$ .

The de Rham cohomology  $H_{\mathrm{dR}}^i(X/S)$  comes equipped with two spectral sequences degenerating to it; the Hodge spectral sequence

$$E_{\mathrm{Hdg},1}^{a,b} = R^b f_* \Omega_{X/S}^a \implies H_{\mathrm{dR}}^{a+b}(X/S)$$

and the conjugate spectral sequence<sup>2</sup>

$$E_{\mathrm{conj},2}^{a,b} \cong R^a f_*^{(p)} \Omega_{X^{(p)}/S}^b \implies H_{\mathrm{dR}}^{a+b}(X/S).$$

We make the following assumptions.

**dR.1** For all  $a, b \geq 0$ , the sheaves  $R^a \pi_* \Omega_{X/S}^b$  are locally free;

**dR.2** the Hodge spectral sequence degenerates at  $E_1$ ; and

**dR.3** the sheaf  $R^n f_* \mathcal{O}_X$  is a line bundle.

<sup>2</sup>The conjugate spectral sequence as written here is not the one obtained from definition of  $H_{\mathrm{dR}}$  as a hypercohomology. Rather, we use here the Cartier isomorphisms.

The assumptions **dR.1** and **dR.2** above implies that the construction of the sheaves  $R^a\Omega_{X/S}^b$  commutes with arbitrary base change and that the conjugate spectral sequence degenerates at the second page. A list of families satisfying assumption **dR.1** and **dR.2** is copied from [MW04]:

- (a) Any abelian scheme  $A \rightarrow S$ .
- (b) Any smooth proper curve  $C \rightarrow S$ .
- (c) Any K3-surface  $X \rightarrow S$ .
- (d) Every smooth complete intersection in  $\mathbb{P}_S^n$ .

In particular, in the applications to Shimura varieties the assumptions are satisfied.

Let  $\text{Fil}_{\text{Hdg}}^\bullet$  denote the decreasing Hodge filtration on  $H_{\text{dR}}^n(X/S)$  and let  $\text{Fil}_{\text{conj},\bullet}$  denote the increasing conjugate filtration. For all  $i$  the Cartier isomorphism induces an isomorphism

$$\varphi_i: (\text{Gr}^i \text{Fil}_{\text{Hdg}}^\bullet)^{(p)} \rightarrow \text{Gr}_i \text{Fil}_{\text{conj},\bullet}.$$

See [Kat70] for details.

**2.1. The Hasse invariant and the conjugate line position.** The *Hasse invariant* is defined as the composition

$$\text{Ha}_S: (\text{Gr}^0 \text{Fil}_{\text{Hdg}}^\bullet)^{(p)} \xrightarrow{\varphi_0} \text{Gr}_0 \text{Fil}_{\text{conj},\bullet} \hookrightarrow H_{\text{dR}}^n(X/S) \twoheadrightarrow \text{Gr}^0 \text{Fil}_{\text{Hdg}}^\bullet.$$

By **dR.3**  $\text{Gr}_0 \text{Fil}_{\text{Hdg}}^\bullet = R^n f_* \mathcal{O}_X$  is a line bundle and thus  $\text{Ha}_S$  can be seen as a section of  $\omega^{p-1}$ , where  $\omega := \det f_* \Omega_{X/S}$ .

*Remark 2.1.1.* If  $X \rightarrow S$  is the universal principally polarized abelian variety (with some level structure), then the non-vanishing set of the Hasse invariant consists exactly of the ordinary abelian varieties (i.e., those whose  $p$ -rank is maximal). For cases of Shimura varieties, the Hasse invariant also satisfies other desirable properties such as being compatible with varying the prime to  $p$  level. Several authors have constructed generalisations of the classical Hasse invariant enjoying similar properties to those described above (see e.g., [GN17], [KW18], [Gor01]). Thanks to these properties, the Hasse invariant and its generalisations have been used successfully to produce congruences between automorphic forms and automorphic Galois representations in the Langlands program (see e.g., [ERX17], [Tay91], [GK19a]). The idea of using the Hasse invariant to this end goes back to the work of Deligne–Serre in the 1970’s on modular forms of weight 1 ([DS74]). The theory of Hasse invariants has also been used to study the geometry of Shimura varieties mod  $p$ . For instance, in [GK19a] it is shown that the Ekedahl–Oort (E–O) stratification of a Hodge-type Shimura variety is uniformly principally pure (see [GK19b] for a definition), and that each stratum in the E–O stratification of the minimal compactification is affine.

