

On numerical range and polar decomposition of the Aluthge transformation

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ABSTRACT

Let $T = U|T|$ be the polar decomposition of an operator T . Aluthge defined an operator transformation $\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ of T which is called Aluthge transformation. In this report, firstly, we shall show $w(T) \geq w(\tilde{T})$ for any operator T , where $w(T)$ means numerical radius of T .

Secondly, we shall show the following relations: (i) If T is an $n \times n$ matrix, then $W(T) \supset W(\tilde{T})$. (ii) If $N(T^*) \supset N(T)$, then $\overline{W(T)} \supset \overline{W(\tilde{T})}$, where $W(T)$ means numerical range of T .

Lastly, we shall show that $\tilde{\tilde{T}} = VU|\tilde{T}|$ is also the polar decomposition, where $|T|^{\frac{1}{2}}|T^*|^{\frac{1}{2}} = V| |T|^{\frac{1}{2}}|T^*|^{\frac{1}{2}} |$ is polar decomposition.

1. INTRODUCTION

In what follows, a capital letter means a bounded linear operator on a complex Hilbert space H . An operator T is said to be positive if $(Tx, x) \geq 0$ holds for all $x \in H$. For each $T \in B(H)$, we write $W(T)$ for the *numerical range* of T , that is,

$$W(T) = \{(Tx, x) : \|x\| = 1\}.$$

$\overline{W(T)}$ and $w(T)$ mean the closure of $W(T)$ and the numerical radius of T , respectively. An operator T is said to be *spectraloid* if $w(T) = r(T)$, where $r(T)$ is the spectral radius of T .

Let $T = U|T|$ be the polar decomposition of T . Aluthge defined an operator transformation $\tilde{T}$ of T by $\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ in [1]. We call this transformation *Aluthge transformation*. Many authors have studied Aluthge transformation and there are many result on properties of this transformation. For example “ $\sigma(T) = \sigma(\tilde{T})$ holds for all $T \in B(H)$ ” in [2, 4, 9], and as a nice application of Furuta inequality [6], “if $|T| \geq |T^*|$, then $|\tilde{T}|^2 \geq |\tilde{T}^*|^2$ holds” in [1, 9].

As fundamental properties of Aluthge transformation, $\|T\| \geq \|\tilde{T}\|$ and $r(T) = r(\tilde{T})$ hold, obviously. And a result on the spectrum of Aluthge transformation stated above was shown in [2, 4, 9]. Moreover as a result on the numerical range of Aluthge transformation, I.B.Jung, E.Ko and C. Pearcy showed that “if T is a 2×2 matrix, then $W(T) \supset W(\tilde{T})$ ” in [10].

On the other hand, the polar decomposition of Aluthge transformation has been discussed in [1], but the concrete form has not been obtained, yet.

In this report, firstly, we shall show a property of Aluthge transformation on the numerical radius, that is, $w(T) \geq w(\tilde{T})$ for all $T \in B(H)$. And as an application of this result, we shall show a characterization of spectraloid operators via Aluthge transformation.

Secondly, we shall show the following inclusion relations: (i) If T is an $n \times n$ matrix, then $W(T) \supset W(\tilde{T})$. (ii) If $N(T^*) \supset N(T)$, then $\overline{W(T)} \supset \overline{W(\tilde{T})}$. (i) is an extension of above result by I.B.Jung, E.Ko and C.Pearcy in [10].

Lastly, we shall obtain a concrete form of the polar decomposition of $\tilde{T}$.

2. NUMERICAL RADIUS

By considering the definition of Aluthge transformation, we can obtain the relations $\|T\| \geq \|\tilde{T}\| \geq r(\tilde{T}) = r(T)$, easily. In this section we shall show the following result:

Theorem 1 ([Y]). *Let $T \in B(H)$. Then $w(T) \geq w(\tilde{T})$.*

To prove Theorem 1, we prepare the following results:

Lemma 2 ([Y]). *Let $T = U|T|$ be the polar decomposition of T . If there exists another decomposition $T = V|T|$, then $\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}} = |T|^{\frac{1}{2}}V|T|^{\frac{1}{2}}$.*

Proof. Let $H = N(|T|^{\frac{1}{2}}) \oplus N(|T|^{\frac{1}{2}})^{\perp}$.

