

Monte Carlo Iterative k-Source Method for Fixed-Source Calculations in Nearly Critical Subcritical Systems

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Abstract

In addition to the conventional fixed-source method, the iterative k-source (IKS) method is another Monte Carlo algorithm used for solving fixed-source problems. In contrast to the fixed-source method, the IKS method iterates the fission sources for each generation, which is similar to the k -eigenvalue calculation. The IKS method does not require following long-chain fission reactions until the end, even for a nearly critical subcritical system. The source-multiplication factor, k_s , which is supposed to be less than unity, is calculated for each cycle. However, k_s occasionally or frequently exceeds unity in a nearly critical system owing to statistical fluctuations, in which case the result of the IKS method is biased. To eliminate this bias, the frequency at which k_s exceeds unity should be minimized by increasing the source size per cycle. As demonstrated in this study, the computational efficiency of the IKS method is three to ten times that of the conventional fixed-source method.

Keywords: Monte Carlo; fixed source; subcritical; iterative k-source

1. Introduction

In the fixed-source Monte Carlo calculation of a subcritical multiplying system, a source neutron is emitted from a fixed source. The source neutron and its subsequently produced fission neutrons are monitored until they are annihilated. In a deep subcritical system, the fission chain terminates within a few generations. For a nearly critical subcritical system, the chain continues for several generations. Thus, fixed-source calculations are time consuming and require a large

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1 amount of data storage for the fission-source bank of a long-lasting fission chain. Furthermore, in
2 a nearly critical subcritical system, some fission chains terminate rapidly, whereas others continue
3 for an extremely long time, resulting in a large variance in the tallies. Monte Carlo fixed-source
4 calculations in nearly critical subcritical systems are difficult to manage.

5 By contrast, the power-iteration method is used for k -eigenvalue calculations, where the
6 fission sources in a generation are repeatedly updated until the desired statistics are obtained. The
7 stability of this iterative procedure does not depend on the criticality of the multiplying system but
8 on the dominance ratio. Compared with the fixed-source method, the power-iteration method
9 yields consistently stable results regardless of the criticality. A method that utilizes the power-
10 iteration scheme for fixed-source calculations was developed previously (Brown and Sutton, 1996;
11 Korobeinikov, 2000; Shang, et al., 2018; Mallick et al., 2022). Hereinafter, this iterative method
12 is referred to as the iterative k -source (IKS) method, after Mallick et al. (2022).

13 Shang, et al. (2018) evaluated the accuracy and computational efficiency of the IKS method
14 and it concluded that “much calculation time can be saved through the method.” The highest k_{eff}
15 tested in this study was less than 0.987, which did not necessarily reflect the “nearly critical” state.
16 Mallick et al. (2022) performed numerical tests of the IKS method for subcritical systems, in which
17 the source-multiplication factor reached 0.99988. In a nearly critical subcritical system, the
18 contribution of the fixed source is insignificant relative to that of fission source. This is attributable
19 to the improper sampling of the fixed-source contribution, as reported by Mallick et al. (2022).
20 Furthermore, reference solutions for nearly critical systems could not be provided using the fixed-
21 source method because of the long fission chain. Therefore, the validity and efficiency of the IKS
22 method for nearly critical systems have not been sufficiently demonstrated.

23 Based on previous results pertaining to the IKS method, this study specifically focuses on the
24 feasibility of the IKS method in terms of accuracy and efficiency for nearly critical subcritical
25 systems.

26 The remainder of this paper is organized as follows: In Section 2, the fundamental concepts
27 of the IKS method and the Monte Carlo calculation algorithm are presented. In Section 3,
28 numerical tests of the IKS method for nearly critical subcritical systems are described. Finally,

Section 4 presents the conclusions of this study based on the IKS method.

2. Revisit of Monte Carlo iterative k-source method

2.1 Theory of IKS method

Prior to describing the IKS method, we introduce ordinary fixed-source problems. The neutron transport equation for fixed-source problems is expressed as follows:

$$\mathbf{L}\phi(\mathbf{r}, \mathbf{v}) = \mathbf{F}\phi(\mathbf{r}, \mathbf{v}) + \mathbf{Q}(\mathbf{r}, \mathbf{v}), \quad (1)$$

where $\mathbf{v}$ is the velocity vector of the neutron, $\mathbf{r}$ the position vector, and ϕ the neutron flux.

$\mathbf{L}\phi(\mathbf{r}, \mathbf{v})$ represents the streaming, collision, and scattering of neutrons as follows:

$$\mathbf{L}\phi(\mathbf{r}, \mathbf{v}) = \boldsymbol{\Omega} \cdot \nabla \phi(\mathbf{r}, \mathbf{v}) + \Sigma_t(\mathbf{r}, \mathbf{v})\phi(\mathbf{r}, \mathbf{v}) - \int \Sigma_s(\mathbf{r}, \mathbf{v}' \rightarrow \mathbf{v})\phi(\mathbf{r}, \mathbf{v}')d\mathbf{v}', \quad (2)$$

where $\boldsymbol{\Omega}$ is the neutron direction, Σ_t the total cross-section, and Σ_s the scattering cross-section.

$\mathbf{F}\phi(\mathbf{r}, \mathbf{v})$ represents the fission-neutron production, which is expressed as follows:

$$\mathbf{F}\phi(\mathbf{r}, \mathbf{v}) = \chi_f(\mathbf{r}, \mathbf{v}) \int \nu(\mathbf{v}')\Sigma_f(\mathbf{r}, \mathbf{v}')\phi(\mathbf{r}, \mathbf{v}')d\mathbf{v}', \quad (3)$$

where χ_f is the fission neutron spectrum, ν the number of neutrons per fission, Σ_f the fission cross-section, and $\mathbf{Q}$ a fixed-source term.

In normal fixed-source calculations, a particle is first emitted from a fixed source. The particle undergoes collision, scattering, capture, and fission reactions during the random-walk process until it is killed by escaping from the external boundary or absorption. If the implicit capture option is used, then the particle will be killed via the Russian roulette instead of being absorbed. The neutrons generated by the fission reaction proceed in the same manner.

Because the neutron flux ϕ satisfies the equilibrium state of the fixed-source transport equation (Eq. (1)), we define the source-multiplication factor k_s as follows:

$$k_s = \frac{\langle \mathbf{F}\phi \rangle}{\langle \mathbf{F}\phi \rangle + \langle \mathbf{Q} \rangle}, \quad (4)$$

where the angle brackets denote the integration over the entire phase space, i.e., energy, direction, and position. Based on Eq. (4), we obtain

$$1 - k_s = \frac{\langle \mathbf{Q} \rangle}{\langle \mathbf{F}\phi \rangle + \langle \mathbf{Q} \rangle}. \quad (5)$$

Thus, the following relationship is derived from Eqs. (4) and (5):

$$k_s: 1 - k_s = \langle \mathbf{F}\phi \rangle: \langle \mathbf{Q} \rangle. \quad (6)$$

Next, we consider the procedure for the fixed-source problem in an iterative manner, similar to the power-iteration method for k -eigenvalue. Suppose that k_s is known and N neutrons are started as source neutrons in each cycle. The starting N neutrons are the sum of fission neutrons inherited from the previous cycle and source neutrons from the fixed source. Based on Eq. (6), at the beginning of each cycle, N neutrons are allocated to the fission and fixed sources under probabilities k_s and $1 - k_s$, respectively. All N neutrons are eventually killed via escape or absorption (capture and fission) (or the Russian roulette) by the end of the cycle. Using Eqs. (5) and (1), we obtain

$$\frac{\langle \mathbf{F}\phi \rangle + \langle \mathbf{Q} \rangle}{\langle \mathbf{Q} \rangle} = \frac{\langle \mathbf{L}\phi \rangle}{\langle \mathbf{Q} \rangle} = \frac{1}{1 - k_s}. \quad (7)$$

Here, $\langle \mathbf{L}\phi \rangle$ corresponds to the total neutrons N emitted from the fission and fixed sources, and $\langle \mathbf{Q} \rangle$ corresponds to neutrons from the fixed source. As the tally scores of the IKS method are normalized in each cycle by dividing them by N , Eq. (7) represents the ratio between the tally score obtained by the iterative procedure of the IKS method in each cycle and the tally score normalized by the number of fixed sources. Thus, the tally scores equivalent to the results of the fixed-source method can be obtained by multiplying the tally scores of the IKS method by $1/(1 - k_s)$.

