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A simple expression for discrete Painlevé equations

By

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Abstract

A simple expression of discrete Painlevé equations and their Lax pair is obtained by using an interpolation problem. We discuss mainly the case of q-Painlevé equation of type $E_8^{(1)}$.

§ 1. Structure of discrete Painlevé equations

The second order discrete Painlevé equations were classified by Sakai [13] as follows:

$$
\begin{tikzcd}[row sep=small, column sep=small]
\textbf{elliptic} & E_8^{(1)} & & & & & & & \\
 & & & & & & & & A_1^{(1)} \\
\textbf{multiplicative} & E_8^{(1)} \arrow[r] & E_7^{(1)} \arrow[r] & E_6^{(1)} \arrow[r] & D_5^{(1)} \arrow[r] & A_4^{(1)} \arrow[r] & A_{2+1}^{(1)} \arrow[r] & A_{1+1}^{(1)} \arrow[r] \arrow[ur] & A_1^{(1)} \\
\textbf{additive} & E_8^{(1)} \arrow[r] & E_7^{(1)} \arrow[r] & E_6^{(1)} \arrow[rr] & & D_4^{(1)} \arrow[r] & A_3^{(1)} \arrow[r] \arrow[dr] & A_{1+1}^{(1)} \arrow[r] \arrow[dr] & A_1^{(1)} \\
 & & & & & & & A_2^{(1)} \arrow[r] & A_1^{(1)}.
\end{tikzcd}
$$

Each discrete Painlevé equation, represented as a rational map on $\mathbb{P}^1 \times \mathbb{P}^1$,

$$
T : (f, g) \mapsto (\overline{f}, \overline{g}) = \left(\frac{\psi_1(f,g)}{\psi_0(f,g)}, \frac{\phi_1(f,g)}{\phi_0(f,g)} \right), \tag{1.1}
$$

has eight singular points where $\psi_0 = \psi_1 = 0$ or $\phi_0 = \phi_1 = 0$. Conversely, a configuration of eight points on $\mathbb{P}^1 \times \mathbb{P}^1$ (or nine points on $\mathbb{P}^2$) characterize the Painlevé equation. Let us look at some examples.

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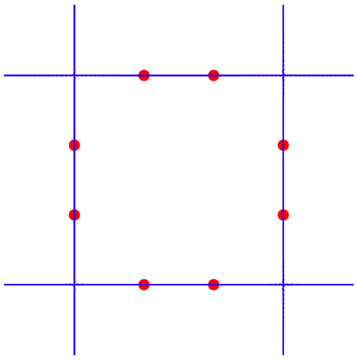
Example 1.1. q-$D_5^{(1)}$[3]: $T(t,f,g)=(qt,\overline{f},\overline{g})$, $q=\dfrac{a_3a_4b_1b_2}{a_1a_2b_3b_4}$.

$$\begin{aligned}\overline{f}f &= \frac{(\overline{g}-b_1t)(\overline{g}-b_2t)}{(\overline{g}-b_3)(\overline{g}-b_4)}a_3a_4,\\ \overline{g}g &= \frac{(f-a_1t)(f-a_2t)}{(f-a_3)(f-a_4)}b_3b_4.\end{aligned}\tag{1.2}$$

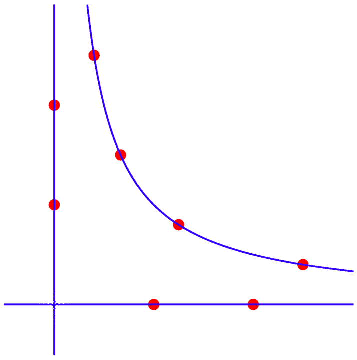
The 8 singular points are on the four lines $f=0, f=\infty, g=0$ and $g=\infty$:



Example 1.2. q-$E_6^{(1)}$ [12][14][11]: $T(t,f,g)=(qt,\overline{f},\overline{g})$, $q=\dfrac{b_5b_6b_7b_8}{b_1b_2b_3b_4}$.

