

Singular parts of moduli spaces for cubic polynomials and quadratic rational maps

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1. Quadratic rational maps

1.1. Moduli space of quadratic rational maps

Let $\overline{\mathbf{C}}$ be the Riemann sphere and $\text{Rat}_2(\mathbf{C})$ the space of all quadratic rational maps from $\overline{\mathbf{C}}$ to itself. The group $\text{PSL}_2(\mathbf{C})$ of Möbius transformations acts on the space $\text{Rat}_2(\mathbf{C})$ by conjugation,

$$g \circ f \circ g^{-1} \in \text{Rat}_2(\mathbf{C}) \quad \text{for } g \in \text{PSL}_2(\mathbf{C}), f \in \text{Rat}_2(\mathbf{C}).$$

Two maps $f_1, f_2 \in \text{Rat}_2(\mathbf{C})$ are **holomorphically conjugate**, denoted by $f_1 \sim f_2$, if and only if there exists $g \in \text{PSL}_2(\mathbf{C})$ with $g \circ f_1 \circ g^{-1} = f_2$. The quotient space of $\text{Rat}_2(\mathbf{C})$ under this action will be denoted by $\mathcal{M}_2(\mathbf{C})$, and called the **moduli space** of holomorphic conjugacy classes $\langle f \rangle$ of quadratic rational maps f .

Milnor introduced coordinates in $\mathcal{M}_2(\mathbf{C})$ as follows; for each $f \in \text{Rat}_2(\mathbf{C})$, let z_1, z_2, z_3 be the fixed points of f and μ_i the multipliers of z_i ; $\mu_i = f'(z_i)$ ($1 \leq i \leq 3$). Consider the elementary symmetric functions of the three multipliers,

$$\sigma_1 = \mu_1 + \mu_2 + \mu_3, \quad \sigma_2 = \mu_1\mu_2 + \mu_2\mu_3 + \mu_3\mu_1, \quad \sigma_3 = \mu_1\mu_2\mu_3.$$

These three multipliers determine f up to holomorphic conjugacy, and are subject only to the restriction that

$$\sigma_3 = \sigma_1 - 2.$$

Hence the moduli space $\mathcal{M}_2(\mathbf{C})$ is canonically isomorphic to $\mathbf{C}^2$ with coordinates σ_1 and σ_2 (Lemma 3.1 in [Mil93]).

For each $\mu \in \mathbf{C}$ let $\text{Per}_n(\mu)$ be the set of all conjugacy classes $\langle f \rangle$ of maps f which having a periodic point of period n and multiplier μ .

Each of $\text{Per}_1(\mu)$ and $\text{Per}_2(\mu)$ forms a straight lines as follows:

$$\begin{aligned} \text{Per}_1(\mu) &= \{ \langle f \rangle \in \mathcal{M}_2(\mathbf{C}); \sigma_2 = (\mu + \mu^{-1})\sigma_1 - (\mu^2 + 2\mu^{-1}) \} \\ \text{Per}_2(\mu) &= \{ \langle f \rangle \in \mathcal{M}_2(\mathbf{C}); \sigma_2 = -2\sigma_1 + \mu \}, \end{aligned}$$

(Lemmas 3.4 and 3.6 in [Mil93]).

Remark $\text{Per}_1(-1) \subseteq \text{Per}_2(1)$ by definition. But, in the case of $\mathcal{M}_2(\mathbb{C})$, it is clear that two families coincide.

By an automorphism of a quadratic rational map f , we will mean $g \in \text{PSL}_2(\mathbb{C})$ which commutes with f . The collection $\text{Aut}(f)$ of all automorphisms of f forms a finite group. It is clear that $\text{Aut}(\tilde{f})$ is isomorphic to $\text{Aut}(f)$ for any $\tilde{f} \in \langle f \rangle$.

The set

$$\mathcal{S} = \{ \langle f \rangle; \text{Aut}(f) \text{ is non-trivial} \} \subset \mathcal{M}_2(\mathbb{C})$$

is called the **symmetry locus**.

