

Algebraic-geometric invariants reflected in counter-examples of the (integral) Hodge conjecture

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Abstract

I shall report my recent results which endows some algebraic-geometric invariant interpretations for the discrepancies reflected in counter-examples of the (integral) Hodge conjecture, ¹⁾

1 Background

1.1 The cycle map and related maps

For a smooth projective complex variety X , we have the usual cycle map $cl_{Hdg}^i(X)$ and related maps, whose cokernels shall be denoted as follows:

$$\begin{array}{ccc}
 CH^i(X) & \xrightarrow{cl_{Hdg}^i(X)} & Hdg^{2i}(X, \mathbb{Z}) := H^{2i}(X, \mathbb{Z}) \cap H^{i,i}(X, \mathbb{C}) \\
 \downarrow cl_{Tot}^i(X) & \searrow cl_{Betti}^i(X) & \downarrow \\
 (MU^*(X) \otimes_{MU^*} \mathbb{Z})^{2i} & \xrightarrow{Thom^{2i}(X)} & H^{2i}(X, \mathbb{Z})
 \end{array}$$

$\Rightarrow$

$$\begin{array}{ccc}
 Z_{Hdg}^{2i}(X)\{tors\} \hookrightarrow Z_{Hdg}^{2i}(X) = \text{Coker}(cl_{Hdg}^i(X)) & & \\
 \parallel & \downarrow & \\
 Z_{Betti}^{2i}(X)\{tors\} \hookrightarrow Z_{Betti}^{2i}(X) = \text{Coker}(cl_{Betti}^i(X)) \twoheadrightarrow Z_{Thom}^{2i}(X) = \text{Coker}(Thom^{2i}(X)) & &
 \end{array}$$

Of course, we have various conjectures concerning these cokernels:

- codimension i integral Hodge conjecture , which predicts $Z_{Hdg}^{2i}(X) = 0$. (This is well known to be false in general, ever since the famous Atiyah-Hirzebruch topological counter-example [AH62].)

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¹⁾This report replaces the author's previous, unfortunately erroneous, report

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which dealt with the same theme.

- codimension i weak integral Hodge conjecture, which predicts $Z_{Hdg}^{2i}(X)\{tors\} \stackrel{\text{easy}}{=} Z_{Betti}^{2i}(X)\{tors\} = 0$. (This is also known to be false in general. Actually, all the known counter-examples of the integral Hodge conjecture are actually counter-examples of this weak conjecture.)
- codimension i Hodge conjecture predicts $Z_{Hdg}^{2i}(X) \otimes \mathbb{Q} = 0$, the famous conjecture.

1.2 The motivation of this work

The Question

Are there any algebro-geometric invariant interpretations of

$$Z_{Hdg}^{2i}(X), Z_{Hdg}^{2i}(X)\{tors\} \stackrel{\text{easy}}{=} Z_{Betti}^{2i}(X)\{tors\}, Z_{Hdg}^{2i}(X) \otimes \mathbb{Q}, \text{ and } Z_{Thom}^{2i}(X)?$$

In this paper, I shall announce my recent results on this question.

2 Known cases and a warning

2.1 The codimension $i = 1$ case

The codimension 1 integral Hodge conjecture is known to hold by Lefschetz and Hodge [KS53, p.876, Theorem 4]. The essence may be summarized by the following commutative diagram arising from the short exact sequence $0 \rightarrow \mathbb{Z} \rightarrow \mathcal{O}_{X^{an}} \xrightarrow{\exp} \mathcal{O}_{X^{an}}^* \rightarrow 0$ of sheaves on the corresponding complex analytic manifold X^{an} :

$$\begin{array}{ccccc} H^1(X, \mathcal{O}_X^*) & \xlongequal{\quad} & CH^1(X) & & \\ \text{Serre's GAGA} \parallel & & \downarrow_{cycl^1} & & \\ H^1(X^{an}, \mathcal{O}_{X^{an}}^*) & \longrightarrow & H^2(X^{an}, \mathbb{Z}) & \longrightarrow & H^2(X^{an}, \mathcal{O}_{X^{an}}) \end{array}$$

This means $Z^2(X) = 0$, and so not interesting from our point of view.

Warning: The above proof made use of analytic technique. Such is the case, the case $i = 1$ of the similarly defined Tate conjecture is still unknown, because of the lack of such analytic technique in its setting.