$$\begin{aligned}\frac{(fg-1)(\overline{f}g-1)}{f\overline{f}} &= \frac{qt^2(b_1g-1)(b_2g-1)(b_3g-1)(b_4g-1)}{b_5b_6(b_7gt-1)(b_8gt-1)},\\ \frac{(\overline{f}g-1)(\overline{f}\overline{g}-1)}{g\overline{g}} &= \frac{q^2t^2(b_1-\overline{f})(b_2-\overline{f})(b_3-\overline{f})(b_4-\overline{f})}{(b_5-\overline{f}qt)(b_6-\overline{f}qt)}.\end{aligned}\tag{1.3}$$

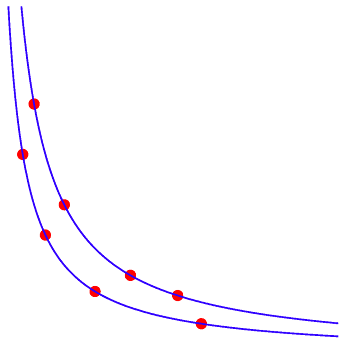
The 8 singular points are on the two lines $f=0, g=0$ and one curve $fg=1$:



Example 1.3. q-$E_7^{(1)}$ [1][11]: $T(t,f,g)=(qt,\overline{f},\overline{g})$, $q=\dfrac{b_5b_6b_7b_8}{b_1b_2b_3b_4}$.

$$\begin{aligned}\frac{(fg-1)(\overline{f}g-1)}{(fgt^2-1)(\overline{f}gqt^2-1)} &= \frac{(b_1g-1)(b_2g-1)(b_3g-1)(b_4g-1)}{(b_5gt-1)(b_6gt-1)(b_7gt-1)(b_8gt-1)},\\ \frac{(\overline{f}g-1)(\overline{f}\overline{g}-1)}{(\overline{f}gqt^2-1)(\overline{f}\overline{g}q^2t^2-1)} &= \frac{(b_1-\overline{f})(b_2-\overline{f})(b_3-\overline{f})(b_4-\overline{f})q}{(b_5-\overline{f}qt)(b_6-\overline{f}qt)(b_7-\overline{f}qt)(b_8-\overline{f}qt)}.\end{aligned}\tag{1.4}$$

The 8 singular points are on the two curves $fg=1$ and $fgt^2=1$:



Example 1.4. q-$E_8^{(1)}$[11]: $T(k,\ell,f,g)=(\frac{k}{q},q\ell,\overline{f},\overline{g})$, $\quad q=\dfrac{k^2\ell^2}{u_1\cdots u_8}$.

$$\frac{(\overline{f}-g)(f-g)-(\frac{k}{q}-\ell)(k-\ell)\frac{1}{\ell}}{(\frac{\overline{f}q}{k}-\frac{g}{\ell})(\frac{f}{k}-\frac{g}{\ell})-(\frac{q}{k}-\frac{1}{\ell})(\frac{1}{k}-\frac{1}{\ell})\ell}=\frac{k^2}{q}\frac{A(\ell,g)}{B(\ell,g)}, \tag{1.5}$$

$$\frac{(\overline{f}-\overline{g})(\overline{f}-g)-(\frac{k}{q}-\ell q)(\frac{k}{q}-\ell)\frac{q}{k}}{(\frac{\overline{f}q}{k}-\frac{\overline{g}}{\ell q})(\frac{\overline{f}q}{k}-\frac{g}{\ell})-(\frac{q}{k}-\frac{1}{\ell q})(\frac{q}{k}-\frac{1}{\ell})\frac{k}{q}}=q\ell^2\frac{A(\frac{k}{q},\overline{f})}{B(\frac{k}{q},\overline{f})}, \tag{1.6}$$

where $A(h,x)$, $B(h,x)$ are polynomials in x of degree 4 given by

$$\begin{aligned}
A(h,x)&=(\frac{m_0}{h^2}-\frac{m_2}{h}+m_4-hm_6+h^2m_8)+(\frac{m_1}{h^2}-m_5+2hm_7)x\\
&\quad+(-\frac{m_0}{h^3}+m_6-3hm_8)x^2-m_7x^3+m_8x^4,\\
B(h,x)&=(\frac{m_0}{h^2}-\frac{m_2}{h}+m_4-hm_6+h^2m_8)+(\frac{2m_1}{h^2}-\frac{m_3}{h}+hm_7)x\\
&\quad+(-\frac{3m_0}{h^3}+\frac{m_2}{h^2}-hm_8)x^2-\frac{m_1}{h^3}x^3+\frac{m_0}{h^4}x^4,\\
U(z)&=\frac{1}{z^4}\prod_{i=1}^{8}(z-u_i)=\frac{1}{z^4}\sum_{i=0}^{8}(-1)^i m_i z^i.
\end{aligned} \tag{1.7}$$