Proposition 1 *The symmetry locus $\mathcal{S}$ of quadratic rational maps forms an irreducible algebraic curve as follows;*

$$S(\sigma_1, \sigma_2) = 2\sigma_1^3 + \sigma_1^2\sigma_2 - \sigma_1^2 - 4\sigma_2^2 - 8\sigma_1\sigma_2 + 12\sigma_1 + 12\sigma_2 - 36 = 0. \quad (1)$$

Proof of Corollary 1.

$\text{Aut}(f)$ coincides with the group consisting of all permutations of the fixed points which preserve the multipliers. In the case of f has the three distinct fixed points, $\text{Aut}(f)$ has order 1, 2, or 6 according as three multipliers are distinct, two are equal, or all the three are equal, respectively, while, if f has multiple fixed points then $\text{Aut}(f)$ is non-trivial if and only if f has a triple fixed point. The multipliers μ_i are the roots of the equation:

$$\mu^3 - \sigma_1\mu^2 + \sigma_2\mu - \sigma_1 + 2 = 0. \quad (2)$$

The equation (2) has multiple roots if and only if its discriminant is equal to zero. Hence we have

$$(\sigma_2 - 2\sigma_1 + 3)(2\sigma_1^3 + \sigma_1^2\sigma_2 - \sigma_1^2 - 4\sigma_2^2 - 8\sigma_1\sigma_2 + 12\sigma_1 + 12\sigma_2 - 36) = 0.$$

The first factor corresponds with $\text{Per}_1(1)$. Considering the line of the first factor ($\text{Per}_1(1)$) tangent to the curve of the second factor ($\mathcal{S}$) with tangency of degree three, the second factor is the required equation. ■

The following result is obtained immediately by the definition of the envelope of the family of curves.

Corollary 1 *The envelope of $\{\text{Per}_1(\mu)\}_\mu$ coincides with the symmetry locus.*

Remark (Theorem 5.1. of [Mil93]) A quadratic rational map has a non-trivial automorphism if and only if it is conjugate to a map in the unique normal form $f(z) = k(z + \frac{1}{z})$ with $k \in \mathbb{C} \setminus \{0\}$.

1.2. Real moduli space

Let $\text{Rat}_2(\mathbf{R})$ be the set of real quadratic rational maps. Then the parameters σ_i ($1 \leq i \leq 3$) are all real, because the three fixed points and the corresponding multipliers are either all real or one real and a pair of complex conjugate numbers. According to J. Milnor, we define the real moduli space $\mathcal{M}_2(\mathbf{R})$ for $\text{Rat}_2(\mathbf{R})$ to be simply the real (σ_1, σ_2) -plane. This notation needs some care when used: if we put $\mathcal{S}_{\mathbf{R}} = \mathcal{S} \cap \mathcal{M}_2(\mathbf{R})$, and denote by $\langle \rangle_{\mathbf{R}}$ the real conjugacy class, then $(\text{Rat}_2(\mathbf{R})/\text{PGL}_2(\mathbf{R})) \setminus \{\langle a(x + \frac{1}{x}) \rangle_{\mathbf{R}}, \langle a(x - \frac{1}{x}) \rangle_{\mathbf{R}}\}_{a \in \mathbf{R}^\times}$ is canonically isomorphic to $\mathbf{R}^2 \setminus \mathcal{S}_{\mathbf{R}}$, whereas there is a canonical two-to-one correspondence between $\{\langle a(x \pm \frac{1}{x}) \rangle_{\mathbf{R}}\}_{a \in \mathbf{R}^\times}$ and $\mathcal{S}_{\mathbf{R}}$.

For map $f \in \mathcal{M}_2(\mathbf{R})$, the two critical points of f are two real numbers or a pair of complex conjugate numbers. If f has a pair of complex conjugate critical points, this map is two-to-one covering map on $S^1 = \mathbf{R} \cup \{\infty\}$. In this case, if $f' > 0$ then f is called the map of degree +2, else $f' < 0$ then the map of degree -2.

While a map f with real critical points is called monotone (resp. unimodal, bimodal) if the interval $I = \text{int}(f(S))$ contains no (resp. one, two) critical points ([Mil93]).

