

The Global Weak Solutions of the Compressible Euler Equation with Spherical Symmetry

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1 Introduction

The compressible Euler equation for an isentropic gas in $\mathbf{R}^n$ is given by

$$(1.1) \quad \begin{aligned} \rho_t + \nabla \cdot (\rho \vec{u}) &= 0, \\ (\rho \vec{u})_t + \nabla \cdot (\rho \vec{u} \otimes \vec{u} + p) &= 0, \end{aligned}$$

with the equation of state

$$(1.2) \quad p = a^2 \rho^\gamma,$$

where density ρ , velocity $\vec{u}$ and pressure p are functions of $x \in \mathbf{R}^n$ and $t \geq 0$, while $a > 0$ and $\gamma \geq 1$ are given constants.

For one dimensional case ($n=1$), the Cauchy problem for (1.1) with (1.2) has been studied by many authors. Nishida [10] established the existence of global weak solutions, for the first time, for the case $\gamma = 1$ with arbitrary initial data, and Nishida and Smoller [11] for $\gamma \geq 1$ but with small initial data, both using Glimm's method. DiPerna [3] extended the latter result to the case of large initial data, using the theory of compensated compactness under the restriction $\gamma = 1 + 2/(2m + 1)$, $m \geq 2$ integers. Ding et al [1], [2] removed this restriction and established the existence of global weak solutions for $1 < \gamma \leq 5/3$.

On the other hand, little is known for the case $n \geq 2$. No global solutions have been known to exist, but only local classical solutions ([5], [6], [8] and [9]).

In this paper, we will present global weak solutions first for the case $n \geq 2$. We will do this, however, only for the case of spherical symmetry with $\gamma = 1$. As will be seen below, our proof does not work without these restrictions.

Thus, we look for solutions of the form

$$(1.3) \quad \rho = \rho(t, |x|), \quad \vec{u} = \frac{x}{|x|} \cdot u(t, |x|).$$

Then, denoting $r = |x|$, (1.1) becomes

$$(1.4) \quad \begin{aligned} \rho_t + \frac{1}{r^{n-1}} (r^{n-1} \rho u)_r &= 0, \\ \rho (u_t + u u_r) + p_r &= 0, \end{aligned}$$

This equation has a singularity at $r=0$. To avoid the difficulty caused by this singularity, we simply deal with the boundary value problem for (1.4) in the domain $1 \leq r < \infty$ (the exterior of a sphere) with the boundary condition $u(t, 1) = 0$, which is identical, under the assumption (1.3), to the usual boundary condition $\vec{n} \cdot \vec{u} = 0$ for (1.1) where $\vec{n}$ is the unit normal to the boundary.

Put $\tilde{\rho} = r^{n-1} \rho$. Then we get from (1.4)

$$(1.5) \quad \begin{aligned} \tilde{\rho}_t + (\tilde{\rho} u)_r &= 0, \\ u_t + u u_r + \frac{a^2 \gamma \tilde{\rho}_r}{\tilde{\rho}^{2-\gamma} r^{(n-1)(\gamma-1)}} &= \frac{a^2 \gamma (n-1) \tilde{\rho}^{\gamma-1}}{r^n \cdot r^{(n-1)(\gamma-2)}}. \end{aligned}$$

Introduce the Lagrangean mass coordinates

$$(1.6) \quad \tau = t, \quad \xi = \int_1^r \tilde{\rho}(t, r) dr.$$

Then $\xi > 0$ as long as $\tilde{\rho} > 0$ for $r > 1$, and (1.5) is reformulated as

$$(1.7) \quad \begin{aligned} \tilde{\rho}_\tau + \tilde{\rho}^2 u_\xi &= 0, \\ u_\tau + \frac{a^2 \gamma \tilde{\rho}_\xi}{\tilde{\rho}^{1-\gamma} r^{2\gamma-2}} &= \frac{a^2 \gamma (n-1) \tilde{\rho}^{\gamma-1}}{r^n \cdot r^{(n-1)(\gamma-2)}}. \end{aligned}$$

Put $v = 1/\tilde{\rho}$ and note that the inverse transformation to (1.6) is given by

$$(1.8) \quad t = \tau, \quad r = 1 + \int_0^\xi v(\zeta, t) d\zeta.$$

Then after changing τ to t and ξ to x , (1.7) is written as

$$(1.9) \quad \begin{aligned} v_t - u_x &= 0, \\ u_t + \left(\frac{a^2}{v^\gamma} \right)_x \cdot \frac{1}{r^{(n-1)(\gamma-1)}} &= \frac{a^2 \gamma (n-1) v^{1-\gamma}}{r^n \cdot r^{(n-1)(\gamma-2)}}, \end{aligned}$$

where r is now defined by $r = 1 + \int_0^x v(t, \zeta) d\zeta$.

Now we restrict ourselves to the case $\gamma = 1$. Then (1.7) becomes

$$(1.10) \quad \begin{aligned} v_t - u_x &= 0, \\ u_t + \left(\frac{a^2}{v} \right)_x &= \frac{K}{1 + \int_0^x v(t, \zeta) d\zeta}. \end{aligned}$$

where $K = a^2(n-1)$.

Let us consider the initial boundary value problem for (1.10) in $t \geq 0, x \geq 0$ with the following boundary and initial conditions.

$$(1.11) \quad u(0, x) = u_0(x), \quad v(0, x) = v_0(x), \quad \text{for } x > 0,$$

$$(1.12) \quad u(t, 0) = 0, \quad \text{for } t > 0.$$

Let $BV(\mathbf{R}_+)$ denote the space of functions of bounded variation on $\mathbf{R}_+ = (0, \infty)$. Our main result is as follows.

Theorem (Main Result) *Suppose that $u_0(x), v_0(x) \in BV(\mathbf{R}_+)$, and that $v_0(x) \geq \delta_0 > 0$ for all $x > 0$ with some positive constant δ_0 . Then (1.10), (1.11) and (1.12) have a global weak solution which belongs to the class*

$$u, v \in L^\infty(0, T; BV(\mathbf{R}_+)) \cap Lip([0, T]; L^1_{loc}(\mathbf{R}_+))$$

for any $T > 0$.

The definition of the weak solution will be given in section 4. This theorem can be proved by following Nishida's argument [10] based on Glimm's

method. Indeed this can be seen from the following two simple observations. First, the homogeneous equation corresponding to (1.10),

$$(1.13) \quad \begin{aligned} v_t - u_x &= 0, \\ u_t + \left(\frac{a^2}{v} \right)_x &= 0, \end{aligned}$$

is just the same equation as solved by Nishida [10] using Glimm's method both on the Cauchy problem and the initial boundary value problem. Note that if $\gamma > 1$, the homogeneous equation for (1.9) has a variable coefficient and hence does not coincide with the one dimensional Euler equation.

The second observation is that, as long as $v \geq 0$, the right hand side of (1.10),

$$(1.14) \quad \frac{K}{1 + \int_0^x v(t, \zeta) d\zeta},$$

is monotone decreasing in x and has an a priori estimate

$$(1.15) \quad T.V. \left(\frac{K}{1 + \int_0^x v(t, \zeta) d\zeta} \right) \leq K,$$

independent of v . The one dimensional inhomogeneous Euler equation has been studied in [12]. However, the conditions imposed therein on the inhomogeneous term are not applicable to our (1.14).

These observations allow us to use Nishida's argument [10] to construct global weak solutions to (1.10), (1.11) and (1.12). More precisely, we will first construct, in section 2, approximate solutions of the form

$$\{\text{solution of Riemann problem for (1.13)}\} + \{\text{nonhomogeneous term}\} \times t.$$

This is the main idea of [12]. Then in section 3, we will estimate the total variation of the approximate solutions. Thanks to (1.15), this can be done with a slight modification of Nishida's argument [10]. In section 4, we will show that there exists a subsequence of approximate solutions which converges strongly in L^1_{loc} for any finite time interval. Finally, for the sake of completeness, we give in Appendix a detailed proof of two lemmas used in section 3. These lemmas are due to Nishida [10], but their proofs are not found in the literature.