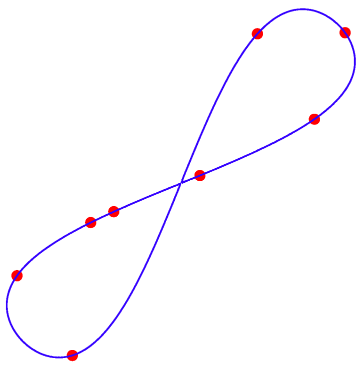
Though it is not so obvious in this form, the singularity of this equation are given by $(f,g)=(F(u_i),G(u_i))$, $(i=1,\cdots,8)$ where

$$F(u)=u+\frac{k}{u},\quad G(u)=u+\frac{\ell}{u}. \tag{1.8}$$

These points are on the curve of bi-degree $(2,2)$

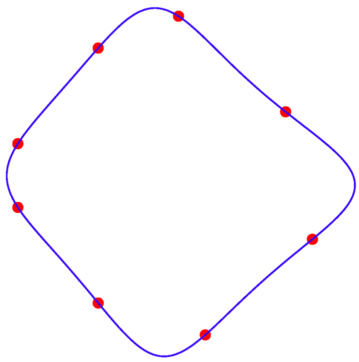
$$(f-g)(\frac{f}{k}-\frac{g}{\ell})-(k-\ell)(\frac{1}{k}-\frac{1}{\ell})=0, \tag{1.9}$$

which has a node at (∞,∞).



In the next section, we will rewrite the q-$E_8^{(1)}$ equation in simpler form where the singularity structure is manifest (see Theorem.2.2).

Example 1.5. The most generic equation, the elliptic-$E_8^{(1)}$ [13][11], is more complicated than q-$E_8^{(1)}$ case. The 8 points are on a smooth bi-degree $(2,2)$ curve (i.e. an elliptic curve):



There have been many challenges to obtain an explicit expression of the elliptic $E_8^{(1)}$ equation (e.g.[4][7][8]). We will give one simple expression (Theorem.3.1) which was obtained in [10] by a similar method as the q-case discussed below.

§ 2. An approach from the Padé interpolation

There exists a simple method to derive a Lax pair for Painlevé equations[15]. Using a discrete version of it, we will derive a simple form of q-Painlevé equation of type $E_8^{(1)}$. Here, we use parameters $a_1, a_2, a_3, b_1, b_2, b_3, k, \ell \in \mathbb{C}$ and $m, n \in \mathbb{Z}_{\geq 0}$ with constraints

$$q^{-1}k\ell = q^{-n}a_1a_2a_3 = q^{-m}b_1b_2b_3. \tag{2.1}$$

The time evolution is $\overline{k} = \frac{k}{q}$, $\overline{\ell} = q\ell$ (and $\overline{x} = x$ for $x = a_i, b_i, m, n$). The parameters u_i in Example.1.4 are related to the parameters a_i, b_i, k, ℓ, m, n by $(u_1, \cdots, u_8) = (a_1, a_2, a_3, q^{-m-n}, b_1, b_2, b_3, q)$.