For any  $k$ -point  $s$  in  $S$ , the *conjugate line position*<sup>3</sup> is defined as

$$\text{clp}_s(H_{\text{dR}}^n(X/S)) := \min\{i : \text{Fil}_{\text{conj},0,s} \subset \text{Fil}_{\text{Hdg},s}^i\}.$$

**2.1.2. Ogus’ principle for families mod  $p$ .** We say that Ogus’ principle holds for  $X \rightarrow S$  if, for all  $k$ -points  $s$  in  $S$ , we have that

$$\text{ord}_s(\text{Ha}_S) = \text{clp}_s(H_{\text{dR}}^n(X/S)).$$

Ogus proves that this holds for families of Calabi-Yau varieties satisfying the following assumptions.

- (a) Same as assumption **dR.2**.
- (b) The Kodaira-Spencer map<sup>4</sup>  $\Theta: T_{S/k} = (\Omega_{S/k}^1)^\vee \rightarrow R^1 \pi_* T_{X/S}$  is surjective.

<sup>3</sup>This is sometimes referred to as the  $a$ -number. We prefer the notion conjugate line position as it is more descriptive.

<sup>4</sup>Defined e.g. as the coboundary map of  $R^1 \pi_*$  applied to the short exact sequence  $0 \rightarrow T_{X/S} \rightarrow T_{X/k} \rightarrow \pi^* T_{S/k} \rightarrow 0$ .

(c) For all pair of integers  $j \leq m < p$  the maps

$$\mathrm{Sym}^j R^1 \pi_* T_{X/S} \otimes \pi_* \Omega_{X/S}^1 \rightarrow R^j \pi_* \Omega_{X/S}^{n-j}$$

induced by cup product and interior multiplication, are surjective.

### 3. GROUP THEORETIC OGUS' PRINCIPLE

**3.1. Group theoretic notation.** Fix a connected, reductive  $\mathbf{F}_p$ -group  $G$  and a cocharacter  $\mu: \mathbf{G}_m \rightarrow G_k$ . It determines a pair of opposite parabolics  $P, P^+$  of  $G_k$  intersecting in a common Levi factor  $L := P \cap P^+ = \mathrm{Cent}_{G_k}(\mu)$ . Let  $Q := (P^+)^{(p)}$  with Levi quotient  $M$ .

Let  $\sigma: k \rightarrow k$  denote the arithmetic Frobenius  $a \mapsto a^p$ . Given a  $k$ -scheme  $X$ , let  $X^{(p)} := X \otimes_{k, \sigma} k$  be its Frobenius twist and let  $\varphi: X \rightarrow X^{(p)}$  be the relative Frobenius morphism. The unipotent radical of an algebraic  $k$ -group  $H$  is denoted  $R_u(H)$ . If  $g \in G$  and  $H \subset G$ , then let  ${}^g H := gHg^{-1}$ .

**3.2.  $G$ -zips.** The theory of  $G$ -zips were developed by Pink-Wedhorn-Ziegler in [PWZ11] and [PWZ15] and builds on the notion of  $F$ -zips, introduced and studied by Moonen-Wedhorn in [MW04]. Thereafter, Goldring-Koskivirta further developed the theory in [GK19b]. We present here the main definitions needed to state our results.