2 The Difference Scheme

To construct the approximate solutions, we shall use the difference scheme developed in [10]. For $l, h > 0$, define

$$(2.1) \quad \begin{aligned} Y &= \{ (n, m); n = 1, 2, 3, \dots, m = 1, 3, 5, \dots \}, \\ A &= \prod_{(m,n) \in Y} [\{nh\} \times ((m-1)l, (m+1)l)] , \end{aligned}$$

where l/h will be determined later. Choose a point $\{a_{nm}\} \in A$ randomly, and write $a_{nm} = (nh, c_{nm})$. For $n = 0$, we put $c_{0m} = ml$. We denote approximate solutions by u^l and v^l . Mesh lengths l and h are chosen so that $l/h > a/(\inf v^l)$, for any given $T > 0$. We shall show later that there exists a $\delta > 0$ such that $\inf v^l \geq \delta > 0$.

For $0 \leq t < h$, $ml \leq x < (m+2)l$, m : odd, we define

$$(2.2) \quad \begin{aligned} u^l(t, x) &= u_0^l(t, x) + U^l(t, x)t, \\ v^l(t, x) &= v_0^l(t, x), \end{aligned}$$

where u_0^l and v_0^l are the solutions of

$$(2.3) \quad \begin{aligned} v_t - u_x &= 0, \\ u_t + \left(\frac{a^2}{v} \right)_x &= 0, \end{aligned}$$

with initial data

$$(2.4) \quad \begin{aligned} u_0^l(0, x) &= \begin{cases} u_0(ml), & x < (m+1)l, \\ u_0((m+2)l), & x > (m+1)l, \end{cases} \\ v_0^l(0, x) &= \begin{cases} v_0(ml), & x < (m+1)l, \\ v_0((m+2)l), & x > (m+1)l, \end{cases} \end{aligned}$$

and

$$(2.5) \quad U^l(t, x) = \frac{K}{1 + \sum_{j=1}^{\frac{m+1}{2}} v_0((2j-1)l) \cdot 2l} .$$

For $0 \leq t < h$, $0 \leq x < l$, we define u^l and v^l by (2.2) where u_0^l and v_0^l are the solutions of (2.3) with initial boundary data

$$(2.6) \quad u_0^l(0, x) = u_0(l), \quad v_0^l(0, x) = v_0(l), \quad x > 0,$$

$$(2.7) \quad u(t, 0) = 0, \quad t > 0,$$

and

$$(2.8) \quad U^l(t, x) = K.$$

Suppose that u^l and v^l are defined for $0 \leq t < nh$. For $nh \leq t < (n+1)h$, $ml \leq x < (m+2)l$, m : odd, we define

$$(2.9) \quad \begin{aligned} u^l(t, x) &= u_0^l(t, x) + U^l(t, x) \cdot (t - nh), \\ v^l(t, x) &= v_0^l(t, x), \end{aligned}$$

where u_0^l and v_0^l are the solutions of (2.3) with initial data ($t=nh$)

$$(2.10) \quad \begin{aligned} u_0^l(nh, x) &= \begin{cases} u^l(nh - 0, c_{nm}), & x < (m+1)l, \\ u^l(nh - 0, c_{n, m+2}), & x > (m+1)l, \end{cases} \\ v_0^l(nh, x) &= \begin{cases} v^l(nh - 0, c_{nm}), & x < (m+1)l, \\ v^l(nh - 0, c_{n, m+2}), & x > (m+1)l, \end{cases} \end{aligned}$$

and

$$(2.11) \quad U^l(t, x) = \frac{K}{1 + \sum_{j=1}^{\frac{m+1}{2}} v^l(nh - 0, c_{n, 2j-1}) \cdot 2l}.$$

For $nh \leq t < (n+1)h$, $0 \leq x < l$, we define u^l and v^l as (2.9) where u_0^l and v_0^l are the solutions of (2.3) with initial ($t=nh$) boundary data

$$(2.12) \quad u_0^l(nh, x) = u^l(nh - 0, c_{n1}), \quad v_0^l(nh, x) = v^l(nh - 0, c_{n1}), \quad x > 0,$$

$$(2.13) \quad u(t, 0) = 0, \quad t > nh,$$

and $U^l(t, x)$ is as (2.8).

3 Bounds for Approximate Solutions

System (1.6) is hyperbolic provided $v > 0$, with the characteristic roots and Riemann invariants given by

$$(3.1) \quad \begin{aligned} \lambda &= -\frac{a}{v}, & r &= u + a \log v, \\ \mu &= \frac{a}{v}, & s &= u - a \log v. \end{aligned}$$

It is well-known, [10], that all shock wave curves in the (r,s) -plane have the same figure. (See Figure 1.) The 1-shock wave curve S_1 , starting from (r_0, s_0) can be expressed in the form

$$(3.2) \quad s - s_0 = f(r - r_0) \quad \text{for } r \leq r_0,$$

and the 2-shock wave curve S_2 can also be expressed in the form

$$(3.3) \quad r - r_0 = f(s - s_0) \quad \text{for } s \leq s_0,$$

where

$$0 \leq f'(x) < 1, \quad f''(x) \leq 0, \quad \lim_{x \rightarrow -\infty} f'(x) = 1.$$