Our staring point is the following Padé interpolation

$$
\begin{aligned}
&Y_s = \frac{q^{ns}(\frac{k}{a_1}, \frac{k}{a_2}, \frac{k}{a_3}, b_1, b_2, b_3)_s}{q^{ms}(a_1, a_2, a_3, \frac{k}{b_1}, \frac{k}{b_2}, \frac{k}{b_3})_s} = \frac{P_m}{Q_n}, \quad x = x_s, \quad (s = 0, 1, \cdots, m+n), \\
&(a)_i = \prod_{j=0}^{i-1}(1 - aq^j), \quad (a, \cdots, b)_s = (a)_s \cdots (b)_s,
\end{aligned}
\tag{2.2}
$$

where P_m, Q_n are polynomials of degree m, n in variable x. In the folowings, we use variable z such that $x = z + \frac{k}{qz}$, hence P_m, Q_n are Laurent polynomials of the form

$$
P_m(z) = \sum_{i=0}^{m} u_i (z + \frac{k}{qz})^i, \quad Q_n(z) = \sum_{i=0}^{n} v_i (z + \frac{k}{qz})^i. \tag{2.3}
$$

The interpolating points are $x_s = q^{-s} + kq^{s-1}$ (q-quadratic grid) in variable x, and hence $z_s = q^{-s}$ (q-grid) in variable z.

The main ingredients are the contiguous relations satisfied by $u(z) = P_m(z)$ and $v(z) = Y(z)Q_n(z)$ where $Y(z)$ is a function such as $Y(q^{-s}) = Y_s$. For instance, the relation between $y(z), y(\frac{z}{q}), \overline{y}(\frac{z}{q})$ is obtained by evaluating the Casorati determinant

$$
\begin{vmatrix}
y(z) & y(\frac{z}{q}) & \overline{y}(\frac{z}{q}) \\
u(z) & u(\frac{z}{q}) & \overline{u}(\frac{z}{q}) \\
v(z) & v(\frac{z}{q}) & \overline{v}(\frac{z}{q})
\end{vmatrix} = 0. \tag{2.4}
$$

This determinant divided by $Y(z)$ is a Laurent polynomial in z and has many known zeros due to the interpolating condition $u(z_s) = v(z_s)$. Hence one can determine the structure of the contiguous relations without knowing the explicit form of P_m and Q_n.

Proposition 2.1. *The following relations hold for $y(z) = P_m(z), Y(z)Q_n(z)$:*

$$
\begin{aligned}
L_2(z): \quad & B_2(z)\Big\{g - G(\frac{k}{z})\Big\}y(z) - B_2(\frac{k}{z})\Big\{g - G(z)\Big\}y(\frac{z}{q}) \\
& + c\Big\{f - F(z)\Big\}(z - \frac{k}{z})\overline{y}(\frac{z}{q}) = 0,
\end{aligned}
\tag{2.5}
$$

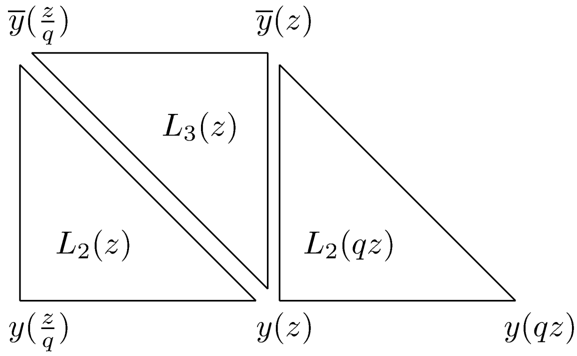
$$
\begin{aligned}
L_3(z): \quad & B_1(z)\Big\{g - G(\frac{k}{qz})\Big\}\overline{y}(\frac{z}{q}) - B_1(\frac{k}{qz})\Big\{g - G(z)\Big\}\overline{y}(z) \\
& + \frac{w}{c}\Big\{\overline{f} - \overline{F}(z)\Big\}(z - \frac{k}{qz})y(z) = 0,
\end{aligned}
\tag{2.6}
$$

where

$$
\begin{aligned}
&B_1(z) = \frac{1}{z^2}\prod_{i=1}^{4}(z - u_i), \quad B_2(z) = \frac{1}{z^2}\prod_{i=5}^{8}(z - u_i), \\
&F(z) = z + \frac{k}{z}, \quad G(z) = z + \frac{\ell}{z},
\end{aligned}
\tag{2.7}
$$

and f, g, c, w are some constants (independent of z).

Combining L_2 and L_3, one can obtain the three term relation L_1 between $y(qz), y(z), y(z/q)$.