Unlike the conventional fixed-source method, the IKS method features fission chain reactions generated by a source neutron, which spans multiple generations, that need not be tracked until they are all terminated. In the fixed-source method, a new source neutron cannot be started until all fission neutrons generated by the previous source neutron are all killed if a sequential calculation is employed. In the IKS method, the fission chain can be aborted before all progenies are killed. The IKS method can be implemented by slightly modifying the k -eigenvalue calculation algorithm of existing Monte Carlo codes.

2.2 Monte Carlo algorithm of IKS

The Monte Carlo algorithm of the IKS method adopted in this study is the same as that used

by Mallick et al. (2022). The algorithm is modified slightly as follows:

(1) For each cycle, the nominal source size, which includes both fission and fixed sources, is set to N . However, in the first cycle, the fission-source distribution is unknown. Thus, all starting neutrons N are emitted from the fixed source.

(2) The neutrons from both the fission and fixed sources are tracked via the random-walk process in the same manner as in the ordinary k -eigenvalue criticality calculation.

(3) After all the source neutrons are killed via escape and absorption (or the Russian roulette), the source-multiplication factor k_s is calculated as follows:

$$k_s = \frac{N_f}{N}, \quad (8)$$

where N_f denotes the number of fission neutrons generated during the cycle, and N_f is calculated in the same manner as in the ordinary k -eigenvalue calculation.

(4) The number of source neutrons from the fixed source N_s in the next cycle is determined as follows:

$$N_s = \left\lfloor \left(\frac{1}{k_s} - 1 \right) N_f + \xi \right\rfloor, \quad (9)$$

where ξ is a uniform pseudorandom number in the range $[0,1)$.

(5) If $k_s > 1$ because of statistical fluctuations of the Monte Carlo calculation, then $N_s = 0$.

(6) The number of starting neutrons in the next cycle N' is expressed as

$$N' = N_s + N_f. \quad (10)$$

The weights of the source particle are set to N/N' such that the total source weight is a constant N for each cycle.

(7) In the next cycle, N_f fission neutrons are started from the fission-source sites calculated in the previous cycle and N_s neutrons are started from the fixed source. The energy spectrum of the source is determined from the spectra of either the fission or fixed source, depending on the source type.

(8) Steps (2)–(7) are repeated until the end of the total cycles. The inactive cycles before the convergence of k_s and the fission-source distribution are discarded from the tally score estimates. The number of inactive cycles should be determined based on an appropriate diagnostic method for k -eigenvalue calculations (e.g., Ueki and Brown, 2005; Brown and

Martin, 2018).

(9) Using k_s averaged over the active cycles, $\overline{k_s}$, the tally scores is multiplied by $1/(1 - \overline{k_s})$.

For example, the subcritical multiplication factor, which is defined as the number of fission neutrons (including all progenies) per fixed-source neutron, is calculated as

$$M_s = \frac{\overline{k_s}}{1 - \overline{k_s}}. \quad (11)$$

The flow chart of this algorithm is illustrated in Fig.1. Unlike the method proposed by Mallick et al. (2022), which fixes the number of sources per cycle, the number of sources N' varies from cycle to cycle. Instead, a constant effective source size is maintained by adjusting the starting weight as outlined in step (6). In addition, for cases where $k_s > 1$, a setting where $N_s = 0$ has been newly added in step (5).

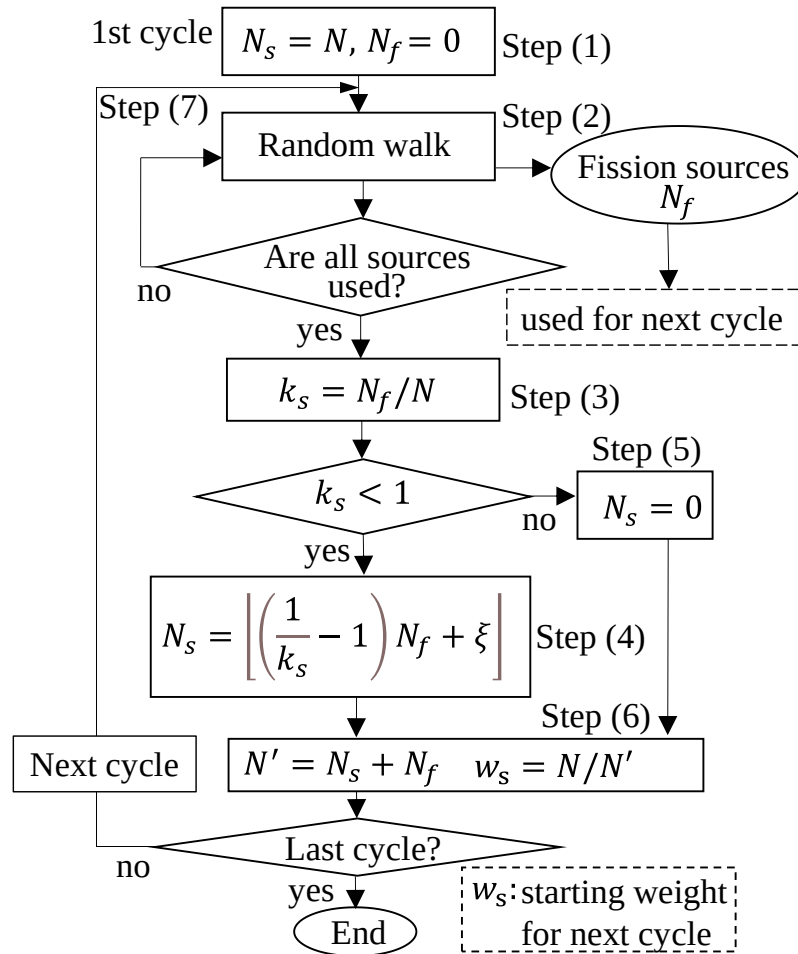


Fig. 1 Flow chart of IKS method

The standard deviation of M_s is calculated using the error-propagation method as follows:

$$\sigma_A(M_s) = \frac{\sigma_A(\overline{k_s})}{(1 - \overline{k_s})^2}, \quad (12)$$

where $\sigma_A(\overline{k_s})$ is the standard deviation of $\overline{k_s}$ obtained via normal statistical processing. It does not account for the intercycle correlation effect and is not a “real” standard deviation. The subscript “A” denotes the “apparent” standard deviation.

As mentioned in Step (5), if k_s is greater than unity, then the number of fixed sources N_s is inevitably set to zero in the next cycle. This can occur when k_s is slightly lower than unity in a nearly critical subcritical system. In addition, if the nominal source size N is not sufficiently large, k_s tends to become larger than unity owing to the enlarged fluctuation of k_s . The fraction of number of fixed sources N_s affects the calculation results via the difference in the importance and neutron spectra between the fixed and fission sources. Setting $N_s = 0$ yields biased results, which should be avoided. Thus, the source size N in each cycle should be set as large as possible such that the frequency of k_s exceeding unity can be minimized by reducing the statistical fluctuation of k_s . However, completely eliminating the frequency of k_s exceeding unity in a nearly critical subcritical system is almost impossible. In the next section, the effect of the frequency of k_s exceeding unity on the results is examined based on numerical tests.

Notably, even if k_s exceeds unity, a value exceeding zero can be assigned to N_s . Alternatively, the cycle that yields k_s exceeding unity can be discarded and the cycle can be repeated until k_s is less than unity. However, these methods yield biased results, similar to setting $N_s = 0$, which should be avoided.