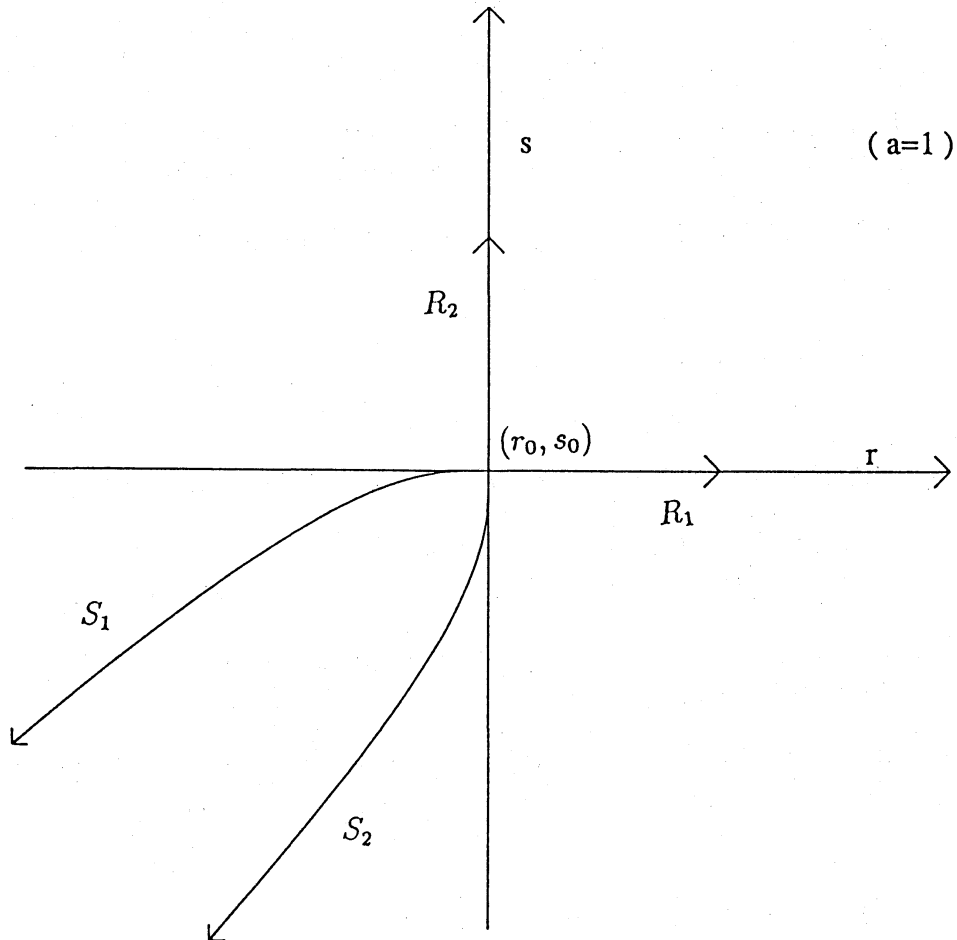


Figure.1

The 1-rarefaction wave curve R_1 can be expressed in the form

$$(3.4) \quad s - s_0 = 0 \quad \text{for } r \geq r_0,$$

and the corresponding expression for the 2-rarefaction wave curve R_2 is

$$(3.5) \quad r - r_0 = 0 \quad \text{for } s \geq s_0.$$

Now we must prepare some lemmas to estimate Riemann invariants. First, let us consider (2.3) with following initial data

$$(3.6) \quad u_0(x) = \begin{cases} u_l, & x < 0, \\ u_r, & x > 0. \end{cases} \quad v_0(x) = \begin{cases} v_l, & x < 0, \\ v_r, & x > 0. \end{cases}$$

Lemma 3.1 *Let u and v are the solutions of (2.3) and (3.6). Then,*

$$(3.7) \quad \begin{cases} r(t, x) \equiv r(u(t, x), v(t, x)) \geq r_0 \equiv \min(r(u_r, v_r), r(u_l, v_l)), \\ s(t, x) \equiv s(u(t, x), v(t, x)) \leq s_0 \equiv \max(s(u_r, v_r), s(u_l, v_l)). \end{cases}$$

Next consider (2.3) in $t \geq 0, x \geq 0$ with following initial and boundary conditions

$$(3.8) \quad u(0, x) = u_0^+, \quad v(0, x) = v_0^+, \quad \text{for } x > 0,$$

$$(3.9) \quad u(t, 0) = 0, \quad \text{for } t > 0.$$

Lemma 3.2 *Let u and v are the solutions of (2.3), (3.8) and (3.9). Then,*

$$(3.10) \quad \begin{cases} r(t, x) \equiv r(u(t, x), s(t, x)) \geq r(u_0^+, v_0^+), \\ s(t, x) \equiv s(u(t, x), s(t, x)) \leq \max(-r(u_0^+, v_0^+), s(u_0^+, v_0^+)). \end{cases}$$

The above two lemmas were proved in [10]. Using these two lemmas, we can get the following lemma.

Lemma 3.3 *Let u^l and v^l be the approximate solutions defined in section 2 and put $r_0 = \min r(u_0(x), v_0(x))$ and $s_0 = \max s(u_0(x), v_0(x))$. Then, for $0 < t < T$,*

$$(3.11) \quad \begin{cases} r^l(t, x) \equiv r(u^l(t, x), s^l(t, x)) \geq r_0, \\ s^l(t, x) \equiv s(u^l(t, x), s^l(t, x)) \leq \max(-r_0, s_0) + KT \end{cases}$$

Let us consider Riemann problem (2.3) and (3.6). Denote by Δr (resp Δs) the absolute value of the variation of the Riemann invariant r (resp s) in the first (resp second) shock wave.

Definition 3.4 *We denote*

$$P(u_l, v_l, u_r, v_r) = \Delta r + \Delta s.$$

Then we have the following lemma.

Lemma 3.5

$$(3.12) \quad P(u_1, v_1, u_3, v_3) \leq P(u_1, v_1, u_2, v_2) + P(u_2, v_2, u_3, v_3),$$

where u_1, u_2 and u_3 are arbitrary constants and v_1, v_2 and v_3 are arbitrary positive constants.

We shall prove Lemma 3.5 in the Appendix A.

Denote by $i_0^{n\pm}$ the straight line segments joining the points $(0, (n \pm \frac{1}{2})h)$ and a_{1n} . Let $F(i_0^{n\pm})$ be the absolute value of the variation of the Riemann invariants for all shocks on $i_0^{n\pm}$. Then we also have the following Lemma.

Lemma 3.6

$$(3.13) \quad F(i_0^{n+}) \leq F(i_0^{n-}).$$

This lemma 3.6 will be proved in the Appendix B.

We denote

$$\begin{aligned} Z_1 &= \{l - 0, l + 0, 3l - 0, \dots, (2m - 1)l - 0, (2m - 1)l + 0, \dots\}, \\ Z_2 &= \{2l, 4l, 6l \dots 2ml, \dots\}. \end{aligned}$$

Let $Z_{(n)} = Z_1 \cup Z_2 \cup \{c_{nm}\}$ and line up the elements $z_{n,i}$ of $Z_{(n)}$ so that $z_{n,i} \leq z_{n,i+1}$. (We regard $(2m - 1)l - 0 < (2m - 1)l + 0$ for m : integer.)
Let

$$\begin{aligned} F(nh - 0, u^l, v^l) &= \frac{1}{2} F(i_0^{n-}) \\ &+ \sum_{z_{n,i} \in Z_{(n)}} P(u^l(nh - 0, z_{n,i}), v^l(nh - 0, z_{n,i}), u^l(nh - 0, z_{n,i+1}), v^l(nh - 0, z_{n,i+1})), \end{aligned}$$

$$F(nh+0, u^l, v^l) = \frac{1}{2}F(i_0^{n+}) + \sum_{m: \text{odd}} P(u^l(a_{nm}), v^l(a_{nm}), u^l(a_{nm+2}), v^l(a_{nm+2})).$$

Using Lemma 3.5 and Lemma 3.6, we get

$$(3.14) \quad F((n+1)h+0, u^l, v^l) \leq F((n+1)h-0, u^l, v^l).$$

The following equality is obvious from the definition of F , u^l and v^l .

$$(3.15) \quad F((n+1)h-0, u_0^l, v_0^l) = F(nh+0, u^l, v^l).$$

We also get

$$\begin{aligned} F((n+1)h-0, u^l, v^l) &= F((n+1)h-0, u_0^l, v_0^l) \\ &+ \sum_{m: \text{odd}} P(u^l((n+1)h-0, ml-0), v^l((n+1)h-0, ml-0), \\ &u^l((n+1)h-0, ml+0), v^l((n+1)h-0, ml+0)). \end{aligned}$$

Lemma 3.7

$$(3.16) \quad \begin{aligned} &P(u^l((n+1)h-0, ml-0), v^l((n+1)h-0, ml-0), \\ &u^l((n+1)h-0, ml+0), v^l((n+1)h-0, ml+0)) \\ &\leq 2h \left\{ U^l(nh, (m-1)l) - U^l(nh, (m+1)l) \right\}, m : \text{odd}. \end{aligned}$$

Proof. From the definition,

$$\begin{aligned} u^l((n+1)h-0, ml-0) &= u_0^l(nh, ml) + U^l(nh, (m-1)l) \cdot h, \\ u^l((n+1)h-0, ml+0) &= u_0^l(nh, ml) + U^l(nh, (m+1)l) \cdot h, \\ v^l((n+1)h-0, ml-0) &= v^l((n+1)h-0, ml+0) = v_0^l(nh, ml). \end{aligned}$$

Therefore we get

$$(3.17) \quad \begin{aligned} &r^l((n+1)h-0, ml-0) - r^l((n+1)h-0, ml+0) \\ &= s^l((n+1)h-0, ml-0) - s^l((n+1)h-0, ml+0) \cdot l \\ &= h \times \left\{ U^l(nh, (m-1)l) - U^l(nh, (m+1)l) \right\} \geq 0 \end{aligned}$$

Thus the following inequality holds.

$$(3.18) \quad \Delta r, \Delta s \leq h \left\{ U^l(nh, (m-1)l) - U^l(nh, (m+1)l) \right\} \leq \Delta r + \Delta s.$$

From (3.18), we get (3.16). $\square$

Using Lemma 3.7, we get

$$(3.19) \quad F((n+1)h - 0, u^l, v^l) - F((n+1)h - 0, u_0^l, v_0^l) \leq 2h \sum_{m: \text{odd}} \left\{ U^l(nh, (m-1)l) - U^l(nh, (m+1)l) \right\} \leq 2Kh$$

From (3.14), (3.15) and (3.19), we get

$$(3.20) \quad F((n+1)h + 0, u^l, v^l) \leq F(nh + 0, u^l, v^l) + 2Kh$$

Thus we obtain the following lemma.

