

Subordinate families to convolution kernels
of logarithmic type

Masayuki ITÔ (Nagoya Univ.)

伊藤正之 (名大. 教養)

§ 1. Let X be a locally compact abelian group with countable basis, $M_K(X)$ the usual topological vector space of all real Radon measures on X with compact support and let $M_K^0(X) = \{\mu \in M_K(X); \int d\mu = 0\}$. A real convolution kernel N on X means a real Radon measure on X . We say that N is a convolution kernel of logarithmic type (resp. a Hunt convolution kernel) if for any $\mu \in M_K^0(X)$ (resp. $\mu \in M_K(X)$), $N*\mu$ is of form

$$(1.1) \quad N*\mu = \int_0^\infty \alpha_t * \mu dt,$$

where $(\alpha_t)_{t \geq 0}$ is a semi-transient convolution semi-group (resp. a transient convolution semi-group) on X . In this case, $(\alpha_t)_{t \geq 0}$ is uniquely determined (see [1] and [6]) and called the convolution semi-group of N . Evidently Hunt convolution kernels are always convolution kernels of logarithmic type.

Let $H_b(R^+)$ be the set of all Hunt convolution kernels on R supported by $R^+ = \{t \in R; t \geq 0\}$ whose convolution semi-groups are sub-Markovian, where R denotes the 1-dimensional Euclidean space.

For any Hunt convolution kernel $N = \int_0^\infty \alpha_t dt$ on X and for any $\kappa \in H_b(R^+)$, $\int \alpha_t d\kappa(t)$ defines a convolution kernel on X and is a Hunt convolution kernel (see [3]). Put

$$(1.2) \quad H(N; X) = \{N_{(\kappa)} = \int \alpha_t d\kappa(t); \kappa \in H_b(R^+)\}.$$

We call $H(N; X)$ the subordinate family to N . To determine Hunt convolution kernels as a kind of the perturbation of N , $H(N; X)$ plays a fundamental role.

Let N be a convolution kernel of logarithmic type, $(\alpha_t)_{t \geq 0}$ the convolution semi-group of N and let $\kappa \in H_b(R^+)$. It was a question if there exists a convolution kernel $N_{(\kappa)}$ of logarithmic type on X satisfying

$$(1.3) \quad N_{(\kappa)} * \mu = \lim_{a \rightarrow \infty} \int_0^a \alpha_t * \mu d\kappa(t) \quad (= \int \alpha_t * \mu d\kappa(t))$$

for every $\mu \in M_K^0(X)$ (see [5]). In the previous paper [8], we gave an affirmative solution:

Theorem. Let N be a convolution kernel of logarithmic type on X and $(\alpha_t)_{t \geq 0}$ the convolution semi-group of N . Then for any $\kappa \in H_b(R^+)$, there exists a convolution kernel $N_{(\kappa)}$ of logarithmic type on X satisfying (1.3) for every $\mu \in M_K^0(X)$.

The purpose of this note is to give a sketch of the proof and to discuss an application of this theorem.

§ 2. Let $C_K(X)$ denote the usual topological vector space of all finite continuous functions with compact support, $C_b(X)$ the usual Banach space of all bounded continuous functions on X and $M(X)$ the topological vector space of all real Radon measures on X with the vague topology. We denote by $C_K^+(X)$, $C_b^+(X)$, $M_K^+(X)$ and by $M^+(X)$ their subsets of all non-negative elements, respectively.

A family $(\alpha_t)_{t \geq 0}$ in $M^+(X)$ is called a convolution semi-group on X if $\alpha_0 =$ the unit measure ε at the origin 0 of X , $\alpha_t * \alpha_s = \alpha_{t+s}$ for all $t, s \in R^+$ and if $R^+ \ni t \rightarrow \alpha_t \in M(X)$ is continuous.

We say that:

$(\alpha_t)_{t \geq 0}$ is sub-Markovian (resp. Markovian) if for any $t \geq 0$,
 $\int d\alpha_t \leq 1$ (resp. $\int d\alpha_t = 1$).

$(\alpha_t)_{t \geq 0}$ is transient if for any $f \in C_K^+(X)$, $\int_0^\infty \int f d\alpha_t dt < \infty$.