Though the explicit form of the L_1 equation is complicated, it can be characterized by the following properties [16]: (1) As a polynomial in (f, g), it is of bi-degree $(3, 2)$. (2) It vanishes when $f = F(u)$, $g = G(u)$ with $u = u_1, \cdots, u_8, qz, \frac{k}{z}$, and $f = F(u)$, $\dfrac{(g - G(\frac{k}{u})y(u)}{(g - G(u))y(\frac{u}{q})} = \dfrac{B_2(\frac{k}{u})}{B_2(u)}$ with $u = z, qz$. Hence, the L_1 equation is equivalent with the L_1 equation in [17] up to some gauge transformations, and the equations L_2, L_3 (or L_1) can be considered as a Lax pair for q-$E_8^{(1)}$.

Theorem 2.2. *The compatibility of the equations L_2, L_3 (2.5)(2.6) is equivalent to the relations*

$$\frac{\{f - F(z)\}\{\overline{f} - \overline{F}(z)\}}{\{f - F(\frac{\ell}{z})\}\{\overline{f} - \overline{F}(\frac{\ell}{z})\}} = \frac{U(z)}{U(\frac{\ell}{z})}, \quad \text{for} \quad g = G(z), \tag{2.8}$$

$$\frac{\{g - G(z)\}\{\overline{g} - \overline{G}(z)\}}{\{g - G(\frac{k}{qz})\}\{\overline{g} - \overline{G}(\frac{k}{qz})\}} = \frac{U(z)}{U(\frac{k}{qz})}, \quad \text{for} \quad \overline{f} = \overline{F}(z), \tag{2.9}$$

along with an additional relation

$$w = \frac{(k - \ell)(k - q\ell)U(z)}{k^2\{f - F(z)\}\{\overline{f} - \overline{F}(z)\}}, \quad \text{for} \quad g = G(z). \tag{2.10}$$

where $U(z) = B_1(z)B_2(z) = \dfrac{1}{z^4}\displaystyle\prod_{i=1}^{8}(z - u_i)$.

Proof. Putting $g = G(z)$ in equations $L_2(z)$ and $L_3(z)$, we have the relation (2.10). Since $G(z) = G(\frac{\ell}{z})$, the relation (2.10) holds also when z is replaced by $\frac{\ell}{z}$. Taking the ratio of these two relations, we obtain the equation (2.8). Putting $\overline{f} = \overline{F}(z)$ in equations $\overline{L_2(z)}$ and $L_3(z)$, we get the equation (2.9). Sufficiency of the equations (2.8) (2.9) (2.10) for the compatibility can be checked by a direct computation. □

The equations (2.8)(2.9) are the desired simple expression for the q-$E_8^{(1)}$. In fact, by eliminating the variable z, they correctly reproduce the equations (1.5)(1.6) , where the polynomials A, B are given by

$$A(h, z+\frac{h}{z}) = \frac{zU(z) - \frac{h}{z}U(\frac{h}{z})}{z-\frac{h}{z}}, \quad B(h, z+\frac{h}{z}) = \frac{zU(\frac{h}{z}) - \frac{h}{z}U(z)}{z-\frac{h}{z}}. \tag{2.11}$$

Remark. Up to now, we used only the defining relation $Y_s = \dfrac{P_m(x_s)}{Q_n(x_s)}$ for $P_m(x)$ and $Q_n(x)$. If we know the explicit forms of $P_m(x)$, $Q_n(x)$, then we can determine the Painlevé variables f, g explicitly. For the interpolation problem with general Y_s and x_s, the following formula has been classically known by Cauchy and Jacobi

$$\begin{aligned} &P_m(x) = f(x)\det\left(W_{i,j}^{(-)}\right)_{i,j=0}^{n}, \quad Q_n(x) = \det\left(W_{i,j}^{(+)}\right)_{i,j=0}^{n-1}, \\ &W_{i,j}^{(\pm)} = \sum_{s=0}^{m+n}\frac{Y_s}{f'(x_s)}x_s^{i+j}(x-x_s)^{\pm 1}, \quad f(x) = \prod_{s=0}^{m+n}(x-x_s). \end{aligned} \tag{2.12}$$

Applying this for the q-quadratic grid: $x = z + \dfrac{k}{qz}$, $x_s = q^{-s} + kq^{s-1}$, we have

$$W_{i,j}^{(\pm)} = \sum_{s=0}^{m+n}\frac{(1-kq^{2s-1})}{(1-kq^{-1})}\frac{(k/q, q^{-m-n})_s}{(q, kq^{m+n})_s}Y_s q^{(n-m)s}x_s^{i+j}(x-x_s)^{\pm 1}. \tag{2.13}$$

These expressions give the special solutions for q-$E_8^{(1)}$ Painlevé equation in terms of the ${}_{10}W_9$ hypergeometric functions and their determinants (c.f.[5][6]).