Lemma 3.8

$$(3.21) \quad F(nh + 0, u^l, v^l) \leq F(+0, u^l, v^l) + 2KT \equiv F_0 + 2KT$$

Denote by $G(\tau)$ the absolute value of the sum of negative variation of r^l and s^l for $t = \tau$. Then for $nh \leq \tau < (n+1)h$, we get

$$(3.22) \quad \begin{aligned} G(\tau) &\leq G(nh) + 2h \sum_{m: \text{odd}} \left\{ U^l(nh, (m-1)l) - U^l(nh, (m+1)l) \right\} \\ &\leq G(nh) + 2Kh. \end{aligned}$$

Lemma 3.9

$$(3.23) \quad G(nh) \leq 2F(nh + 0, u^l, v^l).$$

Proof. Denote by δs (resp δr) the absolute value of the Riemann invariant s (resp r) in the first (resp second) shock wave. By (3.2) and (3.3), $\Delta r + \delta s < 2\Delta r$ on the first shock and $\delta r + \Delta s < 2\Delta s$ on the second shock. So from (3.17), (3.18) and above arguments, we get (3.23). $\square$

From (3.23), (3.24) and (3.25), for any τ ($nh \leq \tau < (n+1)h$),

$$(3.24) \quad \begin{aligned} G(\tau) &\leq G(nh) + 2Kh \leq 2F(nh + 0, u^l, v^l) + 2Kh \\ &\leq 2F_0 + 6KT \equiv M_1. \end{aligned}$$

Now we can establish a priori estimates of u^l and v^l . Denote by T.V.u the total variation of u .

Theorem 3.10 *For any $T > 0$, the variation of u^l and v^l is bounded uniformly for h and $\{a_{mn}\}$. Their upper bound and lower bound, especially the positive lower bound of v^l , are also uniformly bounded.*

Proof. Denote by $T.V^+.u$ (resp $T.V^-.u$) the absolute value of the positive (resp negative) variation of u . Put $f^l \equiv 2u^l = r^l + s^l$. Then $0 \leq f^l(t, 0) \leq Kh$. Without loss of generality, we assume that $u_0(x)$ and $v_0(x)$ are constant outside a bounded interval. Let

$$(3.25) \quad f^l(t, \infty) = r^l(t, \infty) + s^l(t, \infty) \equiv M_2.$$

Then from the definition,

$$f^l(t, 0) + T.V^+.f^l - T.V^-.f^l = f^l(t, \infty).$$

Since $T.V^-.f^l(t, \cdot) \leq G(t)$ for any t , (3.24) yields

$$T.V^+.f^l = f^l(t, \infty) + T.V^-.f^l - f^l(t, 0) \leq M_1 + M_2.$$

Thus we get

$$(3.26) \quad T.V.f^l = T.V.2u^l \leq 2M_1 + M_2.$$

From (3.26), we get

$$|f^l| \leq Kh + 2M_1 + M_2 \leq KT + 2M_1 + M_2 \equiv 2M_3.$$

Therefore we get

$$(3.27) \quad |u_l| \leq M_3.$$

Using Lemma 3.2, we get

$$2a \log v^l = r^l - s^l \geq r_0 - (\max(-r_0, s_0) + KT).$$

Thus we get

$$(3.28) \quad v^l \geq \exp \frac{r_0 - (\max(-r_0, s_0) + KT)}{2a} \equiv \frac{1}{M_5}.$$

From the definition,

$$r^l(t, 0) + T.V^+.r^l - T.V^-.r^l = r^l(t, \infty).$$

Using Lemma 3.3 and (3.24),

$$(3.29) \quad T.V^+.r^l = -r^l(0) + T.V^-.r^l + r(t, \infty) \leq -r_0 + M_1 + r(t, \infty).$$

In view of (3.27) and (3.29), there exists a positive constant M_6 such that

$$(3.30) \quad v^l \leq M_6$$

□

Theorem 3.11 *For any interval $[x_1, x_2] \subset [0, \infty)$, we get*

$$(3.31) \quad \int_{x_1}^{x_2} |u^l(t_2, x) - u^l(t_1, x)| + |v^l(t_2, x) - v^l(t_1, x)| dx \\ \leq M \cdot (|t_2 - t_1| + h), \quad 0 \leq t_1, t_2 < T,$$

where M depends on T , x_1 and x_2 , but not on l and h .

Proof. Without loss of generality, we assume that

$$nh \leq t_1 < (n+1)h < \dots < (n+k)h \leq t_2 < (n+k+1)h.$$

Let

$$\int_{x_1}^{x_2} |u^l(t_2, x) - u^l(t_1, x)| dx \\ \leq I_1 + I_2 + \int_{x_1}^{x_2} |u^l(t_2, x) - u^l((n+k)h + 0, x)| + |u^l(t_1, x) - u^l((n+1)h - 0, x)| dx$$

where

$$I_1 = \int_{x_1}^{x_2} \sum_{i=1}^k |u^l((n+i)h + 0, x) - u^l((n+i)h - 0, x)| dx$$

$$I_2 = \int_{x_1}^{x_2} \sum_{i=1}^{k-1} |u^l((n+i+1)h - 0, x) - u^l((n+i)h + 0, x)| dx$$

and

$$k = \left\lceil \frac{t_2 - t_1}{h} \right\rceil$$

Denote by $1_{[\alpha, \beta]}$ the characteristic function of the interval $[\alpha, \beta]$. We regard $T.V._{-l < x < l} = T.V._{0 < x < l}$. Then,

$$\begin{aligned} I_1 &\leq \sum_{i=0}^{k+1} \sum_{m: \text{integer}} \int_{x_1}^{x_2} T.V._{2ml < x < (2m+2)l} u^l((n+i)h - 0, x) \cdot 1_{[2ml, (2m+2)l]} dx, \\ &\leq \left(\left\lceil \frac{t_2 - t_1}{h} \right\rceil + 2 \right) \cdot \left(\sup_{0 \leq t \leq T} T.V. u^l(t, \cdot) \right) \cdot 2l. \end{aligned}$$

$$\begin{aligned} I_2 &\leq \sum_{i=0}^k \sum_m \int_{x_1}^{x_2} \left(T.V._{(2m-1)l < x < (2m+1)l} u_0^l((n+i+1)h - 0, x) \cdot 1_{[(2m-1)l, (2m+1)l]} + Kh \right) dx, \\ &\leq \sum_{i=0}^k 2l \cdot T.V. u_0^l((n+i+1)h - 0, \cdot) + K(x_2 - x_1)h, \\ &\leq \left(\left\lceil \frac{t_2 - t_1}{h} \right\rceil + 1 \right) \cdot \left(2l \sup_{0 \leq t \leq T} T.V. u_0^l(t, \cdot) + K(x_2 - x_1)h \right). \end{aligned}$$