3. Numerical results

3.1 Overview of numerical test problems

Numerical tests for the IKS method are presented in this section. The geometry of the numerical test problems was composed of a two-dimensional square-shaped homogenized UO₂ rod array surrounded by a light-water reflector with a thickness of 5 cm, as shown in Fig. 2. A vacuum boundary condition was imposed on the external boundary of the reflector. Monte Carlo calculations were performed using three energy group constants. The constants were calculated using the standard thermal reactor core analysis code SRAC (Okumura et al., 2007). Tables 1 and 2 list the three energy group constants for the homogenized fuel rod array and light-water reflector,

respectively. Isotropic scattering in the laboratory system was assumed in this study. Fixed-source neutrons were emitted only from the first energy group.

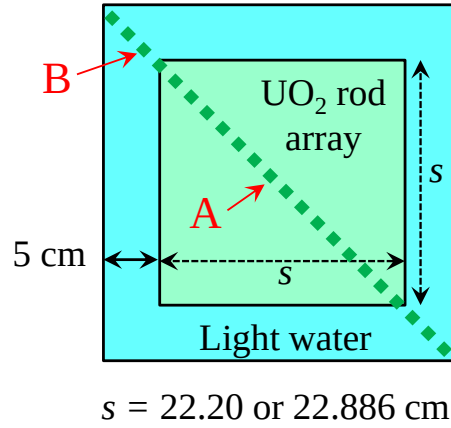


Fig. 2 Geometry of test problem and fixed-source position (not to scale)

The side length s of the fuel rod array in Fig. 2 was varied to adjust the subcriticality to 22.20 or 22.886 cm. The first example (case 1), where $s = 22.20 \text{ cm}$, indicated $k_{eff} \approx 0.987$, which was significantly below the criticality. The second one (case 2), where $s = 22.886 \text{ cm}$, indicated $k_{eff} \approx 0.999$, which was considered to be a nearly critical subcritical multiplying system. A single-point source was placed at point A or B, as shown in Fig. 2. Point A was located at the center of the fuel rod array, whereas point B was located $4.243 \text{ cm} (= 3 \text{ cm} \times \sqrt{2})$ away along the diagonal from the top-left corner. All calculations were performed using an inhouse research purpose Monte Carlo solver that was used in previous studies (e.g., Yamamoto et al., 2022).

Table 1 Three energy group constants of homogenized UO₂ rod array

		1st group	2nd group	3rd group
Σ_t (cm ⁻¹)	Total cross-section	0.298289	0.833341	1.63895
Σ_c (cm ⁻¹)	Capture cross-section	3.74414×10^{-4}	9.77075×10^{-3}	0.0301365
Σ_f (cm ⁻¹)	Fission cross-section	3.01089×10^{-3}	2.12425×10^{-3}	0.0560435
ν	Average number of fission neutrons	2.438	2.438	2.438
χ_f	Fission spectrum	0.87491	0.12509	0.0
χ_s	Source spectrum	1.0	0.0	0.0
$\Sigma_{sg \rightarrow g}$ (cm ⁻¹)	Self-group scattering cross-section	0.221061	0.777643	1.55277
$\Sigma_{sg \rightarrow g+1}$ (cm ⁻¹)	Group transfer scattering cross-section	0.0738427	0.043803	0.0

Table 2 Three energy group constants of light water reflector

		1st group	2nd group	3rd group
Σ_t (cm ⁻¹)	Total cross-section	0.332073	1.12647	2.78124
Σ_c (cm ⁻¹)	Capture cross-section	3.0500×10^{-4}	3.6990×10^{-4}	0.018250
$\Sigma_{sg \rightarrow g}$ (cm ⁻¹)	Self-group scattering cross-section	0.227126	1.02814	2.76278
$\Sigma_{sg \rightarrow g+1}$ (cm ⁻¹)	Group transfer scattering cross-section	0.104642	0.0979597	0.0

3.2 Case 1 for $k_{eff} \approx 0.987$

(1) Source position A (center)

IKS calculations were performed using 300,000 particles per cycle. The fixed source was placed at position A, as shown in Fig. 2 (center). The initial transition of the source-multiplication factor k_s for each cycle up to the 60th cycle is shown in Fig. 3. In the first cycle, all the source neutrons were emitted from the center, thus rendering k_s much larger than the converged k_s until the fifth cycle. The number of fission sources (N_f) and number of fixed sources (N_s) up to the 60th cycle, as calculated using Eqs. (8) and (9), respectively, are shown in Fig 4. As shown in

Figs. 3 and 4, k_s , N_f , and N_s converged by the 50th cycle. The initial 50 cycles, determined by observing the convergence state of k_s , N_f , and N_s , were discarded from the tally scores as they were inactive.

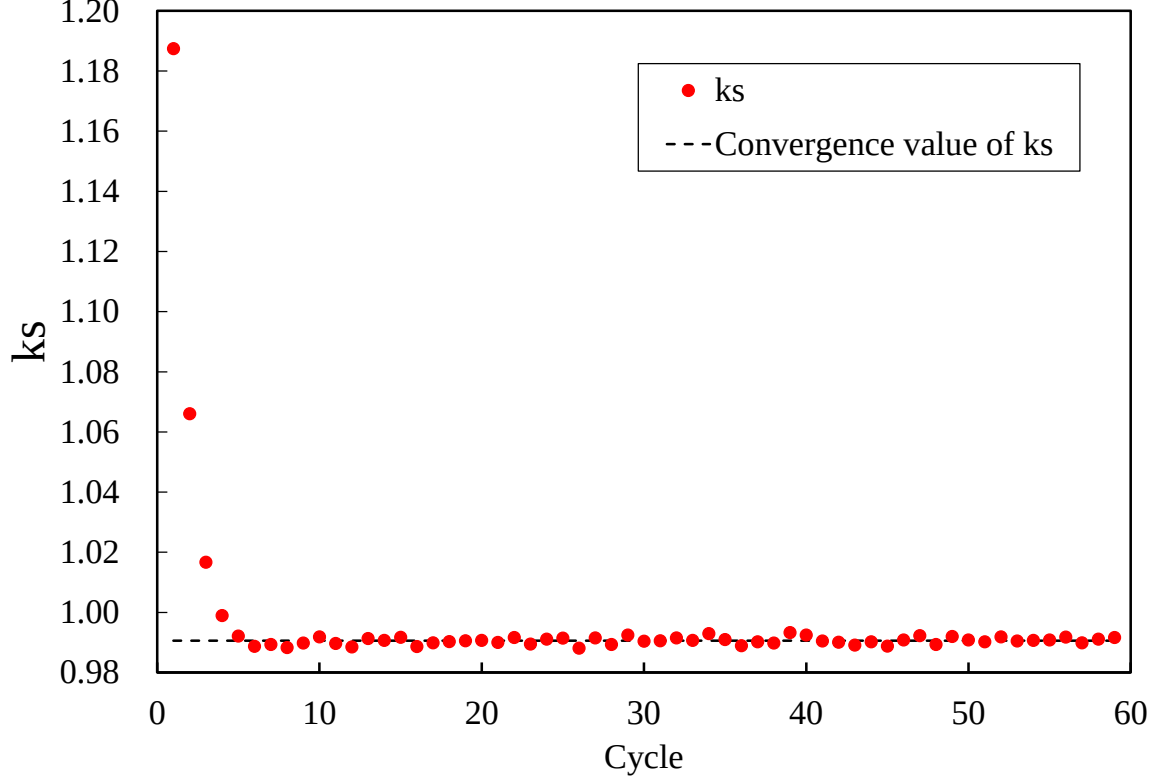


Fig. 3 Initial transition of k_s for source position A (center) ($k_{eff} \approx 0.987$)