In case $x \in N(|T|^{\frac{1}{2}})$. $\tilde{T}x = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}x = 0 = |T|^{\frac{1}{2}}V|T|^{\frac{1}{2}}x.$

In case $x \in N(|T|^{\frac{1}{2}})^{\perp} = \overline{R(|T|^{\frac{1}{2}})}$. There exists $y \in H$ such that $x = |T|^{\frac{1}{2}}y$. Then we have

$$\begin{aligned}\tilde{T}x &= |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}x = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}y = |T|^{\frac{1}{2}}Ty \\ &= |T|^{\frac{1}{2}}V|T|^{\frac{1}{2}}y = |T|^{\frac{1}{2}}V|T|^{\frac{1}{2}}x.\end{aligned}$$

Hence we have $\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}} = |T|^{\frac{1}{2}}V|T|^{\frac{1}{2}}$ on $H = N(|T|^{\frac{1}{2}}) \oplus N(|T|^{\frac{1}{2}})^{\perp}$. $\square$

Lemma 3 ([Y]). *Let A be a positive operator, and $X \in B(H)$. Then the following inequality holds:*

$$\|A^r X A^{1-r} - zI\| \leq \|AX - zI\|^r \|XA - zI\|^{1-r} \quad \text{for all } r \in [0, 1] \text{ and } z \in \mathbb{C}.$$

To prove Lemma 3, the following result is very important:

Theorem A ([7]). *Let A and B be positive operators, and $X \in B(H)$. Then the following inequalities hold:*

- (i) $\|A^r X B^r\| \leq \|AXB\|^r \|X\|^{1-r}$ for $r \in [0, 1]$.
- (ii) $\|A^r X B^r\| \geq \|AXB\|^r \|X\|^{1-r}$ for $r > 1$.

Proof of Lemma 3. We may assume that A is invertible in this proof. Hence we have

$$\begin{aligned}\|A^r X A^{1-r} - zI\| &= \|A^r(XA - zI)A^{-r}\| \\ &\leq \|A(XA - zI)A^{-1}\|^r \|XA - zI\|^{1-r} \quad \text{by (i) of Theorem A} \\ &= \|AX - zI\|^r \|XA - zI\|^{1-r} \quad \text{for all } r \in [0, 1] \text{ and } z \in \mathbb{C}.\end{aligned}$$

$\square$

Let $T = U|T|$ be a decomposition of T . Then by Lemma 2, we have $\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$. So that we obtain the following inequality by putting $A = |T|$, $X = U$ and $r = \frac{1}{2}$ in Lemma 3.

$$(2.1) \quad \|\tilde{T} - zI\| \leq \| |T|U - zI \|^{\frac{1}{2}} \|T - zI\|^{\frac{1}{2}} \quad \text{for all } z \in \mathbb{C}.$$

Theorem B ([3]). *Let $T \in B(H)$. Then $w(T) \leq 1$ is equivalent to*

$$\|T - zI\| \leq 1 + \{1 + |z|^2\}^{\frac{1}{2}} \quad \text{for all } z \in \mathbb{C}.$$

Proof of Theorem 1. Firstly, we shall show that for each $S \in B(H)$,

$$(2.2) \quad w(S) \leq 1 \implies w(\tilde{S}) \leq 1.$$

Let $S = U|S|$ be the polar decomposition of S . Then we have

$$(|S|Ux, x) = (U^*U|S|Ux, x) = \left(S \frac{Ux}{\|Ux\|}, \frac{Ux}{\|Ux\|} \right) (U^*Ux, x).$$