The remaining terms can be evaluated similarly. For

$$\int_{x_1}^{x_2} |v^l(t_2, x) - v^l(t_1, x)| dx,$$

we also have a similar estimate. Combining these results gives (3.31). $\square$

4 Convergence of The Approximate Solution

Let $h_n = T/n$ and $h_n/l_n = \tilde{\delta} < \delta \equiv 1/M_5$. Consider the sequence (u^{l_n}, v^{l_n}) ($n = 1, 2, \dots$). Then from Theorem 3.9 and Theorem 3.10, there exists a subsequence which converges in L^1_{loc} to functions (u, v) uniformly for $t \in [0, T]$. Now we shall prove that $u(x, t)$ and $v(x, t)$ are the weak solutions of initial boundary value problem (1.6), (1.7) and (1.8) provided $\{a_{nm}\}$ is suitably chosen, namely, they satisfy the integral identity

$$\begin{aligned} (4.1) \quad &\int_0^T \int_0^\infty u \phi_t + \left(\frac{a^2}{v} \right) \phi_x + \frac{K}{1 + \int_0^x v(t, \zeta) d\zeta} \cdot \phi dx dt \\ &+ \int_0^\infty u_0(x) \phi(0, x) dx = 0, \end{aligned}$$

$$(4.2) \quad \int_0^T \int_0^\infty v \psi_t - u \psi_x dx dt + \int_0^\infty v_0(x) \psi(0, x) dx = 0,$$

for any smooth functions ϕ and ψ with compact support in the region $\{(t, x) : 0 \leq t < T, 0 \leq x < \infty\}$ and $\phi(t, 0) = 0$. Now we know that u_0^l and v_0^l are weak solutions in each time strip $nh \leq t < (n+1)h$ so that for each test function ϕ satisfying $\phi(t, 0) = 0$,

$$(4.3) \quad \begin{aligned} & \int_{nh}^{(n+1)h} \int_0^\infty u^l \phi_t + \left(\frac{a^2}{v^l} \right) \phi_x + U^l(t, x) \cdot \phi \, dx dt \\ & + \int_0^\infty u^l(nh + 0, x) \phi(nh, x) \\ & - \int_0^\infty u^l((n+1)h - 0, x) \phi((n+1)h, x) dx = 0 \end{aligned}$$

If we sum this over n , we get

$$(4.4) \quad \begin{aligned} & \int_0^T \int_0^\infty u^l \phi_t + \left(\frac{a^2}{v^l} \right) \phi_x + U^l(t, x) \cdot \phi \, dx dt + \int_0^\infty u^l(0, x) \phi(0, x) \\ & = - \sum_{k=1}^N \int_0^\infty \{ u^l(kh + 0, x) - u^l(kh - 0, x) \} \cdot \phi(kh, x) dx \end{aligned}$$

where $N=T/h$. When $N \rightarrow \infty$, the right-hand side of the above equality tends to 0 for almost every $\{a_{nm}\} \in A$ (see [4]). It is immediate to see that

$$\int_0^\infty u^l(0, x) \phi(0, x) dx \rightarrow \int_0^\infty u_0(x) \phi(0, x) dx \quad (N \rightarrow \infty).$$

Lemma 4.1

$$(4.5) \quad U^l(t, x) \rightarrow \frac{K}{1 + \int_0^x v(t, \zeta) d\zeta} \quad (N \rightarrow \infty).$$

locally uniformly for t and x .

Proof. Let $nh \leq t < (n+1)h$, $x \in ((m-1)l, (m+1)l)$, $m : \text{odd}$. Then

$$(4.6) \quad \left| \int_0^x v^l(nh, \zeta) d\zeta - \sum_{j=1}^{\frac{m+1}{2}} v^l(nh, c_{2j-1n}) \right| \leq \|v^l\|_\infty \cdot l.$$

On the other hand

$$(4.7) \quad \int_0^x v^l(t, \zeta) d\zeta \rightarrow \int_0^x v(t, \zeta) d\zeta \quad (N \rightarrow \infty).$$

locally uniformly for t and x .

We get

$$\begin{aligned}
 (4.8) \quad & \left| \int_0^x v^l(t, \zeta) d\zeta - \int_0^x v^l(nh, \zeta) d\zeta \right| \\
 & \leq \int_0^x \sum_{m: \text{odd}} T \cdot V \cdot (m-1)l < \zeta < (m+1)l v^l(nh, \cdot) \cdot 1_{[(m-1)l, (m+1)l]} d\zeta \\
 & \leq \sup_{0 \leq t \leq T} T \cdot V \cdot v^l \cdot 2l.
 \end{aligned}$$

From (4.6), (4.7) and (4.8), we get (4.5). $\square$

For each test function ψ , v^l also satisfies,

$$\begin{aligned}
 (4.9) \quad & \int_0^T \int_0^\infty (v^l \psi_t - u^l \psi_x) dx dt + \int_0^\infty v^l(0, x) \psi(0, x) dx \\
 & = - \sum_{k=1}^N \int_0^\infty \{v^l(kh + 0, x) - v^l(kh - 0, x)\} \cdot \psi(kl, x) dx \\
 & \quad - I_1 - I_2.
 \end{aligned}$$

where

$$I_1 = \sum_{n=0}^{N-1} \int_{nh}^{(n+1)h} U^l(t, 0)(t - nh) \psi(t, 0) dt$$

and

$$I_2 = \sum_{n=0}^{N-1} \sum_{m: \text{odd}} \int_{nh}^{(n+1)h} \{U^l(t, ml + 0) - U^l(t, ml - 0)\} (t - nh) \psi(t, ml) dt.$$

The first term of the the right-hand side of equality (4.9) tends to 0 for almost every $\{a_{nm}\} \in A$ (see [4]). It is also immediate to see that

$$\int_0^\infty v^l(0, x) \psi(0, x) dx \rightarrow \int_0^\infty v_0(x) \psi(0, x) dx \quad (N \rightarrow \infty).$$

We shall show that $I_1, I_2 \rightarrow 0$ as $N \rightarrow \infty$.

$$\begin{aligned}
 (4.10) \quad & I_1 \leq \|\psi\|_\infty \sum_{n=0}^{N-1} \int_{nh}^{(n+1)h} U^l(t, 0)(t - nh) dt \\
 & \leq \|\psi\|_\infty \sum_{n=0}^{N-1} \int_{nh}^{(n+1)h} K(t - nh) dt \\
 & \leq \|\psi\|_\infty ha^2 T.
 \end{aligned}$$

$$\sum_{m: \text{odd}} \int_{nh}^{(n+1)h} \{U^l(t, ml + 0) - U^l(t, ml - 0)\} (t - nh) \psi(t, ml) dt \leq K \|\psi\|_{\infty} h^2.$$

Thus we get

$$(4.11) \quad I_2 \leq \|\psi\|_{\infty} \sum_{n=0}^{N-1} Kh^2 \leq K \|\psi\|_{\infty} hT$$

From above arguments, we can conclude that u and v satisfy (4.1) and (4.2). Thus we obtain our main result.

Theorem 4.2 (Main Result) Suppose that $u_0(x), v_0(x) \in BV(\mathbf{R}_+)$, and that $v_0(x) \geq \delta_0 > 0$ for all $x > 0$ with some positive constant δ_0 . Then (1.10), (1.11) and (1.12) have a global weak solution which belongs to the class

$$u, v \in L^{\infty}(0, T; BV(\mathbf{R}_+)) \cap Lip([0, T]; L^1_{loc}(\mathbf{R}_+))$$

for any $T > 0$.