$(\alpha_t)_{t \geq 0}$ is recurrent if it is not transient.

$(\alpha_t)_{t \geq 0}$ is semi-transient if for any $p > 0$, the convolution semi-group $(\exp(-pt)\alpha_t)_{t \geq 0}$ is transient and if for any $\mu \in M_K^0(X)$, $(\int_0^t \alpha_s * \mu ds)_{t \geq 0}$ is bounded in $M(X)$, where $\int_0^t \alpha_s * \mu ds$ is the real Radon measure on X defined by $C_K(X) \ni f \rightarrow \int_0^t \int f d\alpha_s * \mu ds$.

For a convolution semi-group $(\alpha_t)_{t \geq 0}$ on X , we put

$$(2.1) \quad \Gamma((\alpha_t)) = \overline{\bigcup_{t \geq 0} \text{supp}(\alpha_t)}.$$

Then $\Gamma((\alpha_t))$ is a semi-group in X . Let $(\alpha_t)_{t \geq 0}$ be a convolution semi-group on X such that for any $p > 0$, $(\exp(-pt)\alpha_t)_{t \geq 0}$ is transient. Put $N_p = \int_0^\infty \exp(-pt)\alpha_t dt$ for every $p > 0$, and $(N_p)_{p > 0}$ is called the resolvent defined by $(\alpha_t)_{t \geq 0}$. For any $p > 0$ and $q > 0$, the resolvent equation

$$(2.2) \quad N_p - N_q = (q - p)N_p * N_q$$

holds.

By using the resolvent defined by a convolution semi-group, we obtain the following

Lemma 1. Let $(\alpha_t)_{t \geq 0}$ be a semi-transient convolution semi-group on X . If $(\alpha_t)_{t \geq 0}$ is recurrent, then $\Gamma((\alpha_t)) = X$ and $(\alpha_t)_{t \geq 0}$ is Markovian.

Combining the above lemma and one of the main results in [5], we have the following

Proposition 1. Let $(\alpha_t)_{t \geq 0}$ be a recurrent convolution semi-group

on X . Then $(\alpha_t)_{t \geq 0}$ is semi-transient if and only if $\Gamma((\alpha_t)) = X$, $(\alpha_t)_{t \geq 0}$ is Markovian and for any $p > 0$, N_p is non-singular with respect to a fixed Haar measure ξ on X , where $(N_p)_{p>0}$ is the resolvent defined by $(\alpha_t)_{t \geq 0}$. In this case, $(N_{p,s})_{p>0}$ is bounded in $M(X)$, where $N_{p,s}$ is the singular part of N_p with respect to ξ .

Proposition 2. Let $(\alpha_t)_{t \geq 0}$ be a convolution semi-group on X . Then $(\alpha_t)_{t \geq 0}$ is the convolution semi-group of a convolution kernel of logarithmic type if and only if $(\alpha_t)_{t \geq 0}$ is semi-transient.

This follows from the above lemma and another main theorem in [5].

Let N be a positive convolution kernel on X and $\mu \in M^+(X)$ with $N*\mu \in M^+(X)$. For an open set ω in X , a non-negative Radon measure μ' on X is called an inner N -balayaged measure of μ to ω if $\text{supp}(\mu') \subset \bar{\omega}$, $N*\mu' \leq N*\mu$ in X , $N*\mu' = N*\mu$ in ω and if for any $\nu \in M^+(X)$ with $N*\nu \in M^+(X)$, $N*\nu \geq N*\mu'$ in X whenever $N*\nu \geq N*\mu$ in ω .

Lemma 2 (see, for example, [4]). Let N be a Hunt convolution kernel on X and $\mu \in M^+(X)$ with $N*\mu \in M^+(X)$. Then for any open set ω in X , there exists a uniquely determined inner N -balayaged measure μ'_ω of μ to ω .