Then $W(|S|U) \subset W(S)W(U^*U)$, and we obtain

$$(2.3) \quad w(|S|U) \leq w(S).$$

If $w(S) \leq 1$, then $w(|S|U) \leq w(S) \leq 1$ by (2.3). By Theorem B, we have

$$\|S - zI\| \leq 1 + \{1 + |z|^2\}^{\frac{1}{2}} \quad \text{for all } z \in \mathbb{C}$$

and

$$\||S|U - zI\| \leq 1 + \{1 + |z|^2\}^{\frac{1}{2}} \quad \text{for all } z \in \mathbb{C}.$$

Hence by (2.1) we obtain

$$\|\tilde{S} - zI\| \leq \||S|U - zI\|^{\frac{1}{2}} \|S - zI\|^{\frac{1}{2}} \leq 1 + \{1 + |z|^2\}^{\frac{1}{2}} \quad \text{for all } z \in \mathbb{C}.$$

And $w(\tilde{S}) \leq 1$ follows from Theorem B.

Secondly, put $S = \frac{T}{w(T)}$. Then $\tilde{S} = \frac{\tilde{T}}{w(T)}$ and $w(S) \leq 1$. Then by (2.2), we have

$$w(\tilde{S}) = \frac{w(\tilde{T})}{w(T)} \leq 1.$$

Hence the proof of Theorem 1 is complete. $\square$

Related to Aluthge transformation, for each natural number n , I.B.Jung, E.Ko and C.Pearcy defined n -th Aluthge transformation $\tilde{T}_n$ of T by $\tilde{T}_n = \widetilde{(\tilde{T}_{n-1})}$ and $\tilde{T}_0 = T$ in [10]. By using n -th Aluthge transformation, we showed some properties of Aluthge transformation on operator norms, and pointed out that Aluthge transformation has similar properties to powers of operators in [13, 14, 15]. An operator T is said to be *normaloid* if $\|T\| = r(T)$ which is equivalent to $\|T\| = \|T^n\|^{\frac{1}{n}}$ for all natural number n . It is well known that “every normaloid operator is spectraloid.” Related to normaloid operators and spectral radius, we obtained the following results:

Theorem C ([14]). *Let $T \in B(H)$. Then the following assertions are equivalent:*

- (i) T is normaloid.
- (ii) $\|T\| = \|\tilde{T}_n\|$ for all natural number n .

Theorem D ([15]). *Let $T \in B(H)$. Then $\lim_{n \rightarrow \infty} \|\tilde{T}_n\| = r(T)$.*

As an application of these results, we obtain a characterization of spectraloid operators which is a parallel result to Theorem C as follows:

Corollary 4 ([Y]). *Let $T \in B(H)$. Then the following assertions are equivalent:*

- (i) T is spectraloid.
- (ii) $w(T) = w(\widetilde{T}_n)$ for all natural number n .

Proof of Corollary 4. Since $\|T\| \geq w(T) \geq r(T)$, we have

$$\lim_{n \rightarrow \infty} w(\widetilde{T}_n) = r(T)$$

by Theorem D. And by Theorem 1, we obtain the following inequalities:

$$w(T) \geq w(\widetilde{T}) \geq \cdots \geq w(\widetilde{T}_n).$$

Hence the proof is complete. □

3. NUMERICAL RANGE

In the previous section, we showed a relation between $w(T)$ and $w(\widetilde{T})$, and obtained a characterization of spectraloid operators. On the other hand, I.B.Jung, E.Ko and C.Pearcy obtained an extension of Theorem 1 in case T is 2×2 matrix, and have conjectured a relation between $W(\widetilde{T})$ and $W(T)$ as follows:

Theorem E ([10]). *Let T be a 2×2 matrix. Then $W(T) \supset W(\widetilde{T})$.*

Conjecture ([10]). *For every $T \in B(H)$, $W(T) \supset W(\widetilde{T})$.*

In this section, we shall show a relation between $W(T)$ and $W(\widetilde{T})$ as a partial solution of above conjecture:

Theorem 5 ([Y]). *Let $T = U|T|$ be a decomposition. If U is isometry, then $\overline{W(T)} \supset W(\widetilde{T})$.*