Appendix

A Proof of Lemma 3.5

Let $g(x) = -f(-x)$, and put

$$P(u_1, v_1, u_2, v_2) = \Delta r_1 + \Delta s_1$$

$$P(u_2, v_2, u_3, v_3) = \Delta r_2 + \Delta s_2$$

$$P(u_1, v_1, u_3, v_3) = \Delta r_3 + \Delta s_3$$

Then it is obvious that

$$\begin{aligned} & \Delta r_3 + g(\Delta s_3) + \Delta s_3 + g(\Delta r_3) \\ & \leq \Delta r_1 + \Delta r_2 + \Delta s_1 + \Delta s_2 + g(\Delta r_1) + g(\Delta r_2) + g(\Delta s_1) + g(\Delta s_2) \end{aligned}$$

We notice that $f'' \leq 0$ and hence

$$\leq \Delta r_1 + \Delta r_2 + \Delta s_1 + \Delta s_2 + g(\Delta r_1 + \Delta r_2) + g(\Delta s_1 + \Delta s_2).$$

Let $x + g(x) = h(x)$, $\Delta r_3 = p'$, $\Delta s_3 = q'$, $\Delta r_1 + \Delta r_2 = p$ and $\Delta s_1 + \Delta s_2 = q$.

Then

$$(A.1) \quad h(p') + h(q') \leq h(p) + h(q).$$

Put $K = h(p') + h(q')$. We shall estimate $p + q$ from below under the restriction (A.1). To do this, as h is monotone increasing function, we must estimate $p + q$ from below under the restriction

$$(A.2) \quad h(p) + h(q) = K.$$

We do this by using Lagrange's method of indeterminate coefficients.

Put $G(p, q, \lambda) = p + q + \lambda (h(p) + h(q) - K)$. Then

$$G_p = 1 + \lambda h'(p) = 0, \quad G_q = 1 + \lambda h'(q) = 0.$$

Because $h''(x) > 0$, we get $p = q$. So $p + q$ attains its extremum at $p = q$.

We can show that when $p = q$, $p + q$ is minimum under the restriction (A2).

Therefore

$$h(p) = h(q) = \frac{K}{2} = \frac{h(p') + h(q')}{2} \geq h\left(\frac{p' + q'}{2}\right).$$

Hence it follows that

$$p = q \geq \frac{p' + q'}{2}.$$

Thus we get

$$(A.3) \quad p + q \geq p' + q'.$$

which proves Lemma 3.5.

B Proof of Lemma 3.6

To prove Lemma 3.6, we must check the following 12 cases:

- 1) $c_{1n} < l$,
 - (1) S_2 crosses i_0^{n-} ,
 - (2) R_2 crosses i_0^{n-} ,
 - (3) no wave cross i_0^{n-} .
- 2) $c_{1n} \geq l$,
 - (1) S_2 and S_1 cross i_0^{n-} ,
 - (2) R_2 and S_1 cross i_0^{n-} ,
 - (3) S_2 and R_1 cross i_0^{n-} ,
 - (4) R_2 and R_1 cross i_0^{n-} ,
 - (5) S_1 crosses i_0^{n-} ,
 - (6) R_1 crosses i_0^{n-} ,
 - (7) S_2 crosses i_0^{n-} ,
 - (8) R_2 crosses i_0^{n-} ,
 - (9) no wave cross i_0^{n-} .

Put $r_+^{n-1} = r^l(a_{1\ n-1})$, $s_+^{n-1} = s^l(a_{1\ n-1})$, $r_-^{n-1} = -s_-^{n-1} = r^l((n-1)h + 0, 0)$, and $\delta_{n-1} = U^l(a_{1\ n-1})$.
 Put $r_+^{n-1'} = r^l((n-1)h + 0, 2l)$ and $s_+^{n-1'} = s^l((n-1)h + 0, 2l)$.
 Put $A = (r_-^{n-1}, s_-^{n-1})$, $B = (r_+^{n-1}, s_+^{n-1})$ and $B' = (r_+^{n-1'}, s_+^{n-1'})$.
 Put $C = (r_+^{n-1} + Kh, s_+^{n-1} + Kh)$,
 (resp $= (r_+^{n-1'} + \delta_{n-1}h, s_+^{n-1'} + \delta_{n-1}h)$) if $c_{1n} < l$ (resp $c_{1n} \geq l$).
 If R_2 crosses i_0^{n+} , $F(i_0^{n+}) = 0 \leq F(i_0^{n-})$, so that it is sufficient to consider the cases when S_2 crosses i_0^{n+} .

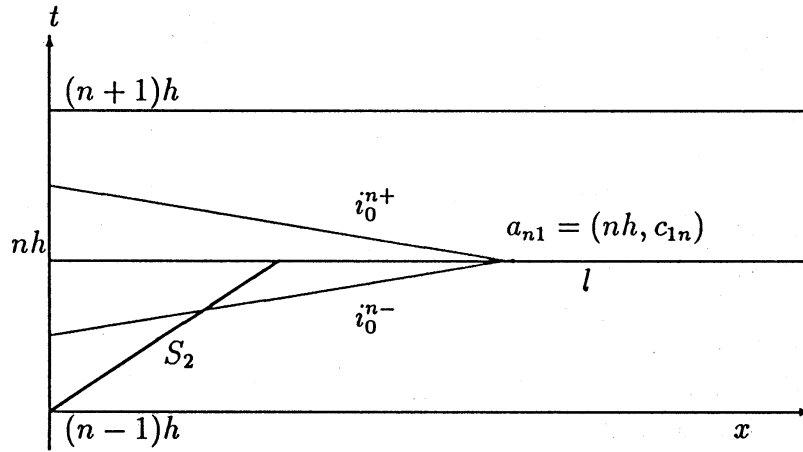


Figure.2

1) $c_{1n} < l$.

(1) S_2 crosses i_0^{n-} (Figure 2). Denote by I (resp II) the halfspace $\{(r, s) | r + s < 0\}$ (resp $\{(r, s) | r + s \geq 0\}$).)

i) $C \in I$.

In this case S_2 crosses i_0^{n+} . Denote by $V(PQ)$ the absolute value of the total variation of r and s by the line segment PQ . From Figure.3,

$$F(i_0^{n+}) = V(A'C) \leq V(A'C') = V(AB) = F(i_0^{n-}).$$

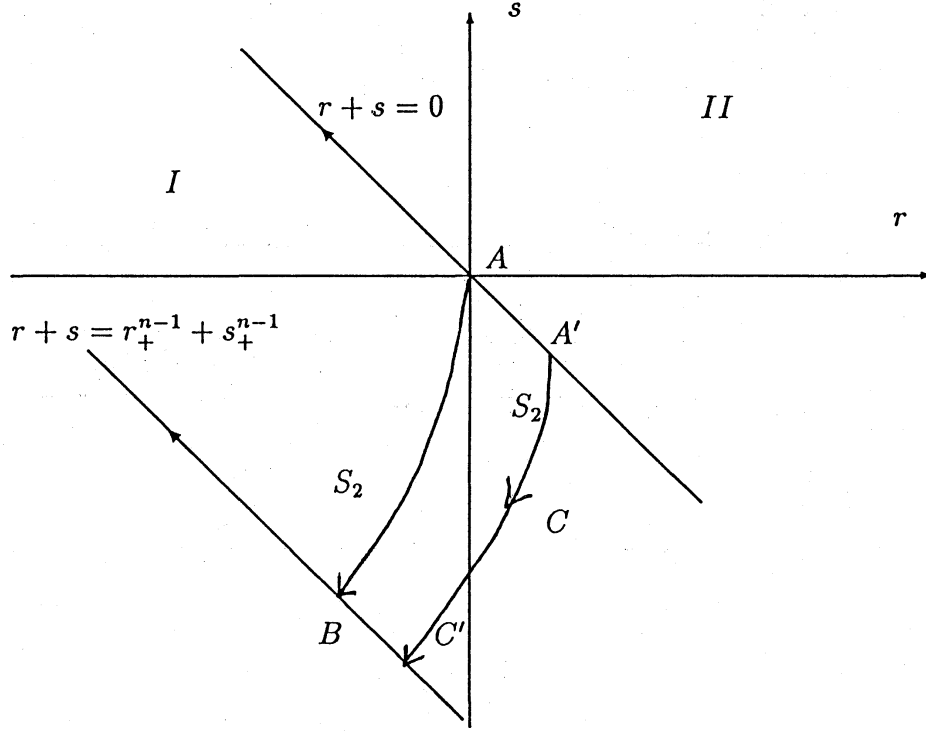


Figure.3

ii) $C \in II$.