**3.2.1.  $F$ -zips.** Let  $\mathcal{V}$  be a locally free sheaf of rank  $n$  over a  $k$ -scheme  $S$ . By a descending filtration  $\mathrm{Fil}^\bullet$  on  $\mathcal{V}$  we mean a sequence of sub  $\mathcal{O}_S$ -modules  $\mathrm{Fil}^\bullet = (\mathrm{Fil}^i)_{i \in \mathbf{Z}}$  such that  $\mathrm{Fil}^i$  is a local direct summand of  $\mathrm{Fil}^{i-1}$  for all  $i$ , and such that  $\mathrm{Fil}^i = \mathcal{V}$  for all  $i$  small enough and  $\mathrm{Fil}^i = 0$  for all  $i$  large enough. We remark in particular that the graded pieces  $\mathrm{Gr}^i = \mathrm{Fil}^i / \mathrm{Fil}^{i-1}$  are locally free sheaves. We define an ascending filtration analogously and write  $\mathrm{Gr}_i$  for its graded pieces. An  $F$ -zip over  $S$  is a tuple  $\underline{\mathcal{Z}} = (\mathcal{V}, \mathrm{Fil}_{\mathrm{Hdg}}^\bullet, \mathrm{Fil}_{\bullet}^{\mathrm{conj}}, \varphi_\bullet)$  where  $\mathrm{Fil}_{\mathrm{Hdg}}^\bullet$  is a descending filtration on  $\mathcal{V}$ ,  $\mathrm{Fil}_{\bullet}^{\mathrm{conj}}$  is an ascending filtration on  $\mathcal{V}$  and for all  $i$ ,  $\varphi_i: (\mathrm{Gr}^i)^{(p)} \xrightarrow{\sim} \mathrm{Gr}_i$  is an isomorphism between graded pieces. Given a function  $\gamma: \mathbf{Z} \rightarrow \mathbf{N}$  we say that an  $F$ -zip has type  $\gamma$  if  $\gamma(i) = \dim \mathrm{Gr}^i$  for all  $i \in \mathbf{Z}$ . To give a type  $\gamma$  is equivalent to give a (conjugacy class of a) cocharacter  $\mu_\gamma$ .

**Example 3.2.2.** As we saw, for all  $i$  the de Rham cohomology  $H_{\mathrm{dR}}^i(X/S)$  has the structure of an  $F$ -zip.

**3.2.3.  $G$ -zips of type  $\mu$ .** A  $G$ -zip of type  $\mu$  over  $S$  is a tuple  $(I, I_P, I_Q, \iota)$  where  $I$  is a  $G_k$ -torsor over  $S$ ,  $I_P \subset I$  is a  $P$ -torsor over  $S$ ,  $I_Q \subset I^{(p)}$  is a  $Q$ -torsor over  $S$  and

$$\iota: I_P^{(p)} / R_u(P)^{(p)} \xrightarrow{\cong} I_Q / R_u(Q)$$

is an isomorphism of  $M$ -torsors. A morphism of  $G$ -zips  $(I, I_P, I_Q, \iota) \rightarrow (I', I'_P, I'_Q, \iota')$  is a morphism of  $G_k$ -torsors  $I \rightarrow I'$  compatible with the reductions to  $P$  and  $Q$  and compatible with the morphisms  $\iota, \iota'$ .

**Example 3.2.4.** If  $G = \mathrm{GL}(n)$  then to give an  $F$ -zip of type  $\gamma$  is equivalent to give a  $G$ -zip of type  $\mu_\gamma$ .

Let  $G\text{-Zip}^\mu(S)$  denote the category of  $G$ -zips (resp.  $G$ -zip flags) of type  $\mu$  over  $S$ . This construction give rise to a smooth algebraic  $k$ -stack  $G\text{-Zip}^\mu$ .

**3.2.5.  $F$ -zips of Calabi-Yau type.** Let  $\underline{\mathcal{Z}}$  be an  $F$ -zip over  $S$ , and let  $i_0$  be the largest integer such that  $\mathrm{Fil}_{\mathrm{Hdg}}^{i_0} = \mathcal{V}$ . We say that  $\underline{\mathcal{Z}}$  is of *CY-type* if  $\mathrm{Gr}_{\mathrm{Hdg}}^{i_0} := \mathrm{Fil}_{\mathrm{Hdg}}^{i_0} / \mathrm{Fil}_{\mathrm{Hdg}}^{i_0+1}$  is a line bundle.