Lemma 3 (see [4]). Let $(\alpha_t)_{t \geq 0}$ be a convolution semi-group on X and assume that for any $p > 0$, $(\exp(-pt)\alpha_t)_{t \geq 0}$ is transient. Let $(N_p)_{p>0}$ be the resolvent defined by $(\alpha_t)_{t \geq 0}$. For any $p > 0$, any open set ω in X and for $\mu \in M^+(X)$ with $N*\mu \in M^+(X)$, we denote by $\mu'_{p,\omega}$ the inner N_p -balayaged measure of μ to ω . Then we have:

(1) The mapping $X \ni x \rightarrow \varepsilon'_{x,p,\omega} \in M(X)$ is Borel measurable, that is, for any $f \in C_K(X)$, the function $\int f d\varepsilon'_{x,p,\omega}$ of x is Borel measurable on X , where ε_x denotes the unit measure at x and where $\varepsilon'_{x,p,\omega} = (\varepsilon_x)'_{p,\omega}$.

(2) For any $\mu \in M^+(X)$ with $N_p * \mu \in M^+(X)$, we have

$$(2.3) \quad \mu'_{p,\omega} = \int \varepsilon'_{x,p,\omega} d\mu(x),$$

that is, $\int f d\mu'_{p,\omega} = \int \int f d\varepsilon'_{x,p,\omega} d\mu(x)$ for every $f \in C_K(X)$.

(3) For any $p > q > 0$ and any $\mu \in M^+(X)$ with $N_q * \mu \in M^+(X)$, we have

$$(2.4) \quad \mu'_{q,\omega} = \mu'_{p,\omega} + (p - q)(N_p * (\mu - \mu'_{p,\omega}))'_{q,\omega}.$$

Let $(\alpha_t)_{t \geq 0}$ be a Markovian convolution semi-group on X and $(N_p)_{p > 0}$ the resolvent defined by $(\alpha_t)_{t \geq 0}$. For any $p > 0$, any $\mu \in M_K^+(X)$ and for any open set ω in X , the positive mass principle of N_p ¹⁾ gives $\int d\mu'_{p,\omega} \leq \int d\mu$. Since $(\mu'_{p,\omega})_{p > 0}$ increases as p decreases, $\lim_{p \rightarrow 0} \mu'_{p,\omega}$ exists in $M^+(X)$. Putting $\mu'_\omega = \lim_{p \rightarrow 0} \mu'_{p,\omega}$, we call it the inner balayaged measure of μ to ω with respect to $(\alpha_t)_{t \geq 0}$. Furthermore, for any $\mu \in M^+(X)$ with $\int d\mu < \infty$, it can be defined by the usual limit process. Lemma 3 gives the following

Lemma 4. Let $(\alpha_t)_{t \geq 0}$ be a Markovian convolution semi-group on X , $(N_p)_{p > 0}$ the resolvent defined by $(\alpha_t)_{t \geq 0}$ and ω an open set in X . Then for any $\mu \in M^+(X)$ with $\int d\mu < \infty$ and any $p > 0$,

$$(2.5) \quad \mu'_\omega = \int \varepsilon'_{x,\omega} d\mu(x),$$

where $\varepsilon'_{x,\omega} = (\varepsilon_x)'_\omega$ and

$$(2.6) \quad \mu'_\omega = \mu'_{p,\omega} + p(N_p * \mu - N_p * \mu'_{p,\omega})'_\omega.$$

1) This means that for any $\mu, \nu \in M_K^+(X)$, $N_p * \mu \leq N_p * \nu$ on X implies $\int d\mu \leq \int d\nu$.

To prove our main theorem, the following lemma will play an essential role.

Lemma 5. Let $(\alpha_t)_{t \geq 0}$ be a semi-transient and recurrent convolution semi-group on X . Then for any $t_0 > 0$ and any $p > 0$, $\text{supp}(\sum_{n=0}^{\infty} \exp(-pnt_0) \alpha_{nt_0}) = X$ and $\sum_{n=0}^{\infty} \exp(-pnt_0) \alpha_{nt_0}$ is non-singular with respect to ξ .

For $\mu \in M^+(X)$, $\text{supp}(\mu)$ denotes the support of μ .

It suffices to show our conclusion in the case of $t_0 = 1$. Let $(N_p)_{p>0}$ be the resolvent defined by $(\alpha_t)_{t \geq 0}$. Put $N'_p = \sum_{n=0}^{\infty} \exp(-pn) \alpha_n$ and $N''_p = \int_0^1 \exp(-pt) \alpha_t dt$. Then

$$(2.7) \quad N'_p * N''_p = N_p.$$

By (2.7), the recurrence of $(\alpha_t)_{t \geq 0}$ and the Choquet-Deny theorem ([2]), we obtain that $\text{supp}(N'_p) = X$ for every $p > 0$.