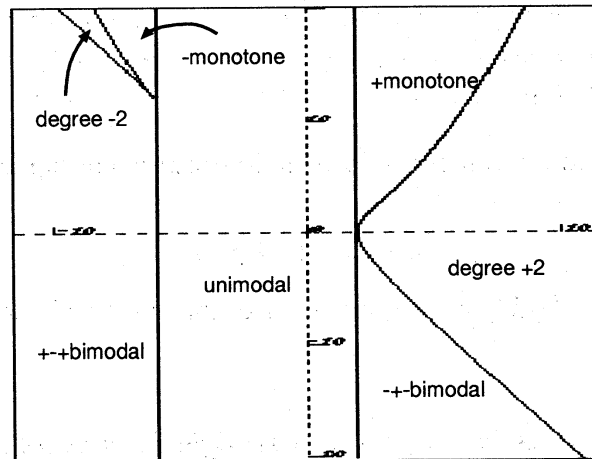


Fig. 1. The topological partition of the $\mathcal{M}_2(\mathbf{R})$.

Boundary curves of Figure 1

$$CD_1 : \sigma_1 = 2$$

$$BC_1 : \sigma_1 = 6$$

$$\text{Symmetry locus} : S(\sigma_1, \sigma_2) = 0$$

where the curves CD_1 ($\text{Per}_1(0)$) and BC_1 are “center curve” defined in [NN93].

Remark Two curves BC_1 and CD_1 are boundary curves of the “unimodal” region.

2. Cubic polynomials

2.1. Moduli space of cubic polynomials

Let $\text{Poly}_3(\mathbf{C})$ be the space of all cubic polynomials from $\mathbf{C}$ to itself. The group $\text{Poly}_1(\mathbf{C})$ of affine transformations acts on the space $\text{Poly}_3(\mathbf{C})$ by conjugation,

$$g \circ p \circ g^{-1} \in \text{Poly}_3(\mathbf{C}) \quad \text{for} \quad g \in \text{Poly}_1(\mathbf{C}), p \in \text{Poly}_3(\mathbf{C}).$$

Two maps $p_1, p_2 \in \text{Poly}_3(\mathbf{C})$ are **holomorphically conjugate**, denoted by $p_1 \sim p_2$, if and only if there exists $g \in \text{Poly}_1(\mathbf{C})$ with $g \circ p_1 \circ g^{-1} = p_2$. The quotient space of $\text{Poly}_3(\mathbf{C})$ under this action will be denoted by $M_3(\mathbf{C})$, and called the **moduli space** of holomorphic conjugacy classes $\langle p \rangle$ of cubic polynomials p .

Doing the same as the case of quadratic rational maps, we introduce coordinates in $M_3(\mathbf{C})$ as follows; for each $p \in \text{Poly}_3(\mathbf{C})$, let $z_1, z_2, z_3, z_4 (= \infty)$ be the fixed points of p and μ_i the multipliers of z_i ; $\mu_i = p'(z_i)$ ($1 \leq i \leq 3$), and $\mu_4 = 0$. Consider the elementary symmetric functions of the four multipliers,

$$\begin{aligned} \sigma_1 &= \mu_1 + \mu_2 + \mu_3 + \mu_4 = \mu_1 + \mu_2 + \mu_3 \\ \sigma_2 &= \mu_1\mu_2 + \mu_1\mu_3 + \mu_1\mu_4 + \mu_2\mu_3 + \mu_2\mu_4 + \mu_3\mu_4 = \mu_1\mu_2 + \mu_1\mu_3 + \mu_2\mu_3 \\ \sigma_3 &= \mu_1\mu_2\mu_3 + \mu_1\mu_2\mu_4 + \mu_1\mu_3\mu_4 + \mu_2\mu_3\mu_4 = \mu_1\mu_2\mu_3 \\ \sigma_4 &= \mu_1\mu_2\mu_3\mu_4 = 0. \end{aligned}$$

These multipliers determine uniquely p up to holomorphic conjugacy, and are subject only to the restriction that

$$3 - 2\sigma_1 + \sigma_2 = 0.$$

Hence the moduli space $M_3(\mathbf{C})$ is canonically isomorphic to $\mathbf{C}^2$ with coordinates σ_1 and σ_3 .