**3.3. The Hasse invariant and conjugate line position of a triple  $(G, \mu, r)$ .** Let  $\underline{\mathcal{Y}}$  be an  $F$ -zip of Calabi-Yau type. We define a Hasse invariant analogously as to the classical case: the Hasse invariant of  $\underline{\mathcal{Y}}$  is defined as

$$\mathrm{Ha}(\underline{\mathcal{Y}}) : (\mathrm{Gr}^{i_0})^{(p)} \xrightarrow{\varphi^i} \mathrm{Gr}_{i_0} \rightarrow \mathcal{V} \rightarrow \mathrm{Gr}^{i_0}.$$

If  $\underline{\mathcal{Y}}$  is not of Calabi-Yau type, then let  $d := \dim \mathrm{Gr}^{i_0}$  and define  $\mathrm{Ha}(\underline{\mathcal{Y}})$  as  $\mathrm{Ha}(\wedge^d \underline{\mathcal{Y}})$ .

The conjugate line position is defined for any  $k$ -point  $s$  as

$$\mathrm{clp}_s(\underline{\mathcal{Y}}) := \max\{j \in \mathbf{Z} \mid \mathrm{Gr}_{i_0, s} \subset \mathrm{Gr}_s^j\} - i_0.$$

**3.3.1.** We say that a representation  $r: G \rightarrow \mathrm{GL}(V)$  is of *Calabi-Yau* type if the highest  $\mu$ -weight of  $V_k$  has multiplicity 1. Let  $\underline{\mathcal{Z}}$  denote the universal  $G$ -zip of type  $\mu$  over the stack  $G\text{-Zip}^\mu$ . By the associated product construction, the representation  $r$  produces an  $F$ -zip  $\underline{\mathcal{Z}}(r)$  on  $G\text{-Zip}^\mu$  which is of Calabi-Yau type since  $r$  is. We thus define the *Hasse invariant* of the triple  $(G, \mu, r)$  as

$$\mathrm{Ha}(G, \mu, r) := \mathrm{Ha}(\underline{\mathcal{Z}}(r)),$$

and similarly the *conjugate line position* of  $(G, \mu, r)$  is defined as

$$\mathrm{clp}_x(G, \mu, r) := \mathrm{clp}_x(\underline{\mathcal{Z}}(r)),$$

for all  $k$ -points  $x$  of  $G\text{-Zip}^\mu$ .

**3.4. The group theoretic Ogus' principle.** Let  $r$  be of CY-type. We say that the triple  $(G, \mu, r)$  satisfies Ogus' principle if for all  $k$ -points  $x$  of  $G\text{-Zip}^\mu$ , we have that

$$\mathrm{ord}_x(\mathrm{Ha}(G, \mu, r)) = \mathrm{clp}_x(\underline{\mathcal{Z}}(r)).$$

**Example 3.4.1.** By definition, the de Rham cohomology on  $X \rightarrow S$  induces a morphism  $\zeta: S \rightarrow G\text{-Zip}^\mu$  where  $G = \mathrm{GL}(\dim H_{\mathrm{dR}}(X/S))$  and  $\mu$  is the character obtained from the Hodge filtration. By construction,  $\mathrm{Ha}_S = \zeta^* \mathrm{Ha}(G, \mu, \mathrm{id})$ . Thus, if  $\zeta$  is smooth, then the group theoretic Ogus' principle implies the Ogus' principle for the family  $X \rightarrow S$ .

**3.5. Relation to Shimura varieties.** Let  $(\mathbf{G}, \mathbf{X})$  be an abelian-type Shimura datum. Assume that  $\mathbf{G}$  is unramified at  $p$ . Let  $K_p \subset \mathbf{G}(\mathbf{Q}_p)$  be a hyperspecial maximal compact subgroup. Let  $K^p$  range over open, compact subgroups of  $\mathbf{G}(\mathbf{A}_f^p)$ . By the work of Kisin [Kis10] and Vasiu [Vas99] the associated projective system of Shimura varieties admits an integral canonical model  $(\mathcal{S}_{K_p K^p}(\mathbf{G}, \mathbf{X}))_{K^p}$  in the sense of Milne [Mil92]. Set  $K := K_p K^p$  and let  $S_K$  be the special  $k$ -fiber of  $\mathcal{S}_{K_p K^p}(\mathbf{G}, \mathbf{X})$ .