In this case R_2 crosses i_0^{n+} . Then

$$(B.1) \quad F(i_0^{n-}) \geq F(i_0^{n+}) = 0.$$

(2) R_2 crosses i_0^{n-} .

In this case $B \in II$ so that R_2 crosses i_0^{n+} . Then

$$(B.2) \quad F(i_0^{n-}) = F(i_0^{n+}) = 0.$$

(3) no wave crosses i_0^{n-} .

In this case (r_+^{n-1}, s_+^{n-1}) is on the line $r + s = 0$. Hence $C \in II$. It is obvious that (B.3) also holds.

2) $c_{1n} \geq l$.

(1) S_2 and S_1 cross i_0^{n-} . (Figure.4)

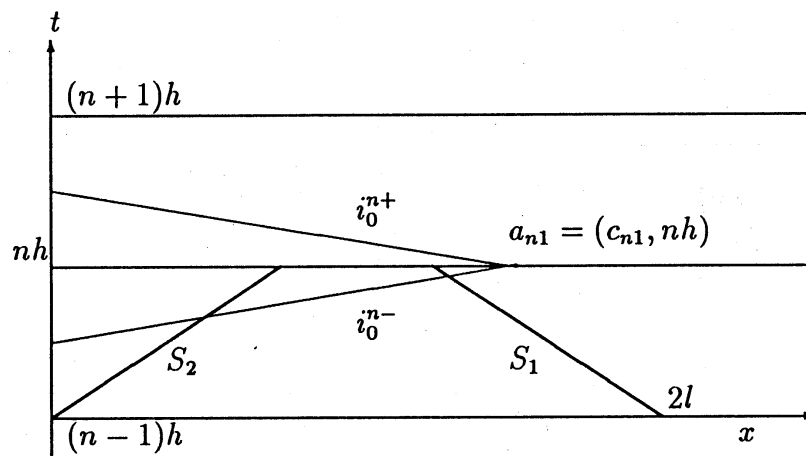


Figure.4

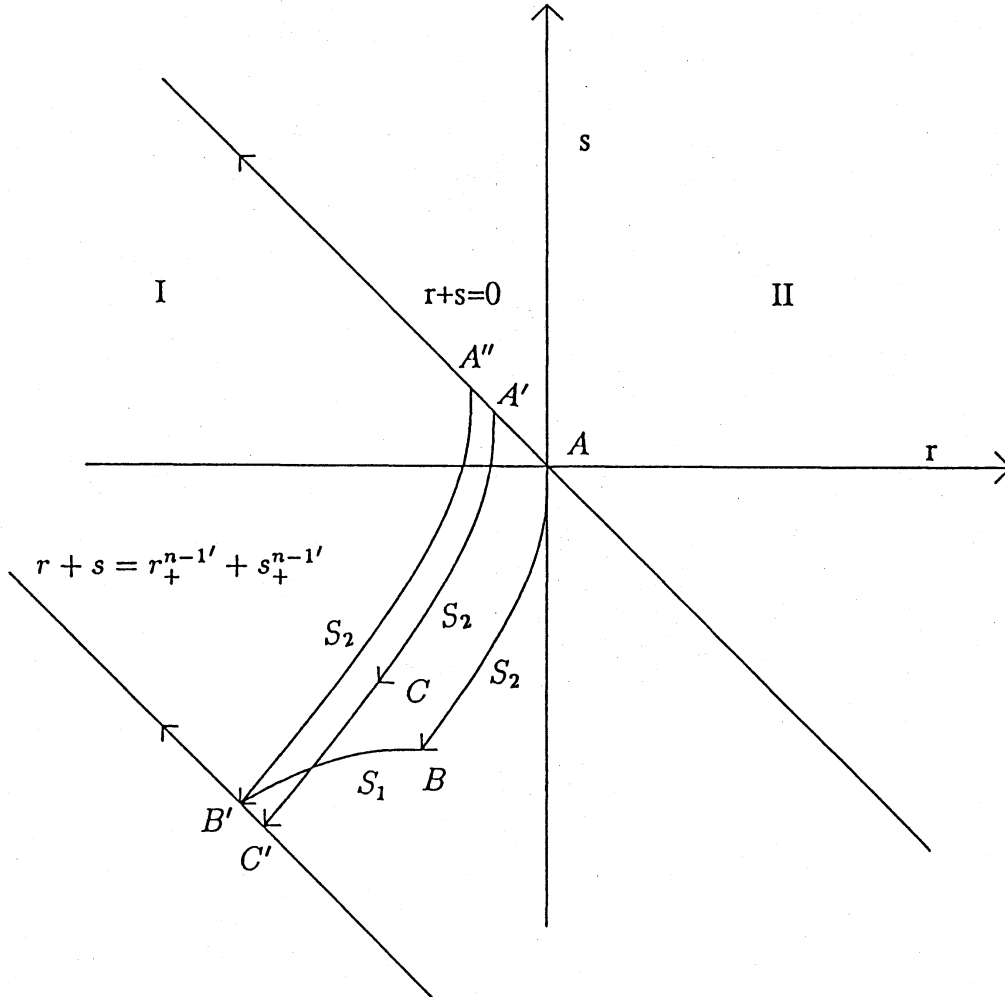


Figure.5

i) $C \in I$.

From Figure.5,

$$F(i_0^{n+}) = V(A'C) \leq V(A'C') = V(A''B') = V(AB') = F(i_0^{n-}).$$

ii) $C \in II$ implies that R_2 crosses i_0^{n+} . So we get (B2).

(2) R_2 and S_1 cross i_0^{n-} .

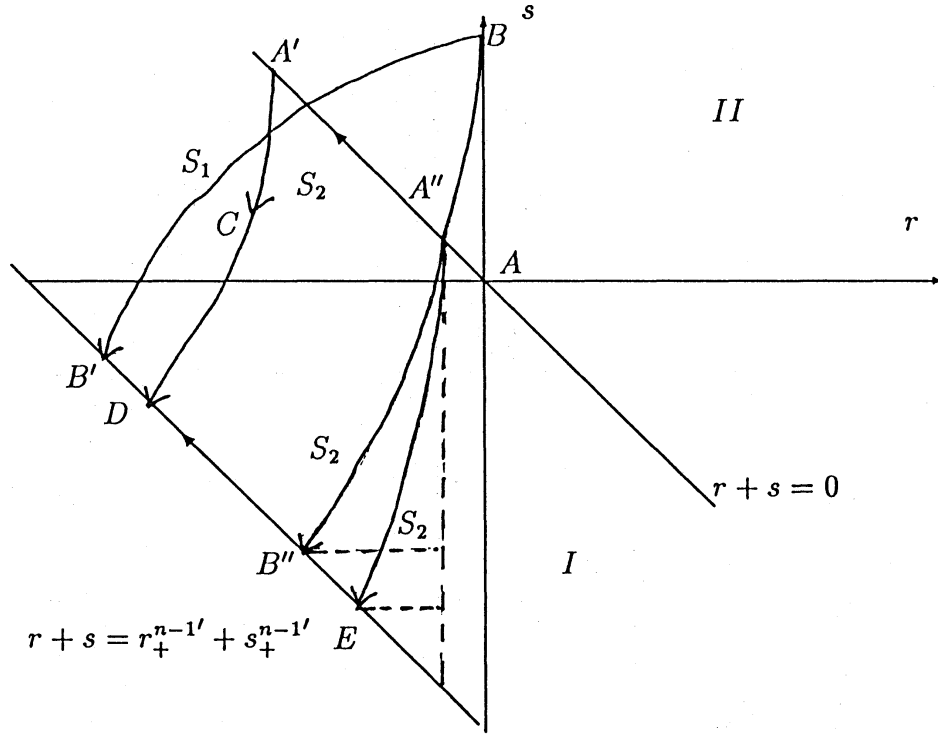


Figure.6

i) $C \in I$.