To prove Theorem 5, we cite the following result:

Theorem F ([5]). *Let $T \in B(H)$. Then*

$$\overline{W(T)} = \bigcap_{\mu \in \mathbb{C}} \{\lambda : |\lambda - \mu| \leq w(T - \mu I)\}.$$

Proof of Theorem 5. First, we shall show the following assertion: If $S = V|S|$ is a decomposition such that V is isometry, then for each $\lambda \in \mathbb{C}$,

$$(3.1) \quad w(S - \lambda I) \leq 1 \implies w(\widetilde{S} - \lambda I) \leq 1.$$

By (2.1), we have the following inequalities:

$$\begin{aligned}
 \|\tilde{S} - zI\| &\leq \| |S|V - zI \|^{1/2} \|S - zI\|^{1/2} \\
 (3.2) \quad &= \|V^*(S - zI)V\|^{1/2} \|S - zI\|^{1/2} \quad \text{by } V^*V = I \\
 &\leq \|S - zI\| \quad \text{for all } z \in \mathbb{C}.
 \end{aligned}$$

Assume that $w(S - \lambda I) \leq 1$. Then by (3.2) and Theorem B, we have

$$\|\tilde{S} - \lambda I - zI\| \leq \|S - \lambda I - zI\| \leq 1 + \{1 + |z|^2\}^{1/2} \quad \text{for all } z \in \mathbb{C}.$$

Hence we obtain $w(\tilde{S} - \lambda I) \leq 1$ by Theorem B.

Next, for each $\mu \in \mathbb{C}$, put $S = \frac{T}{w(T - \mu I)}$ and $\lambda = \frac{\mu}{w(T - \mu I)}$. Then $|S| = \frac{|T|}{w(T - \mu I)}$ holds, and $S = U \frac{|T|}{w(T - \mu I)}$ is a decomposition such that U is isometry, and also $\tilde{S} = \frac{\tilde{T}}{w(T - \mu I)}$. Moreover $w(S - \lambda I) \leq 1$. Then by (3.1), we obtain

$$w(\tilde{S} - \lambda I) = \frac{w(\tilde{T} - \mu I)}{w(T - \mu I)} \leq 1.$$

It is equivalent to

$$w(\tilde{T} - \mu I) \leq w(T - \mu I) \quad \text{for all } \mu \in \mathbb{C}.$$

Hence the proof is complete by Theorem F. □

By Theorem 5, we obtain the following extension of Theorem E.

Corollary 6 ([Y]). *If T is an $n \times n$ matrix, then $W(T) \supset W(\tilde{T})$.*

Proof. Since T is an $n \times n$ matrix, there exists a unitary matrix U such that $T = U|T|$. Since it is in the finite dimensional case, $W(T)$ and $W(\tilde{T})$ are both closed, and the proof is complete by Theorem 5. □

Corollary 7 ([Y]). *Let $T \in B(H)$ with $N(T^*) \supset N(T)$. Then*

$$\overline{W(T)} \supset \overline{W(\tilde{T})} \supset \overline{W(\tilde{T}_2)} \supset \cdots \supset \overline{W(\tilde{T}_n)} \quad \text{hold for all natural number } n.$$

Proof. Since $N(T^*) \supset N(T)$, we can choose an isometry U such that $T = U|T|$. Then we have $\overline{W(T)} \supset \overline{W(\tilde{T})}$ by Theorem 5. So we have only to prove $N(\tilde{T}^*) \supset N(\tilde{T})$ if $N(T^*) \supset N(T)$.

By the definition of Aluthge transformation, $N(\tilde{T}) \supset N(T)$ and $N(\tilde{T}^*) \supset N(T)$ hold, easily. So we shall show $N(T) \supset N(\tilde{T})$.