Assume that  $(\mathbf{G}, \mathbf{X})$  is of Hodge type. For  $g \geq 1$ , let  $(\mathrm{GSp}(2g), \mathbf{X}_g)$  be the Siegel datum, consisting of the  $\mathbf{Q}$ -split symplectic similitude group  $\mathrm{GSp}(2g)$  and the Siegel double half-space  $\mathbf{X}_g$ . Given a symplectic embedding

$$(\mathbf{G}, \mathbf{X}) \hookrightarrow (\mathrm{GSp}(2g), \mathbf{X}_g),$$

for all sufficiently small  $K^p$  there exists a level  $K' \subset \mathrm{GSp}(2g, \mathbf{A}_f)$  and an induced finite map from  $S_K$  to the special  $k$ -fiber of the Siegel-type Shimura variety  $S_{g, K'}$  [Kis10, (2.3.3)]. Let  $Y/S_K$  be the resulting family of abelian schemes. The Zip period map associated to  $H_{\mathrm{dR}}^1(Y/S_K)$  factors through a smooth (Zhang [Zha18]) surjective (Kisin-Madapusi Pera-Shin [KMS]) morphism

$$(3.5.1) \quad \zeta : S_K \rightarrow G\text{-Zip}^\mu,$$

where  $G$  is the reductive  $\mathbf{F}_p$ -group deduced from the  $\mathbf{Q}$ -group  $\mathbf{G}$  and  $\mu \in X_*(G)$  is a representative of the conjugacy class of cocharacters deduced from the Hermitian symmetric space  $\mathbf{X}$ . See [GK19a, Section 4.1-4.2] for more details. When  $(\mathbf{G}, \mathbf{X})$  is not of Hodge-type, there still exists a smooth, surjective morphism (3.5.1) by [SZ22], but it no longer arises from an  $F$ -Zip of the form  $H_{\mathrm{dR}}^1(Y/S_K)$ .

The  $F$ -Zip  $H_{\text{dR}}^g(Y/S_K) = \wedge^g H_{\text{dR}}^1(Y/S_K)$  is of CY-type. It arises from the representation  $r$  which is the  $g$ th exterior power of the dual of  $G \rightarrow \text{GSp}(2g) \rightarrow \text{GL}(2g)$  deduced from  $\varphi$ . Since  $\zeta$  is smooth, we see that Ogus' principle for  $S_K$  is implied by the group theoretic Ogus' principle for  $(G, \mu, r)$ .

#### 4. METHOD OF PROOF

Keep the notation of Section 3.1 and let  $r$  be a representation of CY-type. We compute the vanishing order of  $\text{Ha}(G, \mu, r)$  and  $\text{cl}_x(G, \mu, r)$  separated. The latter is done via [MW04]. The main technical part is the computation of the former. In short, we use the theory of  $G$ -zips to reduce the question of the vanishing order to a study of highest weight sections on the flag variety  $G/B$ . In this section, we sketch how to obtain this reduction and how to compute the vanishing order of the highest weight sections.

**4.1. Root data.** Let  $T$  be a maximal torus contained in a Borel  $B \subset P$ . Let  $(X^*(T), \Phi, X_*(T), \Phi^\vee)$  denote the root system associated to  $(G, T)$ , and let  $\Delta$  denote the simple roots determined by  $B^+$ , the Borel opposite to  $B$ . Let  $W = W(G_k, T_k)$  denote the Weyl group. It is a Coxeter group with length function  $l: W \rightarrow \mathbf{N}$  and longest element  $w_0$ . We denote both the element  $w \in W$  and each suitable<sup>5</sup> lift in  $N_G(T)$  by the same letter  $w$ . For a subset  $K \subset \Delta$ , let  $W_K := \langle s_\alpha : \alpha \in K \rangle \subset W$ . Let  ${}^K W$  (resp.  $W^K$ ) denote the elements of minimal length in the cosets of  $W_K \backslash W$  (resp  $W/W_K$ ). Let  $w_{0,K}$  denote the element of maximal length in  $W_K$ . Let  $I, J \subset \Delta$  be the type of  $P$  and  $Q$  respectively. Let  $z := w_0 w_{0,J}$ , the longest element in  $W^J$ .