Proposition 2 *The locus $\text{Per}_1(\mu)$ forms a straight lines as follows:*

$$\text{Per}_1(\mu) = \{ \langle f \rangle \in M_3(\mathbf{C}); \sigma_3 = (-\mu^2 + 2\mu)\sigma_1 + \mu^3 - 3\mu \}.$$

The locus $\text{Per}_2(\mu)$ forms an algebraic curve of degree three as follows:

$$\begin{aligned} \text{Per}_2(\mu) = \{ \langle f \rangle \in \mathcal{M}_2(\mathbf{C}); & \sigma_3^2 + (4\sigma_1^2 - (\mu + 57)\sigma_1 + 252)\sigma_3 - (4\mu - 16)\sigma_1^3 \\ & + (61\mu - 252)\sigma_1^2 - (4\mu^2 + 246\mu - 1134)\sigma_1 - \mu^3 + 51\mu^2 \\ & - 99\mu - 459 = 0 \}. \end{aligned}$$

Note that this curve is irreducible if and only if $\mu \neq 1$. In the case of $\mu = 1$,

$$\text{Per}_2(1) = \text{Per}_1(-1) \cup \{ \langle f \rangle \in \mathcal{M}_2(\mathbf{C}); \sigma_3 + 4\sigma_1^2 - 61\sigma_1 + 254 = 0 \}.$$

Using conjugation described in above, we can define symmetry locus of this moduli space as one in $\mathcal{M}_2(\mathbf{C})$, and we obtain next results.

Theorem 1 *The symmetry locus $\mathcal{S}$ of cubic polynomials forms an irreducible algebraic curve:*

$$S(\sigma_1, \sigma_3) = 27\sigma_3 + (\sigma_1 - 6)(2\sigma_1 - 3)^2 = 0. \quad (3)$$

The following result is obtained immediately by the definition of the envelope of the family of curves.

Corollary 2 *The envelope of $\{Per_1(\mu)\}_\mu$ coincides with the symmetry locus.*

Remark A cubic polynomial has non-trivial automorphism if and only if it is conjugate to a map in the unique normal form $p(z) = z^3 + az$.

2.2. Real moduli space

Let $\text{Poly}_3(\mathbf{R})$ be the set of real cubic polynomials. By the same reason for the case of $\mathcal{M}_2$, we define the real moduli space $M_3(\mathbf{R})$ for $\text{Poly}_3(\mathbf{R})$ to be simply the real (σ_1, σ_3) -plane. This notation needs some care when used: if we put $\mathcal{S}_{\mathbf{R}} = \mathcal{S} \cap M_3(\mathbf{R})$, and denote by $\langle \cdot \rangle_{\mathbf{R}}$ the real conjugacy class, then $(\text{Poly}_3(\mathbf{R})/\text{Poly}_1(\mathbf{R})) \setminus \{\langle x^3 + ax \rangle_{\mathbf{R}}, \langle -x^3 + ax \rangle_{\mathbf{R}}\}_{a \in \mathbf{R}^\times}$ is canonically isomorphic to $\mathbf{R}^2 \setminus \mathcal{S}_{\mathbf{R}}$, whereas there is a canonical two-to-one correspondence between $\{\langle \pm x^3 + ax \rangle\}_{a \in \mathbf{R}^\times}$ and $\mathcal{S}_{\mathbf{R}}$.

For map $p \in M_3(\mathbf{R})$, if the real filled-in Julia set of p is a single point then it is said that p in the class $\mathcal{R}_0$. Let J be the smallest closed interval which contains the real filled-in Julia set of p . For $p \notin \mathcal{R}_0$, it is said that p belongs to the class $\mathcal{R}_n$ if the graph of p intersected with $J \times J$ has n distinct components ([Mil92]).