Let $\omega \neq \phi$ be an open set in X . By Lemma 4 and the Choquet-Deny theorem ([2]), we obtain that

$$(2.8) \quad \lim_{p \rightarrow 0} N'_p * (\varepsilon - \varepsilon''_{\omega}) = -a_{\omega} \xi \quad \text{in } \omega$$

with some constant $a_{\omega} \geq 0$, where ε''_{ω} is the inner balayaged measure of ε to ω with respect to the convolution semi-group $(\beta_t)_{t \geq 0}$ which defines the resolvent $(\frac{1}{p+1} N'_p \log(p+1))_{p>0}$. We remark here that $(\beta_t)_{t \geq 0}$ is also Markovian and recurrent. Furthermore, putting $\eta_{\omega} = \lim_{p \rightarrow 0} N'_p * (\varepsilon - \varepsilon''_{\omega})$, we obtain that

$$(2.9) \quad \sum_{n=0}^{\infty} \alpha_n * (\varepsilon - \varepsilon''_{\omega}) = \eta_{\omega}$$

and

and that

$$(2.10) \quad N * (\varepsilon - \varepsilon''_{\omega}) = N'_0 * \eta_{\omega},$$

where N is the convolution kernel of logarithmic type on X whose convolution semi-group is equal to $(\alpha_t)_{t \geq 0}$. Assume that there exists $p > 0$ such that N'_p is singular with respect to ξ . Then for any $n \geq 1$, $\alpha_n * (\varepsilon - \varepsilon''_{\omega})$ is singular with respect to ξ . By (2.8) and (2.9), we have $a_{\omega} = 0$, so that $\eta_{\omega} \geq 0$ in X . Therefore (2.10) gives $N * (\varepsilon - \varepsilon''_{\omega}) \geq 0$ in X . Since ω is arbitrary and $\int d\varepsilon''_{\omega} = 1$, we have $N \geq N * \varepsilon_X$ in X for every $x \in X$, so that for any $x \in X$, $N = N * \varepsilon_X$, which is a contradiction. Thus N'_p is non-singular with respect to ξ for every $p > 0$.

Lemma 5 gives the following

Proposition 3. Let $(\alpha_t)_{t \geq 0}$ be a semi-transient and recurrent convolution semi-group on X . Then there exists $t_0 > 0$ such that for any $t \geq t_0$, $\text{supp}(\alpha_t) = X$ and α_t is non-singular with respect to ξ .

In the same manner as in Proposition 32 in [7], we have the following

Lemma 6. Let $(\alpha_t)_{t \geq 0}$ be a semi-transient and recurrent convolution semi-group on X and let $\kappa \in H_b(R^+)$. Denote by $(\gamma_t)_{t \geq 0}$ the convolution semi-group of κ and by $(\kappa_p)_{p > 0}$ the resolvent defined by $(\gamma_t)_{t \geq 0}$. Assume that the convolution semi-group $(\int \alpha_s d\gamma_t(s))_{t \geq 0}$ is semi-transient and recurrent. Let $N_{(\kappa)}$ be the convolution kernel of logarithmic type on X whose convolution semi-group is equal to $(\int \alpha_s d\gamma(s))_{t \geq 0}$. Then for any $\mu \in M_K^0(X)$,

$$(2.11) \quad N_{(\kappa)} * \mu = \lim_{p \rightarrow 0} N_{(\kappa_p)} * \mu = \lim_{t \rightarrow \infty} \int_0^t \alpha_s * \mu d\kappa(s) (= \int \alpha_t * \mu d\kappa(t)).$$