Let $x \in N(\tilde{T})$. Then by $N(|T|^{\frac{1}{2}}) = N(T) \subset N(T^*) = N(|T^*|^{\frac{1}{2}})$, we have

$$\tilde{T}x = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}x = 0 \implies Tx = |T^*|^{\frac{1}{2}}U|T|^{\frac{1}{2}}x = 0.$$

Hence we obtain $N(T) \supset N(\tilde{T})$, and $N(\tilde{T}^*) \supset N(\tilde{T}) = N(T)$. So that the proof is complete by Theorem 5. $\square$

At the end of this section, we would like to summarize some good properties of Aluthge transformation. We remark that in case an operator T is invertible, $\tilde{T}$ can be rewritten as $\tilde{T} = |T|^{\frac{1}{2}}T|T|^{-\frac{1}{2}}$, in other words, $\tilde{T}$ is similar to T . So we would like to summarize some properties of Aluthge transformation, powers of operators and operators which is similar to T as follows:

Results on $\tilde{T}_n$	Results on T^n	Results on $S^{-1}TS$
$\sigma(\tilde{T}_n) = \sigma(T)$ [1, 4, 9] $r(\tilde{T}_n) = r(T)$	$\sigma(T^n) = \sigma(T)^n$ $r(T^n)^{\frac{1}{n}} = r(T)$	$\sigma(S^{-1}TS) = \sigma(T)$ $r(S^{-1}TS) = r(T)$
$w(\tilde{T}_n) \leq w(T)$ Conjecture	$w(T^n)^{\frac{1}{n}} \leq w(T)$	$\bigcap_s \overline{W(S^{-1}TS)} = \text{co}\sigma(T)$ [8]
$\ \tilde{T}_n\ \leq \ T\ $ $\lim_{n \rightarrow \infty} \ \tilde{T}_n\ = r(T)$ [15]	$\ T^n\ ^{\frac{1}{n}} \leq \ T\ $ $\lim_{n \rightarrow \infty} \ T^n\ ^{\frac{1}{n}} = r(T)$	$\inf_s \ S^{-1}TS\ = r(T)$ [12]

Results on $\tilde{T}_n$	There exists a counterexample
$w(\tilde{T}_n) \leq \dots \leq w(\tilde{T}) \leq w(T)$ $\overline{W(\tilde{T}_n)} \subset \dots \subset \overline{W(\tilde{T})} \subset \overline{W(T)}$ if $N(T) \subset N(T^*)$	$w(T^n)^{\frac{1}{n}} \leq w(T^{n-1})^{\frac{1}{n-1}} \leq \dots \leq w(T)$
$\ \tilde{T}_n\ \leq \dots \leq \ \tilde{T}\ \leq \ T\ $	$\ T^n\ ^{\frac{1}{n}} \leq \ T^{n-1}\ ^{\frac{1}{n-1}} \leq \dots \leq \ T\ $

In above tables, $\text{co}\sigma(T)$ means convex hull of $\sigma(T)$.

Counterexample ([11]). Let

$$T = \begin{pmatrix} 0 & a^2 \\ b^2 & 0 \end{pmatrix}, \quad a > b > 0.$$

Then we have $T^{2n} = a^{2n}b^{2n}I$, $T^{2n+1} = a^{2n}b^{2n}T$.

(i) Counterexample of $\|T^n\|^{\frac{1}{n}} \leq \|T^{n-1}\|^{\frac{1}{n-1}} \leq \dots \leq \|T\|$.

By the above matrix T , we have

$$\|T\| = a^2, \quad \|T^{2n}\|^{\frac{1}{2n}} = ab \quad \text{and} \quad \|T^{2n+1}\|^{\frac{1}{2n+1}} = \{a^{2n}b^{2n}\|T\|\}^{\frac{1}{2n+1}} = ab \left(\frac{a}{b}\right)^{\frac{1}{2n+1}}.$$

Hence we obtain $\|T^{2n+1}\|^{\frac{1}{2n+1}} > \|T^{2n}\|^{\frac{1}{2n}}$ by $a > b > 0$.