**4.2. Realization of  $G$ -Zip $^\mu$  as a quotient stack.** Let  $S$  be a  $k$ -scheme and let  $g \in G(S)$ . Let  $I_g$  and  $I_{g,P}$  be the trivial torsors  $S \times G$  respectively  $S \times P$ , let  $I_{g,Q}$  be the image of  $S \times Q$  in  $S \times G$  under left multiplication by  $g$  and let  $\iota_g$  be the isomorphism induced by left multiplication by  $g$ . Let  $\underline{I}_g$  be the  $G$ -zip  $\underline{I}_g = (I_g, I_{g,P}, I_{g,Q}, \iota_g)$ . Let  $\underline{I}$  be an arbitrary  $G$ -zip of type  $\mu$  over  $S$ . Since  $G, P, Q$  and  $L$  are smooth groups, there is a  $g \in G(S)$  such that  $\underline{I}$  is étale-locally isomorphic to the  $G$ -zip  $\underline{I}_g$  (see [PWZ11, Lemma 3.5]).

Let  $E := P \times_M Q$ , and let it act on  $G_k$  by  $(a, b) \cdot g = agb^{-1}$ . Then we have an isomorphism

$$[E \backslash G] \xrightarrow{\cong} G\text{-Zip}^\mu$$

which, roughly speaking, is induced by the map  $g \mapsto \underline{I}_g$  (see [PWZ11, Section 3.4] for details).

**4.2.1. The Ekedahl-Oort stratification.** For each  $w \in W$ , let  $G_w$  denote the  $E$ -orbit of  $wz^{-1}$ . By [PWZ11, Theorem 7.5 and Theorem 11.3], the morphism  $w \mapsto G_w$  induces a homeomorphism of the underlying topological space of  $G\text{-Zip}^\mu$  with  ${}^I W$ . Here  ${}^I W$  is equipped with the topology obtained from the order given by

$$w \preceq w' \iff \text{there exists } x \in W_I \text{ such that } xwz\varphi(x^{-1})z^{-1} \leq w',$$

where  $\varphi$  is the Frobenius action. From this one obtains a stratification whose strata closure relations are given by the order above; the closure of  $w'$  consists of all  $w$  such that  $w \preceq w'$ . For each  $w \in {}^I W$  let  $[E \backslash G_w]$  denote the corresponding stratum of  $G\text{-Zip}^\mu$ . It is a smooth locally closed substack, called the *zip stratum*, or the *Ekedahl-Oort stratum*, corresponding to  $w$ . See [PWZ11] for details.

**4.3.  $G$ -zip flags.** A  $G$ -zip flag of type  $\mu$  over a  $k$ -scheme  $S$  is a tuple  $(\underline{I}, I_B)$ , where  $\underline{I}$  is a  $G$ -zip of type  $\mu$  and  $I_B \subset I$  is a sub  $B$ -torsor. The forgetful map  $(\underline{I}, I_B) \mapsto \underline{I}$  induces a smooth morphism

$$\pi_{\text{Flag, Zip}}: G\text{-ZipFlag}^\mu \rightarrow G\text{-Zip}^\mu.$$

<sup>5</sup>For each  $\dot{w} \in W$  we choose a lift  $w \in N_{G_k}(T_k)$  such that  $w_1 \dot{w}_2 = \dot{w}_1 \dot{w}_2$  whenever  $l(w_1 \dot{w}_2) = l(\dot{w}_1) + l(\dot{w}_2)$ . This is possible by [DG70, Exp. XXIII, Section 6].

4.3.1. *Realization as a quotient stack.* Let  $E' := E \cap (B_k \times G_k)$ . We have that  $G\text{-ZipFlag}^\mu \cong [E' \backslash G]$  where the action of  $E'$  is the one induced from the action of  $E$  (see [GK19b]).

4.4. **Reduction to a flag variety and the Bruhat stratification.** Recall that the classical Bruhat decomposition of  $G$  is given by

$$G = \coprod_{w \in W} BwB.$$

Let  $\text{Sbt} := [B \backslash G / B]$  where  $B \times B$  acts on  $G$  by  $(a, b) \cdot agb^{-1}$ . This is called the *Schubert stack*. It inherits a stratification from the Bruhat decomposition;  $\text{Sbt} = \coprod_{w \in W} \text{Sbt}_w$ , where  $\text{Sbt}_w := [B \backslash BwB / B]$ . The morphism  $g \mapsto gz$  induces an isomorphism  $[B \backslash G / {}^z B] \rightarrow \text{Sbt}$ . Composed with the projection  $[E' \backslash G] \rightarrow [B \backslash G / {}^z B]$  we obtain a smooth morphism