From Figure.6,

$$\begin{aligned} F(i_0^{n+}) &= V(A'C) \leq V(A'D) = V(A''E) = V(A''B'') \\ &\leq V(BB'') = V(BB') = F(i_0^{n+}) \end{aligned}$$

ii) $C \in II$.

Thus R_2 crosses i_0^{n+} , and we get (B2).

(3) S_2 and R_1 cross i_0^{n-} .

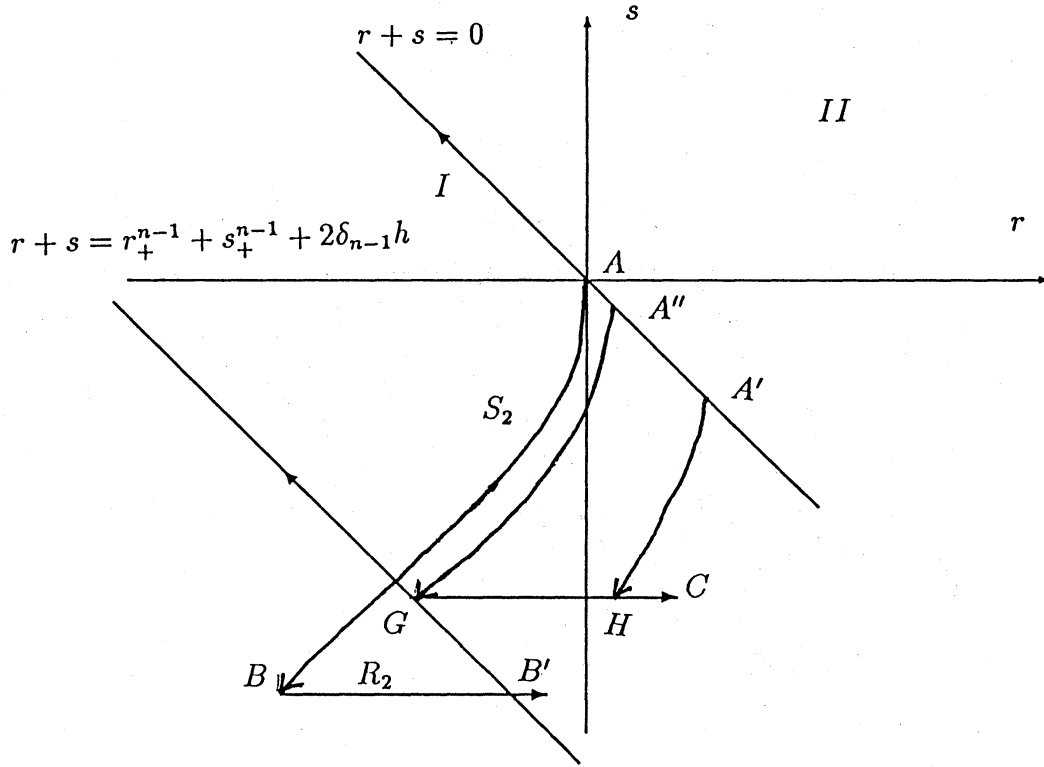


Figure.7

Put $G = (r_+^{n-1} + \delta_{n-1}h, s_+^{n-1} + \delta_{n-1}h)$ and $H = (r^l(a_{1n}), s^l(a_{1n}))$. Then H is on the line CG .

i) $H \in I$.

From Figure.7,

$$F(i_0^{n+}) = V(A'H) \leq V(A''G) \leq V(AB) = F(i_0^{n-}).$$

ii) $H \in II$, so

R_2 crosses i_0^{n+} , and we get (B2).

(4) R_2 and R_1 cross i_0^{n-} .

In this case, R_2 crosses i_0^{n+} . So we get (B3).

(5) S_1 crosses i_0^{n-} .

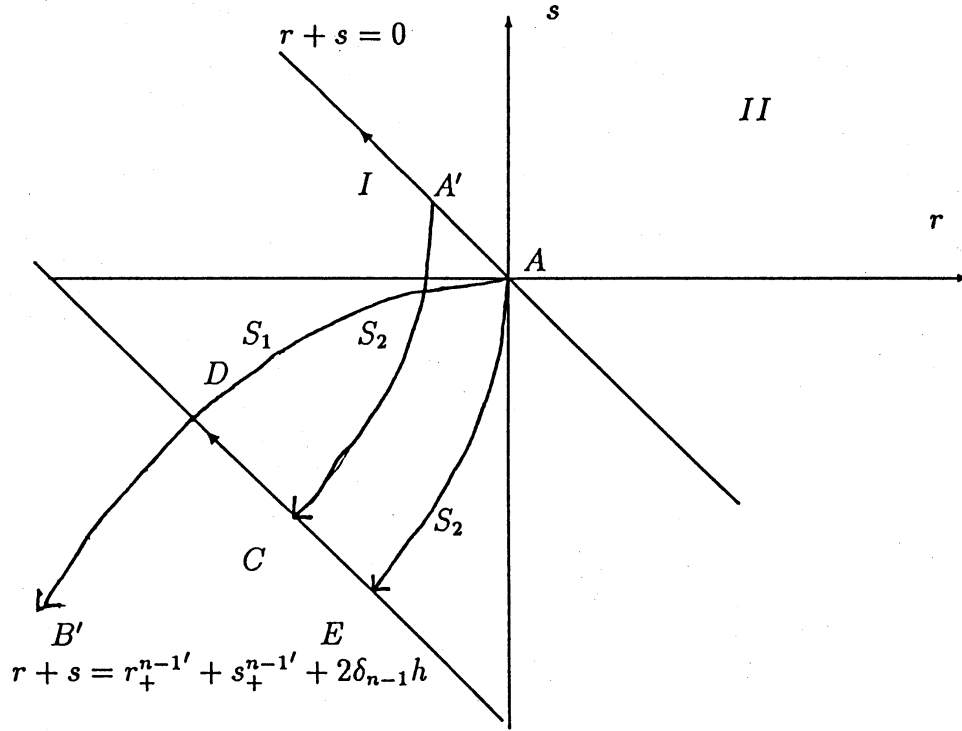


Figure.8

i) $C \in I$.

From Figure.8,

$$\begin{aligned} F(i_0^{n+}) &= V(A'C) = V(AE) = V(AD) \\ &\leq V(AB') = F(i_0^{n-}) \end{aligned}$$

Thus we get (B1).

ii) $C \in II$.

R_2 crosses i_0^{n+} . So we get (B2).

(6) R_1 crosses i_0^{n-} .

In this case, it is obvious that $F(i_0^{n+}) = 0$. Hence we get (B3).

Cases (7), (8) and (9) are almost the same as cases (1), (2) and (3) in 1). Thus, we obtain Lemma 3.6.

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