Remark. The Padé approach to the degenerate cases were studied in [2][9]. The corresponding Padé problems are

q-P	q-$E_7^{(1)}$	q-$E_6^{(1)}$	q-$D_5^{(1)}$	q-$A_4^{(1)}$	q-$A_{2+1}^{(1)}$
Y_s	$\dfrac{(b_1,b_2,b_3)_s}{(a_1,a_2,a_3)_s}$	$\dfrac{(b_1,b_2)_s}{(a_1,a_2)_s}$	$c^s\dfrac{(b)_s}{(a)_s}$	$c^s(a)_s$	$c^s q^{\frac{s(s-1)}{2}}$

with the grid $x_s = q^s$. There is a constraint $a_1a_2a_3q^m = b_1b_2b_3q^n$ for q-$E_7^{(1)}$ case.

§ 3. Elliptic case

As before, we use multiplicative parameters $k, \ell, u_1, \cdots, u_8, q$ $(k^2\ell^2 = qu_1\cdots u_8)$ and p, where q is the base for the q-difference and p is the period of the elliptic functions. Let $[z]$ be a theta function such that $[pz] = [z^{-1}] = -z^{-1}[z]$, and define

$$\begin{aligned} a(z) &= [\frac{\alpha}{z}][\frac{k}{\alpha z}], \quad b(z) = [\frac{\beta}{z}][\frac{k}{\beta z}], \\ c(z) &= [\frac{\alpha}{z}][\frac{\ell}{\alpha z}], \quad d(z) = [\frac{\beta}{z}][\frac{\ell}{\beta z}], \end{aligned} \tag{3.1}$$

$(\alpha \neq \beta)$. Then the functions

$$F(z) = \frac{b(z)}{a(z)}, \quad G(z) = \frac{d(z)}{c(z)}, \tag{3.2}$$

are elliptic functions such that

$$F(pz) = F(z) = F(\frac{k}{z}), \quad G(pz) = G(z) = G(\frac{\ell}{z}), \tag{3.3}$$

which gives the parametrization $f = F(z)$, $g = G(z)$ of the elliptic curve in Example.1.5. By the same method as the q-case in previous section, we have [10]

Theorem 3.1. *The elliptic difference Painlevé equation of type $E_8^{(1)}$ can be written in the form $(k, \ell, f, g) \mapsto (k/q, q\ell, \overline{f}, \overline{g})$, where $\overline{f}$, $\overline{g}$ are given by*

$$\frac{\{a(\tilde{z})f - b(\tilde{z})\}\{\overline{a}(\tilde{z})\overline{f} - \overline{b}(\tilde{z})\}}{\{a(z)f - b(z)\}\{\overline{a}(z)\overline{f} - \overline{b}(z)\}} = \frac{\tilde{z}^2}{z^2}\prod_{i=1}^{8}\frac{[\frac{u_i}{\tilde{z}}]}{[\frac{u_i}{z}]}, \quad \text{for} \quad g = G(z), \quad \tilde{z} = \frac{\ell}{z}, \tag{3.4}$$

$$\frac{\{c(\tilde{z})g - d(\tilde{z})\}\{\overline{c}(\tilde{z})\overline{g} - \overline{d}(\tilde{z})\}}{\{c(z)g - d(z)\}\{\overline{c}(z)\overline{g} - \overline{d}(z)\}} = \frac{\tilde{z}^2}{z^2}\prod_{i=1}^{8}\frac{[\frac{u_i}{\tilde{z}}]}{[\frac{u_i}{z}]}, \quad \text{for} \quad \overline{f} = \overline{F}(z), \quad \tilde{z} = \frac{k}{qz}. \tag{3.5}$$

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