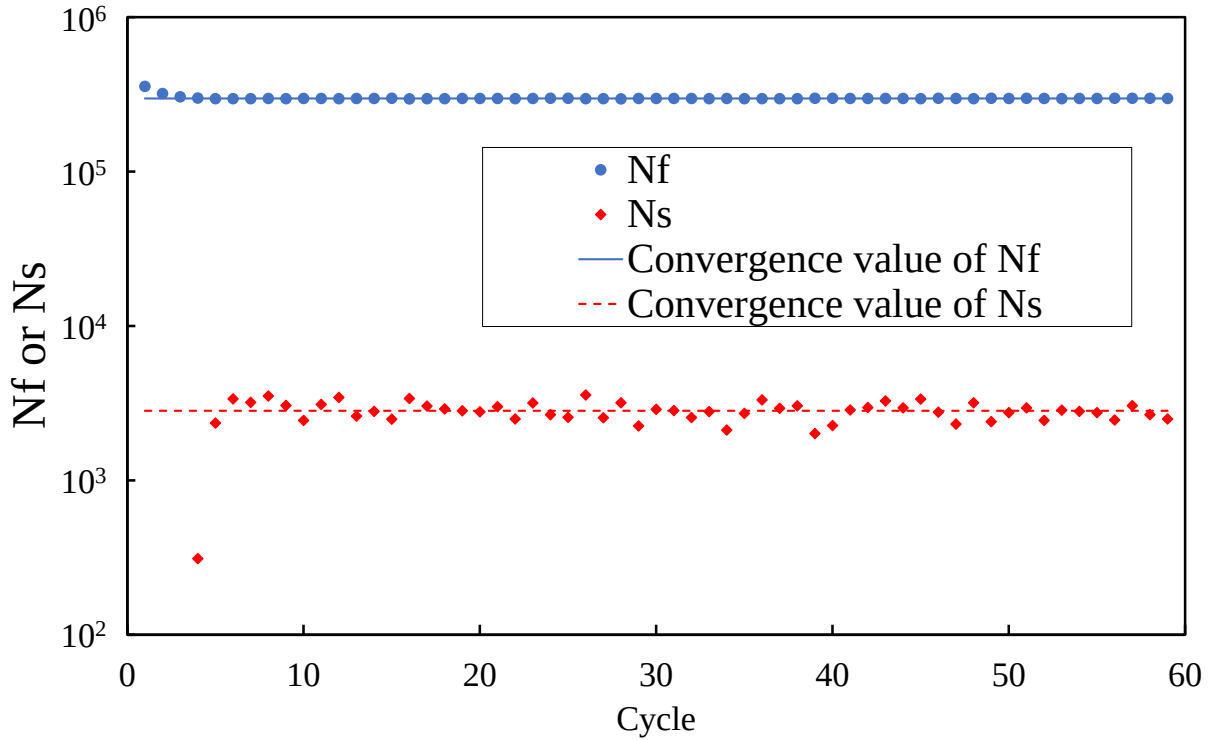


Fig. 4 Initial transition of N_f and N_s for source position A (center) ($k_{eff} \approx 0.987$)

The IKS method inherits fission sources from the previous cycle, which causes biased variances owing to the intercycle correlation of the fission sources (Ueki et al., 1997; Kiedrowski and Brown, 2009; Shim et al, 2014, Park et al., 2016). Therefore, the same IKS calculations were performed repeatedly 20 times using different random number sequences to estimate the “real” variance. The result of one IKS calculation has been divided into five parts to generate totally 100 (= 5 × 20) samples. Each calculation involved 300,000 particles per cycle up to 1,000 cycles. The estimated real variance of the subcritical multiplication factor M_s (defined in Eq. (11)) can be obtained as follows:

$$\sigma_R(M_s)^2 = \frac{1}{L(L-1)} \sum_{i=1}^L \left(M_{s,i} - \frac{1}{L} \sum_{i=1}^L M_{s,i} \right)^2, \quad (13)$$

where L is the number of samples of M_s (in this case, $L=100$), and $M_{s,i}$ is the subcritical multiplication factor of each independent trial. Meanwhile, the “apparent” variance of M_s is expressed as

$$\sigma_A(M_s)^2 = \frac{1}{L} \left(\frac{1}{L} \sum_{i=1}^L \sigma_A(M_{s,i})^2 \right). \quad (14)$$

If an intercorrelation does not exist between the cycles, then the variances calculated using Eqs. (13) and (14) will be similar.

In addition to the IKS method, conventional fixed-source calculations were performed as reference solutions. The k_s and M_s obtained from both methods are listed in Table 3. The apparent and estimated real standard deviations of the IKS method are presented as well. The relative figure-of-merit (FOM), which is defined as $1/(\text{cpu time})/\text{variance}$, is listed in Table 3. The FOM of the IKS method was calculated using both the apparent and estimated real standard deviations.

Table 3 Multiplication factors obtained using fixed-source and IKS methods and FOM (source position A)

	k_s	$\sigma(k_s)$ (pcm)	M_s	$\sigma(M_s)$	CPU time (min)	Relative FOM of M_s [-]
Fixed source	—	—	105.12	0.15	609	1
IKS (300K/cycle)	0.990575	0.7 ^a 0.8 ^b	105.10	0.08 ^a 0.10 ^b	676	3.56 ^a 2.16 ^b

a: Estimated real standard deviation b: Apparent standard deviation

The k -eigenvalue calculation was also performed, and k_{eff} and k_s are compared in Table 4. The relative FOM's of k_{eff} and k_s are shown in Table 4. The FOM of k_s , where the neutron source was placed at the center of the geometry, show nearly equivalent value of k_{eff} 's FOM.

Table 4 Comparison of IKS (source position A) and k -eigenvalue mode methods

	k_s or k_{eff}	$\sigma_R(k_s)$ or $\sigma_R(k_{eff})$ (pcm)	CPU time (min)	Relative FOM of M_s [-]
k -eigenvalue (300K/cycle)	0.987153	0.8	591	1
IKS (300K/cycle)	0.990575	0.7	676	1.06

The subcritical multiplication factor M_s yielded by the IKS method agreed well with that obtained using the fixed-source method within the standard deviation. In Fig. 5, a comparison between the thermal neutron flux (third energy group) distributions along the diagonal line (green line in Fig. 2) obtained using IKS and fixed-source methods is presented. The flux distributions agreed well with each other. As shown in Table 3, $\sigma_A(M_s)$ exceeded $\sigma_R(M_s)$, which is contrary to the usual trend where $\sigma_R(M_s)$ is supposed to exceed $\sigma_A(M_s)$ owing to the intercycle correlation.

Based on $\sigma_R(M_s)$, the computational efficiency of the IKS method was approximately 10 times that of the fixed-source method. Meanwhile, based on $\sigma_A(M_s)$, the efficiency was approximately thrice higher. This tendency is similarly exhibited in the standard deviation of the neutron flux.