2.2 The condimension $i = \dim X - 1$ case (the so-called “curve case”) and the codimension $i = 2$ case

- Thanks to the Hard Lefschetz theorem, we find the usual Hodge conjecture for the curve case follows from the codimension 1 case:

$$Z_{Hdge}^2(X) = 0 \implies Z_{Hdge}^2(X)_{\mathbb{Q}} = 0 \xrightarrow{\text{Hard Lefschetz}} Z_{Hdge}^{2(\dim X - 1)}(X)_{\mathbb{Q}} = 0$$

We should be warned that there is NO integral Hard Lefschetz theorem. Actually, Kollár [K92] found a counter-example to the integral analogue for the curve case:

$$Z_{Hdge}^{2(\dim X - 1)}(X)\{tors\} \neq 0 \quad (\text{with } \dim X = 3)$$

These facts imply whereas $Z_{Hdge}^{2(\dim X-1)}(X)_{\mathbb{Q}} = 0$ is not interesting. $Z_{Hdge}^{2(\dim X-1)}(X) \stackrel{Z_{Hdge}^{2(\dim X-1)}(X)_{\mathbb{Q}}=0}{=} Z_{Hdge}^{2(\dim X-1)}(X)\{\text{tors}\}$ is interesting from our point of view.

- The usual Hodge conjecture for the codimension $i = 2$ case $Z_{Hdge}^4(X) \otimes \mathbb{Q} \stackrel{?}{=} 0$ is still unsolved. However, many counter-examples of its integral analogue $Z_{Hdge}^4(X) \neq 0$ have been constructed ever since the Atiyah-Hirzebruch first counter-example [AH62], including the aforementioned counter-example of Kollár [K92].

Concerning these interesting invariants, Soulé-Voisin and Voisin proved the following important theorem:

Theorem 2.1. [SV05, p.113, Lemma 1][V07, Lemma 15] $Z_{Hdg}^4(X)$ and $Z_{Hdg}^{2\dim X-2}(X)$ are birational invariant.

With the weak factorization theorem at hand, the proof is an immedaite corollary of the blowup formula.

2.3 Warning

Whereas Theorem 2.1 might prompt readers to look after the birational invariance of other $Z_{Hdg}^{2i}(X)$ also. However, for general i , $Z_{Hdg}^{2i}(X)$ is not birational invariant. For instance, the integral Hodge conjecture of a smooth projective X , which is regularly embedded in $\mathbb{P}^N$, is equivalent to the integral Hodge conjecture of the rational smooth projective variety $\widetilde{\mathbb{P}^N}$, obtained as the embedded blowup of $\mathbb{P}^N$, centered at X .

$$\begin{array}{ccc} D \hookrightarrow \widetilde{\mathbb{P}^N} & : & Z_{Hdg}^{2*}(X) = 0 \iff Z_{Hdg}^{2*}(\widetilde{\mathbb{P}^N}) = 0 \\ \downarrow & & \downarrow \\ X \hookrightarrow \mathbb{P}^N & & \end{array}$$

3 The codimension $> c$ birational equivalence and Schreieder's work

In order to generalize Theorem 2.1 to other codimensions, the appropriate equivalence relation is not the birational equivalence, but the following hierarchical analogue:

Definition 3.1. Let us say smooth projective equi-dimensional k -schemes X, Y are codimension $> c$ birational equivalent (or isomorphism in codimension c), if there are closed subsets $Z_X \subset X$ and $Z_Y \subset Y$ s.t.

- $\text{codim}_X Z_X > c, \quad \text{codim}_Y Z_Y > c.$
- $X \setminus Z_X \xrightarrow[f]{} Y \setminus Z_Y.$

(The case $c = 0$ is the usual birational equivalence.)

Now, Stefan Schreieder generalized the torsion analogue of Theorem 2.1 for smaller i 's including the case of $i = 2$:

Theorem 3.2. [S23, Theorem 1.6.(i)] [S22, Corollary 6.12] $Z_{Hdg}^{2i}(X)\{tors\}$ is $\text{codim} > (i-2)$ birational invariant. $\square$

(Schreieder's argument appears to be inapplicable to obtain similar results for $Z_{Hdg}^{2i}(X)$, $Z_{Hdg}^{2i}(X) \otimes \mathbb{Q}$, and $Z_{Thom}^{2i}(X)$. Also not so useful when i is large, e.g. $Z_{Thom}^{2\dim X-2}(X)\{tors\}$.)