$$\pi_{\text{Flag}, \text{Sbt}} : G\text{-ZipFlag}^\mu \rightarrow \text{Sbt}.$$

Analogously to the Schubert stack, the *Bruhat stack* is the double quotient  $\mathcal{B} := [P \backslash G / Q]$ , studied by Wedhorn in depth in [Wed14a], [Wed14b]. It too has a stratification induced by Bruhat decomposition;  $\mathcal{B} = \coprod_{w \in {}^I W^J} \mathcal{B}_w$ . The identity map  $G \rightarrow G$  induces a smooth surjection  $\pi_{\text{Zip}, \mathcal{B}} : G\text{-Zip}^\mu \rightarrow \mathcal{B}$ . By taking preimages, this induces the *Bruhat stratification* on  $G\text{-Zip}^\mu$ , which is coarser than the EO-stratification.

4.4.1. We thus have the following diagram of smooth morphisms

$$(4.4.2) \quad \begin{array}{ccc} G\text{-Zip}^\mu & \xleftarrow{\pi_{\text{Flag}, \text{Zip}}} & G\text{-ZipFlag}^\mu \\ \pi_{\text{Zip}, \mathcal{B}} \downarrow & & \downarrow \pi_{\text{Flag}, \text{Sbt}} \\ \mathcal{B} & & \text{Sbt} \longleftarrow G/B \end{array}$$

4.4.3. *Line bundles on the stacks.* Any pair of characters  $\lambda_1, \lambda_2 : B \rightarrow \mathbf{G}_m$  induce a line bundle  $\mathcal{L}_{\text{Sbt}}(\lambda_1, \lambda_2)$  on  $\text{Sbt}$ . By [GK19a, Theorem 2.2.1]  $\mathcal{L}_{\text{Sbt}}(\lambda_1, \lambda_2)$  admits nontrivial global sections if and only if  $\lambda_1$  is dominant and  $\lambda_2 = -w_0\lambda_1$ . Given a character  $\lambda$  of  $B$ , via the projection  $E' \rightarrow B$  we obtain also a character on  $E'$ , still denoted  $\lambda$ . This yields a line bundle  $\mathcal{L}_{G\text{-ZipFlag}}(\lambda)$  on  $G\text{-ZipFlag}^\mu$ . By [GK19a, Lemma 3.1.1] we have that

$$\pi_{\text{Flag}, \text{Sbt}}^* \mathcal{L}_{\text{Sbt}}(\lambda, -w_0\lambda) = \mathcal{L}_{G\text{-ZipFlag}}(D_{w_0}(\lambda)),$$

where  $D_{w_0} : X^*(T) \rightarrow X^*(T)$  is the map

$$D_{w_0} : \lambda \mapsto \lambda - p^{\sigma^{-1}}(zw_0^{-1}\lambda).$$

If  $\lambda : L \rightarrow \mathbf{G}_m$  is a character of  $L$ , then via the projections  $E \rightarrow P \rightarrow L$ , we obtain a character of  $E$ , which yields a line bundle  $\mathcal{L}_{G\text{-Zip}}(\lambda)$  on  $G\text{-Zip}^\mu$ .

Suppose that  $r : G \rightarrow GL(V)$  is a representation of CY-type with highest weight  $\lambda_r$ . Let

$$\eta_r := -\lambda_r.$$

The composition  $r \circ \mu$  induces a filtration on  $V_k$  whose graded pieces are stable under the action of  $L$  via  $r$ . Since  $r$  is of CY-type,  $\lambda_r$  restricts to a character of  $L$  and we have that  $\text{Ha}(G, \mu, r)$  is a section of the line bundle  $\mathcal{L}_{G\text{-Zip}}((p-1)(\eta_r))$ . When  $(G, \mu, r)$  arises from a Shimura variety of Hodge type, then the character  $\eta_r$  is often called the *Hodge character*. It satisfies the equation  $\omega \cong \zeta_K^* \mathcal{L}(\eta_r)$ .