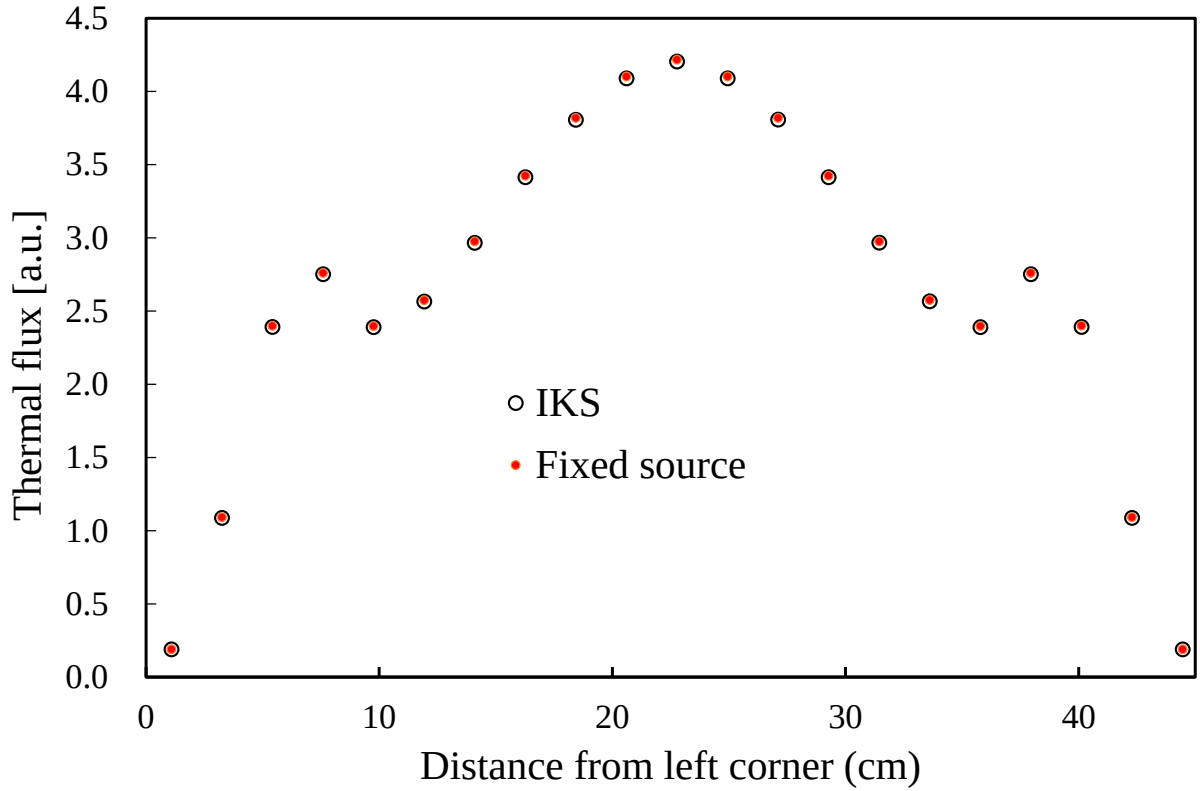


Fig. 5 Thermal neutron-flux distribution yielded by IKS and fixed-source methods along diagonal line

(2) Source position B (corner)

Similar calculations were performed using the fixed source placed near the top-left corner (position B in Fig. 2). The initial transition of the source-multiplication factor k_s for each cycle up to the 60th cycle is shown in Fig. 6. Because the initial source was placed at a low neutron-importance position, k_s in the initial cycles was much smaller than the converged k_s , the convergence was much slower than the previous case. The N_f and N_s up to the 60th cycle are shown in Fig. 7. Similarly, k_s , N_f , and N_s converged by the 50th cycle, and the initial 50 cycles were discarded. The results are shown in Table 5 in the same manner as in Table 3. Similarly, the subcritical multiplication factor M_s yielded by the IKS method agreed well with that obtained using the fixed source method within the standard deviation. The multiplication in this case yielded a much smaller value than that in the previous case because the fixed source was placed at a low-neutron-importance position. The standard deviation of k_s was much larger than that in the previous case. Contrary to the previous case, $\sigma_A(M_s)$ was significantly underestimated compared with $\sigma_R(M_s)$, which indicates that the ratio between $\sigma_A(M_s)$ and $\sigma_R(M_s)$ depends on the fixed-source position.

Table 5 Multiplication factors by fixed-source and IKS methods and FOM (source position B)

	k_s	$\sigma(k_s)$ (pcm)	M_s	$\sigma(M_s)$	CPU time (min)	Relative FOM of M_s [-]
Fixed source	—	—	11.912	0.030	270	1
IKS (300K/cycle)	0.922456	8.1 ^a 2.2 ^b	11.896	0.014 ^a 0.004 ^b	247	4.7 ^a 66.2 ^b
IKS (10K/cycle)	0.922391	7.4 ^a 2.3 ^b	11.885	0.013 ^a 0.004 ^b	358	4.0 ^a 42.7 ^b

a: Estimated real standard deviation b: Apparent standard deviation

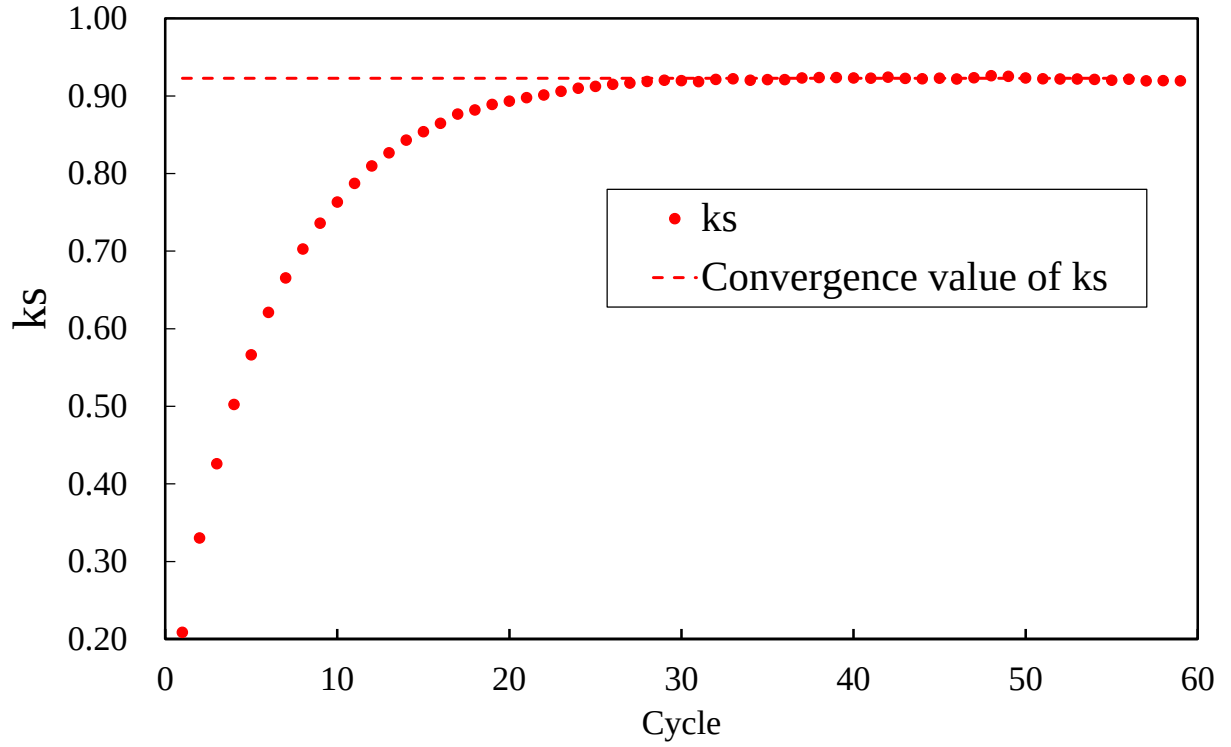


Fig. 6 Initial transition of k_s for source position B (corner) ($k_{eff} \approx 0.987$)