4 Main Theorem

Contrary to Schreieder's Theorem 3.2, my main theorems, which I now state, are especially useful for large i , and, also cover $Z_{Hdg}^{2i}(X)$, $Z_{Hdg}^{2i}(X) \otimes \mathbb{Q}$, and $Z_{Thom}^{2i}(X)$:

Theorem 4.1. (i) $Z_{Hdg}^{2i}(X)$ is $\text{codimension} > (\dim_{\mathbb{C}} X - 1) - i$ birational invariant.

(When $i = \dim_{\mathbb{C}} X - 1$, this recovers the birational invariance of $Z_{Hdg}^{2\dim_{\mathbb{C}} X-2}(X)$, which was first observed by Soulé-Voisin and Voisin.)

(ii) $Z_{Hdg}^{2i}(X) \otimes \mathbb{Q}$ is $\text{codimension} > (\dim X_{\mathbb{C}} - 2) - i$ birational invariant.

(When $i = \dim_{\mathbb{C}} X - 2$, this recovers the birational invariance of $Z_{Hdg}^{2\dim_{\mathbb{C}} X-4}(X)_{\mathbb{Q}}$, which should be well-known to experts as this too can be immediately observed from the blowup formula, under the weak factorization theorem.)

(iii) Each p -primary component $Z_{Thom}^j(X)_{(p)}$ of the finite abelian group $Z_{Thom}^j(X)$ is $\text{codimension} > \left[\dim_{\mathbb{C}} X - \frac{(2p+1)+j}{2} \right]$ birational invariant.

(When $j = 2\dim_{\mathbb{C}} X - 2p$, $2\dim_{\mathbb{C}} X - (2p+1)$, this recovers the birational invariance of $Z_{Thom}^{2\dim_{\mathbb{C}} X-2p}(X)_{(p)}$, $Z_{Thom}^{2\dim_{\mathbb{C}} X-(2p+1)}(X)_{(p)}$, which appear to be new.)

The above Theorem 4.1(iii) turns out to possess a “chromatic hierarchy.”

Theorem 4.2. $Z_{Thom}^j(X)_{(p)}$ admits a “chromatic” filtration of $\text{codim} > \left[\dim_{\mathbb{C}} X - \frac{(2p+1)+j}{2} \right]$ bir. inv.:

$$\begin{aligned} Z_{Thom}^j(X)_{(p)} &= Z_{Thom}^j(X)_{(p)}\langle 0 \rangle \supseteq Z_{Thom}^j(X)_{(p)}\langle 1 \rangle \\ &\supseteq Z_{Thom}^j(X)_{(p)}\langle 2 \rangle \supseteq Z_{Thom}^j(X)_{(p)}\langle 3 \rangle \supseteq \cdots, \end{aligned}$$

arising from the “(cohomological) chromatic tower”, given by the Johnson-Wilson spectra $BP\langle n \rangle$:

$$\begin{array}{ccc} MU^*(X) \otimes_{MU^*} H^* & \xrightarrow{\hspace{15cm}} & H^*(X, \mathbb{Z}) \\ \uparrow & & \downarrow \\ MU^*(X) & \longrightarrow BP^*(X) \longrightarrow \cdots \longrightarrow BP\langle n+1 \rangle^*(X) \longrightarrow BP\langle n \rangle^*(X) \longrightarrow \cdots \longrightarrow BP\langle 0 \rangle^*(X) = H^*(X, \mathbb{Z})_{(p)} \end{array}$$

by setting

$$Z_{Thom}^j(X)_{(p)}\langle n \rangle := \frac{\text{Im} \left(BP\langle n \rangle^*(X) \rightarrow H^*(X, \mathbb{Z})_{(p)} \right)}{\text{Im} \left(Thom^{2j}(X)_{(p)} : (MU^*(X) \otimes_{MU^*} \mathbb{Z})^{2i} \rightarrow H^*(X, \mathbb{Z})_{(p)} \right)}$$

My proof of the above theorems is some very careful analysis of the correspondences together with Hironaka.

In particular, my analysis also leads to some other higher codimensional birational invariants also.

The detail will appear elsewhere.

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