Solving the equation

$$(p-1)\eta_r = D_{w_0}(\lambda)$$

gives a character  $\lambda$  such that

$$(4.4.4) \quad \pi_{\text{Flag}, \text{Zip}}^* \mathcal{L}_{G\text{-Zip}}((p-1)\eta_r) = \pi_{\text{Flag}, \text{Sbt}}^* \mathcal{L}_{\text{Sbt}}(\lambda, -w_0\lambda).$$

If  $\text{Ha}(G, \mu, r)$  is not identically zero, then there is a global section  $\text{Ha}_{\text{Sbt}}$  of  $\mathcal{L}(\lambda, -w_0\lambda)$  such that  $\pi_{\text{Flag}, \text{Sbt}}^* \text{Ha}_{\text{Sbt}} = \pi_{\text{Flag}, \text{Zip}}^* \text{Ha}(G, \mu, r)$ . Hence, for any  $w \in W$ , the vanishing order of  $\text{Ha}_{\text{Sbt}}$  on  $\text{Sbt}_w$  equals the vanishing order of  $\text{Ha}(G, \mu, r)$  on all Ekedahl-Oort strata contained in  $\pi_{\text{Flag}, \text{Zip}}^{-1}(\pi_{\text{Flag}, \text{Sbt}}^{-1}(\text{Sbt}_w))$ .

4.4.5. *Reduction to a flag variety.* Let  $\lambda \in X^*(T)$  be a dominant character. Let  $H^0(\lambda) := H^0(G/B, \mathcal{L}_{G/B}(\lambda))$  and let  $f_\lambda$  be a highest weight vector of  $H^0(\lambda)$ .

4.4.6. *A classical construction of  $f_\lambda$ .* For all  $u^+tu \in U^+B$  define let  $f_\lambda(u^+tu) = \lambda(t)^{-1}$ . For any  $\alpha \in \Delta$ ,  $s_\alpha$  normalizes  $\prod_{\beta \in \Phi^+ \setminus \{\alpha\}} U_\beta$ . Hence, by using relations regarding root group morphisms we can coordinate shift any element in  $s_\alpha U^+B$  to lie  $U^+B$ . We obtain thus an extension of  $f_\lambda$  to  $s_\alpha U^+B$ . The subscheme  $\bigcup_{\alpha \in \Delta} s_\alpha U^+B$  has codimension greater than 1, hence  $f_\lambda$  extends to  $G$  (see [Jan03, Section II.2.6.] for details).

4.4.7. *Extending the classical construction.* Suppose that  $(G, \mu)$  arises from a Shimura variety of abelian type as in Section 3.5. We extend the classical construction of  $f_\lambda$  to subsets of the form  $wU^+B$  for all  $w \in {}^I W$ . Hence, we can study the vanishing order of  $f_\lambda$  on all points in  $wU^+B$ .

We identify  $wU^+B$  with  $k^{|\Phi^+|} \times B$ . Under this identification,  $B^+wB$  is the zero-locus of the last  $l(w)$  coordinates of  $k^{|\Phi^+|}$ . Since  $Bw_0wB = w_0B^+wB \subset wU^+B$ , we see that the vanish order of  $f_\lambda$  on  $B^+wB$  equals that of  $w_0 \cdot f_\lambda$  on  $Bw_0wB$ .

4.4.8. By a direct computation, one finds that for all dominant characters  $\lambda \in X^*(T)$ , we have that

$$H^0(\text{Sbt}, \mathcal{L}_{\text{Sbt}}(\lambda, -w_0\lambda) = H^0(-w_0\lambda)_{-\lambda}$$

Let  $\lambda$  be such that Equation (4.4.4) holds. The order of vanishing of  $f_{-w_0\lambda}$  on any point in  $B^+wB$  gives the vanishing order of  $w_0 \cdot f_{-w_0\lambda} = \text{Ha}_{\text{Sbt}}$  on any point in  $Bw_0wB$ . We obtain thus the vanishing order of  $\text{Ha}(G, \mu, r)$  on all points in  $\pi_{\text{Flag}, \text{Zip}}^{-1}(\pi_{\text{Flag}, \text{Sbt}}^{-1}(\text{Sbt}_{w_0w}))$ .

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