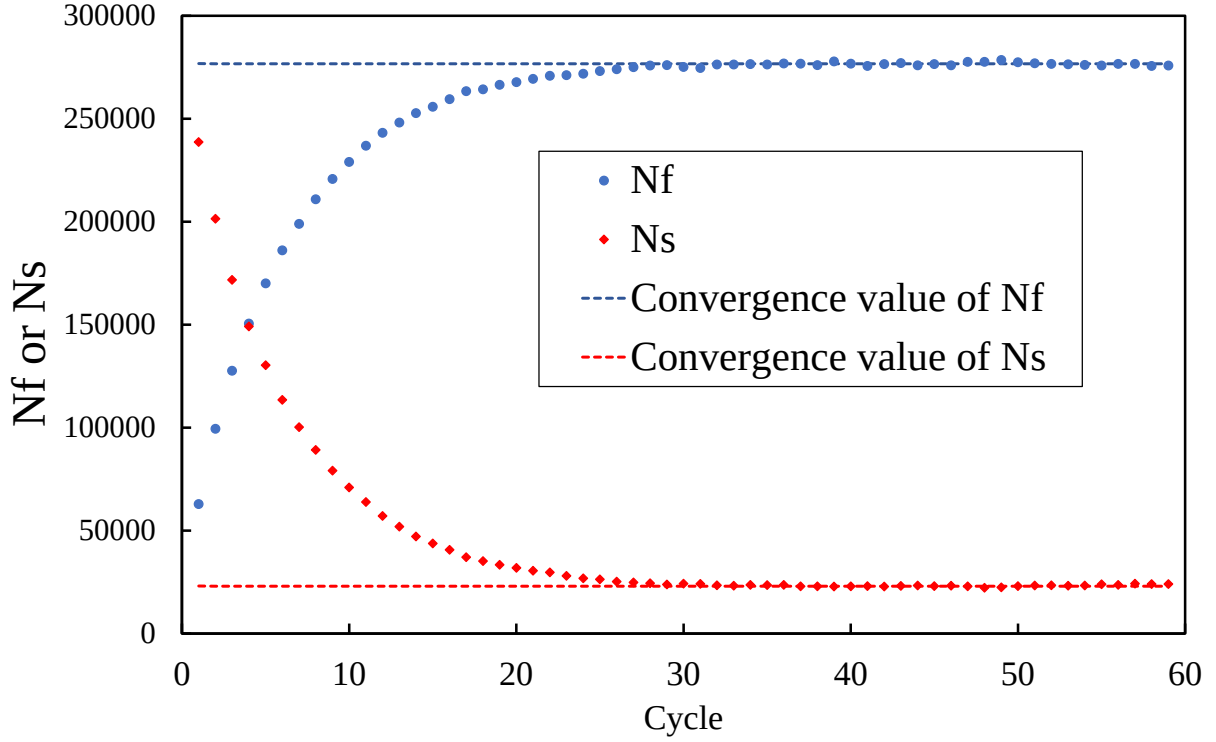


Fig. 7 Initial transition of N_f and N_s for source position B (corner) ($k_{eff} \approx 0.987$)

To investigate the effect of the number of particles per cycle, similar calculations were performed with a significantly reduced number of particles per cycle (10,000 particles/cycle). The results are listed in Table 5, which shows that the number of particles per cycle did not significantly affect the results. Even in this case, k_s did not exceed unity; thus, the source size did not affect the results.

The neutron-flux distributions obtained via both methods are omitted herein; however, they agreed well with each other.

3.3 Case 2 for $k_{eff} \approx 0.999$

(1) Source position A (center)

In this case, the fixed source was placed at the center of the subcritical system. The source multiplication k_s was approximately 0.9993, which was slightly greater than the value of k_{eff} . Thus, unless the nominal source size N is a sufficiently large number, the k_s of each cycle typically exceeds unity, thus resulting in biased results, as mentioned at the end of Section 2.2. The IKS calculations were performed using 2.5 million particles per cycle up to the 300th cycle. The initial 50 cycles were discarded from the tally scores as they were inactive. The same IKS

calculation was performed 20 times using different random number sequences. Again, 100 samples of M_s were taken from 20 independent IKS calculations in the same manner as in Sec. 3.2.

The source-multiplication factor k_s vs. the cycles is shown in Fig. 8. The N_s in each cycle are shown in Fig. 9. Despite using a relatively large number of particles per cycle (2.5 million), k_s exceeded unity in 13 of the 250 active cycles, which is also shown in the frequency distribution of k_s in Fig. 10, and $N_s = 0$ was set as the fixed source for the next cycle. The results for k_s and M_s and their standard deviations (estimated real and apparent) are listed in Table 6. The comparison between k_s and k_{eff} is shown in Table 7. In terms of FOM, the IKS method outperforms the k -eigenvalue calculation by 1.5 times. In this case, $\sigma_A(M_s)$ was greater than $\sigma_R(M_s)$. Similar to Fig. 5 in Section 3.2, the neutron-flux distributions obtained using the IKS and fixed-source methods in this nearly critical system agreed well with each other.

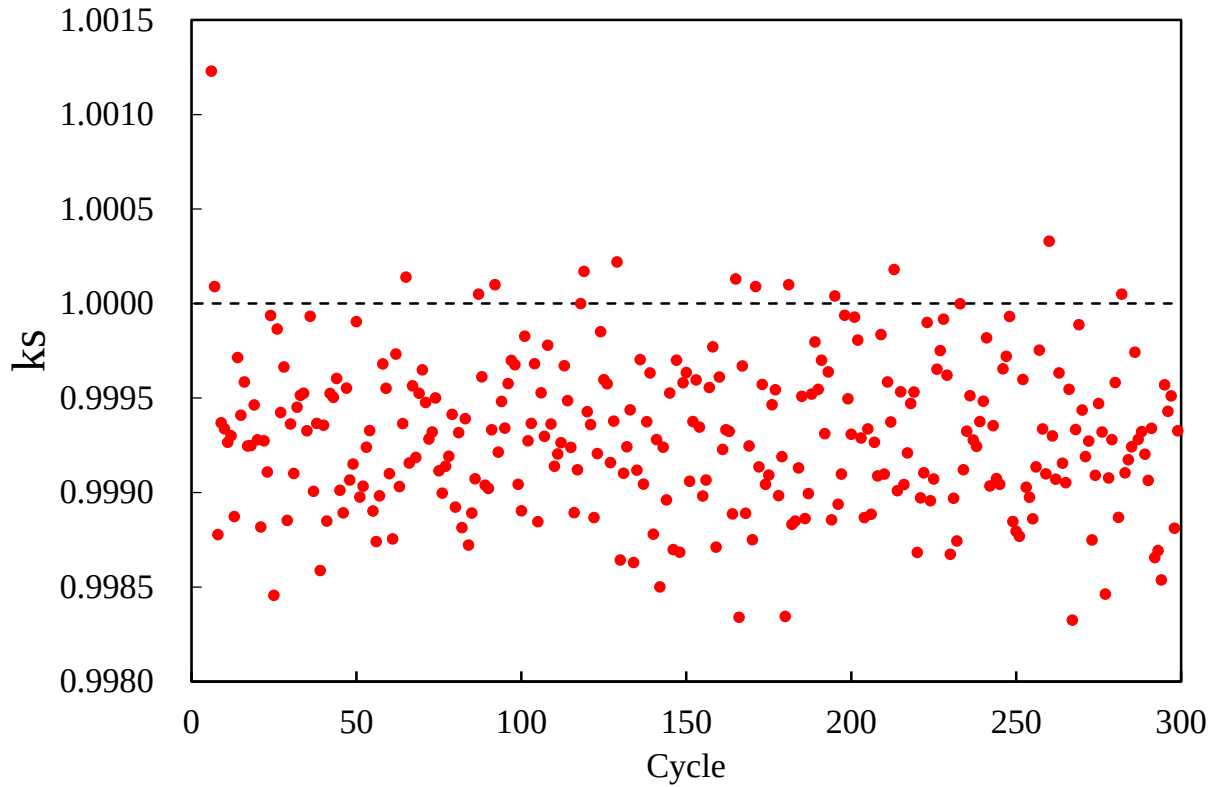


Fig. 8 k_s vs. cycle for source position A (center) ($k_{eff} \approx 0.999$)