§ 3. Let us give a sketch of the proof of our theorem. We may assume that $(\alpha_t)_{t \geq 0}$ is semi-transient and recurrent. Let κ be in $H_b(R^+)$, $(\gamma_t)_{t \geq 0}$ the convolution semi-group of κ and $(\kappa_p)_{p > 0}$ the resolvent defined by $(\gamma_t)_{t \geq 0}$. It suffices to show our conclusion in the case that $(\int \alpha_s d\gamma_t(s))_{t \geq 0}$ is recurrent. Hence it suffices to show that $(\int \alpha_s d\gamma_t(s))_{t \geq 0}$ is semi-transient (see Lemma 6). By the usual method, it suffices to assume that κ is elementary, that is, $\kappa = c(\varepsilon + \sum_{n=1}^{\infty} (\sigma)^n)$, where c is a positive constant, $\sigma \in M^+(R)$ with $\text{supp}(\sigma) \subset R^+$ and with $\int d\sigma \leq 1$, $(\sigma)^1 = \sigma$ and where $(\sigma)^n = (\sigma)^{n-1} * \sigma$ for every $n \geq 1$. By the recurrence of $(\int \alpha_s d\gamma_t(s))_{t \geq 0}$, $\int d\sigma = 1$ and $\gamma((0, \infty)) > 0$. Hence there exists $t_0 > 0$ such that denoting by σ_0 the restriction of σ to $[t_0, \infty)$, we have $\sigma_0 \neq 0$, $\int d\sigma_0 < 1$ and $\text{supp}(\sigma_0) \supset t_0$. Put

$$(3.1) \quad \kappa_{0,p} = \frac{c}{pc+1} (\varepsilon + \sum_{n=1}^{\infty} (\frac{1}{pc+1} \sigma_0)^n)$$

for every $p > 0$. Since $\kappa_p \geq \kappa_{0,p}$, it suffices to show that $N_{(\kappa_{0,p})}$ is non-singular with respect to ξ (see Proposition 1). Since for any $n \geq 1$,

$$(3.2) \quad \text{supp}((\int \alpha_t d\sigma_0(t))^n) \supset \text{supp}(\alpha_{nt_0}),$$

Lemma 5 shows that $\text{supp}(N_{(\kappa_{0,p})}) = X$ for every $p > 0$. Furthermore, Proposition 3 shows that $N_{(\kappa_{0,p})}$ is non-singular with respect to ξ . Thus $(\int \alpha_s d\gamma_t(s))_{t \geq 0}$ is semi-transient, which proves our main theorem.

§ 4. Let N be a convolution kernel of logarithmic type on X and $(\alpha_t)_{t \geq 0}$ the convolution semi-group of N . Put

$$H_1(N; X) = \{N_{(\kappa)} = \int \alpha_t d\kappa(t); \kappa \in H_b(R^+)\},$$

$$\begin{aligned}
 (4.1) \quad & H_2(N; X) = \{(N_1)_{(\kappa)}; N_1 \in H_1(N; X), \kappa \in H_b(R^+)\}, \\
 & \dots\dots\dots, \\
 & H_n(N; X) = \{(N_{n-1})_{(\kappa)}; N_{n-1} \in H_{n-1}(N; X), \kappa \in H_b(R^+)\}, \\
 & \dots\dots\dots,
 \end{aligned}$$

Then for any $n \geq 1$, $H_n(N; X)$ is a family of convolution kernels of logarithmic type on X and $H_n(N; X) \subset H_{n+1}(N; X)$. Put

$$(4.2) \quad H_\infty(N; X) = \overline{\bigcup_{n=1}^{\infty} H_n(N; X)},$$

where the closure is in the sense of the vague topology. We call $H_\infty(N; X)$ the generalized subordinate family to N .

Proposition 4. Let $M \in H_\infty(N; X)$. Then M is a convolution kernel of logarithmic type if and only if M is non-periodic, that is, for any $0 \neq x \in X$, $M * \varepsilon_x \neq M$.

Let M be a given convolution kernel on X . To get a conclusion that M is of logarithmic type, it is very useful to find out a certain convolution kernel N of logarithmic type satisfying $M \in H_\infty(N; X)$. For example, let R^n be the n -dimensional Euclidean space ($n \geq 1$) and put

$$N_0 = \begin{cases} -|x|dx & \text{on } R^1 \\ (-\log|x|)dx & \text{on } R^2 \\ |x|^{2-n}dx & \text{on } R^n \quad (n \geq 3) \end{cases}$$

where $|x|$ denotes the distance between x and 0 and where dx denote the Lebesgue measure. By discussing precisely $H_\infty(N_0; R^n)$, we can obtain several concrete conditions to be Hunt convolution kernels on R^n or convolution kernels of logarithmic type on R^n .

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