(ii) Counterexample of $w(T^n)^{\frac{1}{n}} \leq w(T^{n-1})^{\frac{1}{n-1}} \leq \dots \leq w(T)$.

By the same way to (i), we have

$$w(T) = \frac{a^2 + b^2}{2}, \quad w(T^{2n})^{\frac{1}{2n}} = ab$$

and

$$w(T^{2n+1})^{\frac{1}{2n+1}} = \{a^{2n}b^{2n}w(T)\}^{\frac{1}{2n+1}} = ab \left(\frac{a^2 + b^2}{2ab} \right)^{\frac{1}{2n+1}}.$$

Hence we obtain $w(T^{2n+1})^{\frac{1}{2n+1}} > w(T^{2n})^{\frac{1}{2n}}$ by $a > b > 0$ and $\frac{a^2 + b^2}{2} > ab$.

We can understand that $\tilde{T}$ has some good properties related to T^n and $S^{-1}TS$, and has some better properties than T^n . Moreover we conjecture the following assertion.

Conjecture. For any operator T , does $\bigcap_n \overline{W(\tilde{T}_n)} = \cos(T)$ hold?

4. POLAR DECOMPOSITION

In this section, we shall obtain the polar decomposition of $\tilde{T}$ as follows:

Theorem 8. Let $T = U|T|$ and

$$(4.1) \quad |T|^{\frac{1}{2}}|T^*|^{\frac{1}{2}} = V \left| |T|^{\frac{1}{2}}|T^*|^{\frac{1}{2}} \right|$$

be the polar decompositions. Then $\tilde{T} = VU|\tilde{T}|$ is also the polar decomposition.

By Theorem 8, we can obtain the polar decomposition of n -th Aluthge transformation for any natural number n , because the partial isometry which appears in the polar decomposition of $\tilde{T}$ is a product of two partial isometries.

Proof. (i) Proof of $\tilde{T} = VU|\tilde{T}|$.

$$\begin{aligned} VU|\tilde{T}| &= VU(|T|^{\frac{1}{2}}U^*|T|U|T|^{\frac{1}{2}})^{\frac{1}{2}}U^*U \\ &= V(|T^*|^{\frac{1}{2}}|T||T^*|^{\frac{1}{2}})^{\frac{1}{2}}U \\ &= V \left| |T|^{\frac{1}{2}}|T^*|^{\frac{1}{2}} \right| U \\ &= |T|^{\frac{1}{2}}|T^*|^{\frac{1}{2}}U \quad \text{by (4.1)} \\ &= |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}} = \tilde{T}. \end{aligned}$$

(ii) We shall show $N(\tilde{T}) = N(VU)$. Since $N(|T|^{\frac{1}{2}}|T^*|^{\frac{1}{2}}) = N(V)$, we have

$$N(\tilde{T}) = N(|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}) = N(|T|^{\frac{1}{2}}|T^*|^{\frac{1}{2}}U) = N(VU).$$

(iii) We shall prove that VU is a partial isometry. Since $N(VU)^\perp = N(|\tilde{T}|)^\perp = \overline{R(|\tilde{T}|)}$ hold by (ii), for any $x \in N(VU)^\perp = \overline{R(|\tilde{T}|)}$, there exists $\{y_n\} \subset H$ such that $x = \lim_{n \rightarrow \infty} |\tilde{T}|y_n$. Then we have

$$\begin{aligned} \|VUx\| &= \|VU \lim_{n \rightarrow \infty} |\tilde{T}|y_n\| = \|\lim_{n \rightarrow \infty} VU|\tilde{T}|y_n\| = \|\lim_{n \rightarrow \infty} \tilde{T}y_n\| \quad \text{by (i)} \\ &= \lim_{n \rightarrow \infty} \|\tilde{T}y_n\| = \lim_{n \rightarrow \infty} \| |\tilde{T}|y_n \| = \|\lim_{n \rightarrow \infty} |\tilde{T}|y_n\| = \|x\|, \end{aligned}$$

that is, VU is partial isometry.

Therefore the proof is complete. $\square$

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