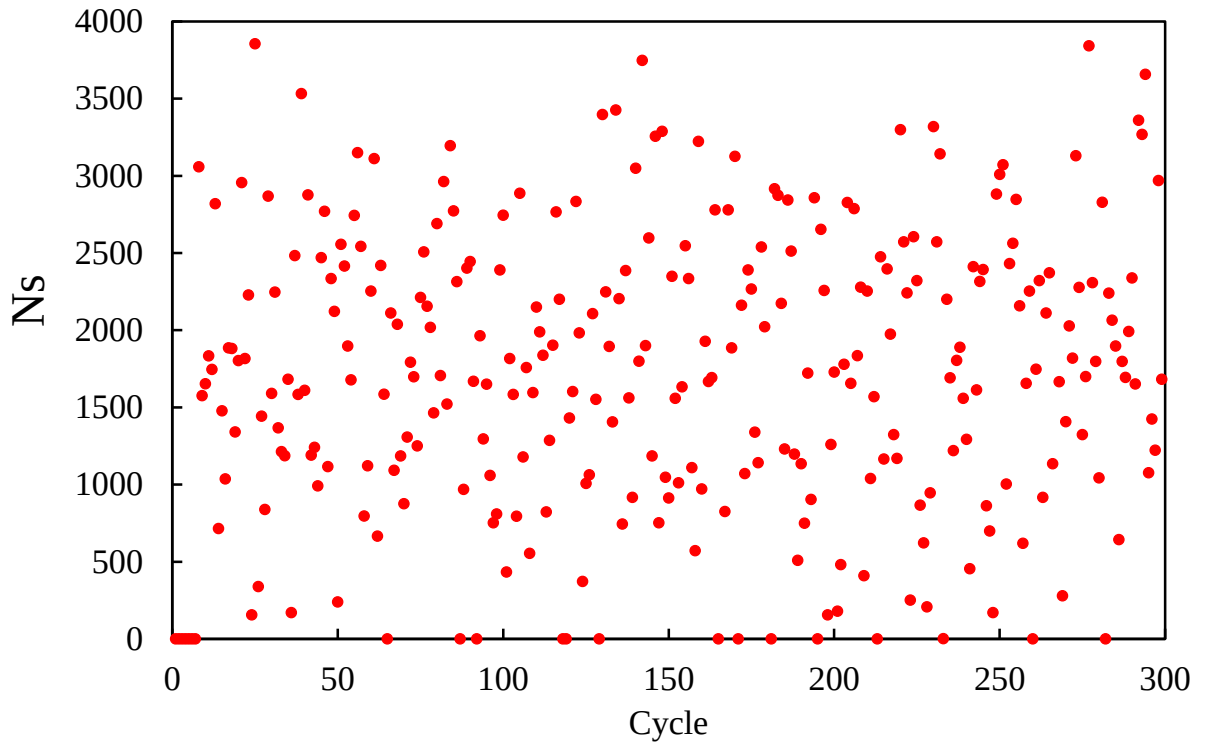


Fig. 9 N_s for source position A (center) ($k_{eff} \approx 0.999$)

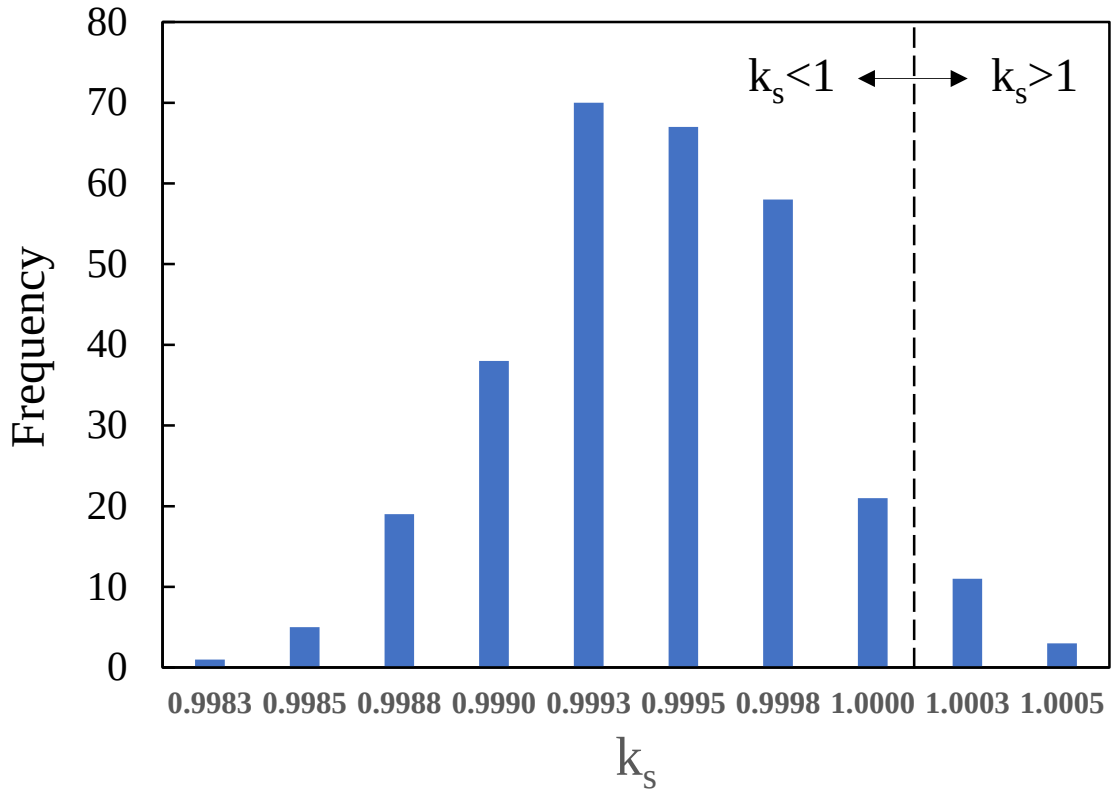


Fig. 10 Frequency of k_s for 2.5 million particles per cycle

Table 6 Multiplication factors by fixed-source and IKS methods and FOM (source position A, 2.5 million particles/cycle)

	k_s	$\sigma(k_s)$ (pcm)	M_s	$\sigma(M_s)$	CPU time (min)	Relative FOM of M_s [-]
Fixed source	—	—	1421.9	22.3	923	1
IKS (2.5M/cycle)	0.999308	0.4 ^a 0.6 ^b	1444.4	9.3 ^a 11.6 ^b	1558	3.4 ^a 2.2 ^b

a: Estimated real standard deviation b: Apparent standard deviation

Table 7 Comparison of IKS (source position A) and k -eigenvalue mode methods

	k_s or k_{eff}	$\sigma_R(k_s)$ or $\sigma_R(k_{eff})$ (pcm)	CPU time (min)	Relative FOM of M_s [-]
k -eigenvalue (250K/cycle)	0.999007	0.9	595	1
IKS (2.5M/cycle)	0.999308	0.4	1558	1.56

Although the frequency of k_s exceeding unity was approximately 3%, the subcritical multiplication factor M_s yielded by the IKS method agreed well with the fixed-source method within the standard deviation. The bias extent of the result yielded by the IKS method was unclear because the statistical errors of the fixed-source method were not sufficiently small in this nearly critical subcritical system. The fixed-source method requires a considerably longer computation time to reduce the standard deviation to the same level as that of the IKS method.

The IKS calculations were performed using smaller source sizes ($N = 1$ million, 0.5 million, 0.3 million, and 0.2 million) to investigate the source-size effect. The total CPU time for each case remained almost constant after adjusting the total number of cycles. The dependence of k_s and M_s on the source size N is listed in Table 8. As the source size decreased, the frequency of k_s exceeding unity increased because of the amplified statistical fluctuations of k_s in each cycle, and k_s and M_s were increasingly overestimated.

Table 8 Dependence of k_s and M_s on source size N

Source size/cycle N	Frequency of $k_s > 1$ (%)	k_s	$\sigma_R(k_s)$ (pcm)	M_s	$\sigma_R(M_s)$
2.5 million	3.3	0.999308	0.4	1443.4	9.3
1 million	12.2	0.999316	0.4	1461.4	8.4
0.5 million	21.3	0.999322	0.4	1473.3	9.5
0.3 million	27.5	0.999353	0.5	1544.5	10.9
0.2 million	32.0	0.999378	0.5	1605.5	13.7

Therefore, to obtain accurate results in a nearly critical subcritical system, the source size N must be increased and k_s does not exceed unity as much as possible. Furthermore, as the subcritical system approaches criticality, the source size N must be increased based on the criticality. Determining the source size to ensure that k_s does not exceed unity, or establishing an acceptable range for the frequency of k_s exceeding unity, is not straightforward due to its dependence of the confidence required for the calculation.

(2) Source position B (corner)

The fixed source was placed at position B (near the top-left corner). The source-multiplication factor k_s was approximately 0.993, which reduced the likelihood of k_s exceeding unity compared with the previous case. The IKS calculations were performed using 300,000 particles per cycle up to the 1000th cycle. The N_f and N_s up to the 200th cycle are shown in Fig. 11. The calculations were performed under the same conditions as in the previous case. The results are summarized in Table 9. The frequency of k_s exceeding unity was 0.13%. The subcritical multiplication factor M_s obtained using the IKS method agreed with that obtained using the fixed-source method within the standard deviation. Similar to Section 3.2 (2), $\sigma_A(M_s)$ was underestimated compared with $\sigma_R(M_s)$.

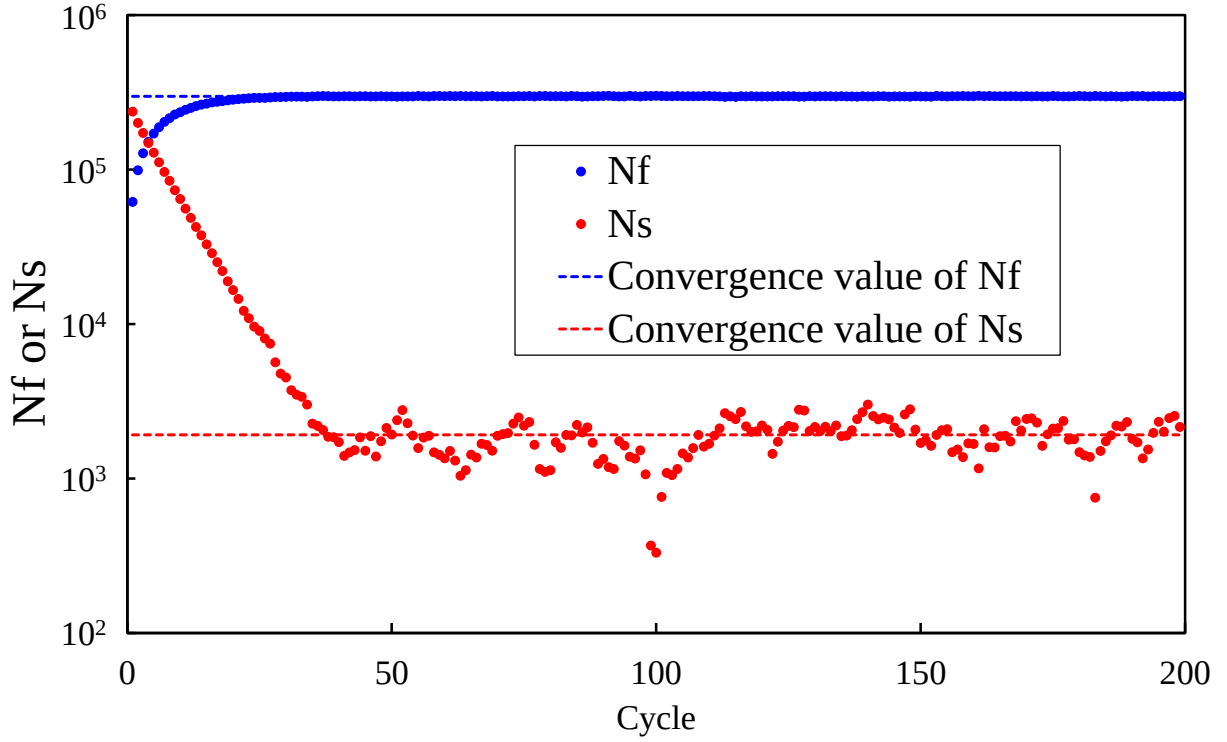


Fig. 11 Initial transition of N_f and N_s for source position B (corner) ($k_{eff} \approx 0.999$)

Table 9 Multiplication factors yielded by fixed-source and IKS methods and FOM (source position B, 300,000 particles/cycle)

	k_s	$\sigma(k_s)$ (pcm)	M_s	$\sigma(M_s)$	CPU time (min)	Relative FOM of M_s [-]
Fixed source	—	—	153.91	1.98	1410	1
IKS (300K/cycle)	0.993602	4.4 ^a 1.2 ^b	155.30	1.08 ^a 0.28 ^b	1090	4.3 ^a 65.9 ^b

a: Estimated real standard deviation b: Apparent standard deviation

The IKS calculations were performed using other source sizes ($N = 2.5$ million, 0.1 million, and 0.05 million) to investigate the effect of the source size. The frequencies at which k_s exceeded unity, k_s , and M_s are listed in Table 10. When the source size per cycle was 100,000 and 50,000, the frequencies of k_s exceeding unity were 2.79% and 6.48%, respectively. Contrary to the previous case where the fixed source was placed at the center, k_s and M_s were underestimated as the frequency of k_s exceeding unity increased.

Table 10 Dependence of k_s and M_s on source size N

Source size/cycle N	Frequency of $k_s > 1$ (%)	k_s	$\sigma_R(k_s)$ (pcm)	M_s	$\sigma_R(M_s)$
2.5 million	0.0	0.993585	3.9	154.89	0.94
0.3 million	0.13	0.993602	4.4	155.30	1.08
0.1 million	2.79	0.993490	3.9	152.61	0.93
0.05 million	6.48	0.992962	4.1	141.08	0.83
Fixed source	—	—	—	153.91	1.98

4. Conclusions

Although the IKS method uses an algorithm that differs completely from the algorithm used in the conventional fixed-source method, it can precisely reproduce the fixed-source method in most cases. The advantage of the IKS method is that it does not require following the long chain of fission reactions until the end, even when the subcritical system is nearly critical. Therefore, a large fission-source bank, which is required for the case of a nearly critical system, need not be stored.

However, owing to the statistical fluctuations of Monte Carlo calculations, the source-multiplication factor k_s occasionally or frequently exceeds unity, particularly in a nearly critical subcritical system. If the source-multiplication factor k_s exceeds unity in a cycle, then the number of source neutrons from the fixed source N_s in the next cycle cannot be defined, and a value of zero must be assigned to N_s in the next cycle. Therefore, the frequency of k_s exceeding unity should be minimized. If the frequency becomes high, then the results of the IKS method will be biased.

To prevent k_s from exceeding unity, the source size per cycle must be increased sufficiently. As shown in Section 3.3, for the subcritical system with $k_{eff} \approx 0.999$, the frequency of k_s exceeding unity was approximately 3% even when the source size per cycle was increased up to 2.5 million. Here, k_s depends on the fixed-source position. Placing the fixed source at a location with high neutron importance (e.g., the center of a core) causes k_s to exceed unity, thus necessitating a larger source size per cycle. However, when the fixed source is placed at a location with low neutron importance (e.g., the reflector region), the source size can be smaller. To date,

no criteria have been established for an acceptable frequency of k_s exceeding unity, or for the source size per cycle required for the acceptable frequency.

In the numerical example, the estimated real standard deviation of k_s or M_s was smaller than the apparent standard deviation when the fixed source was placed at the center of the core. This is contrary to the typical relationship between the estimated real and apparent standard deviations. However, when the fixed source was placed in the reflector region, the apparent standard deviation underestimated the estimated real standard deviation. The mechanism underlying this phenomenon is yet to be elucidated.

Based on the estimated real standard deviation and computation time, the FOM of the IKS method was 3–10 times that of the conventional fixed-source method. Therefore, the introduction of the IKS method does not necessarily improve the computational efficiency, as expected.

To render the IKS method more versatile, an advanced algorithm must be developed that can obtain unbiased solutions even when k_s exceeds unity. For example, the introduction of negative weights can be considered a promising solution and shall be investigated in future studies.

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Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.