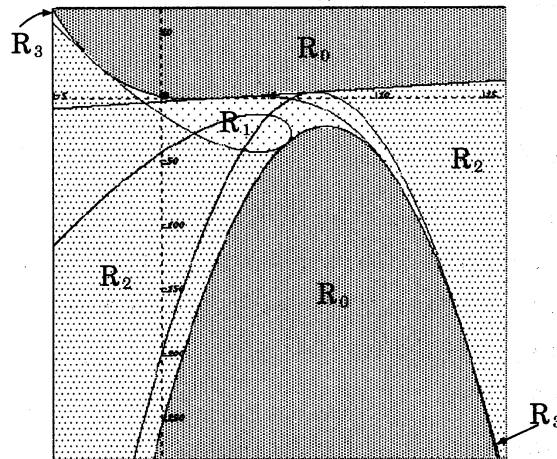


Fig. 2. The topological partition of the $M_3(\mathbf{R})$.

Boundary curves of Figure 2

$$\begin{aligned}
\text{Per}_1(1) &: \sigma_1 - \sigma_3 - 2 = 0 \\
\text{Preper}_{(1)}1 &: -4\sigma_1^2 + 57\sigma_1 - \sigma_3 - 198 = 0 \\
\text{Symmetry locus} &: S(\sigma_1, \sigma_3) = 0 \\
\text{Per}_2(1) &: -8\sigma_1^3 + 180\sigma_1^2 - 1809\sigma_1 - 27\sigma_3 + 6966 = 0 \\
\text{Preper}_{(1)}2 &: 64\sigma_1^6 - 1152\sigma_1^5 + 7776\sigma_1^4 + (432\sigma_3 - 25056)\sigma_1^3 \\
&\quad + (-3888\sigma_3 + 41796)\sigma_1^2 + (8748\sigma_3 - 34992)\sigma_1 + 729\sigma_3^2 \\
&\quad - 45198\sigma_3 + 543105 = 0
\end{aligned}$$

3. Polynomials of degree n

3.1. Moduli space of polynomials of degree n

Now we discuss about the moduli space $M_n(\mathbb{C})$ for the space, $\text{Poly}_n(\mathbb{C})$, of polynomials of degree n .

Doing the same as the case of cubic polynomials, we try introducing coordinates in $M_n(\mathbb{C})$ as follows; for each $p(z) \in \text{Poly}_n(\mathbb{C})$, let $z_1, \dots, z_n, z_{n+1}(=\infty)$ be the fixed points of p and μ_i the multipliers of z_i ; $\mu_i = p'(z_i)$ ($1 \leq i \leq n$), and $\mu_{n+1} = 0$. Consider the elementary symmetric functions of the n multipliers,

$$\begin{aligned}
\sigma_{n,1} &= \mu_1 + \dots + \mu_n, \\
\sigma_{n,2} &= \mu_1\mu_2 + \dots + \mu_{n-1}\mu_n = \sum_{i=1}^{n-1} \mu_i \sum_{j>i}^n \mu_j, \\
&\dots \\
\sigma_{n,n} &= \mu_1\mu_2 \dots \mu_n, \\
\sigma_{n,n+1} &= 0.
\end{aligned}$$

Example 1 For example, we assume $p(z) \in \text{Poly}_4(\mathbb{C})$;

- fixed points: $z_1, z_2, z_3, z_4, \infty$
- multiplier: $\mu_1, \mu_2, \mu_3, \mu_4, 0$
- elementary symmetric functions:

$$\begin{cases}
\sigma_{4,1} = \mu_1 + \mu_2 + \mu_3 + \mu_4 \\
\sigma_{4,2} = \mu_1\mu_2 + \mu_1\mu_3 + \mu_1\mu_4 + \mu_2\mu_3 + \mu_2\mu_4 + \mu_3\mu_4 \\
\sigma_{4,3} = \mu_1\mu_2\mu_3 + \mu_1\mu_2\mu_4 + \mu_1\mu_3\mu_4 + \mu_2\mu_3\mu_4 \\
\sigma_{4,4} = \mu_1\mu_2\mu_3\mu_4 \\
\sigma_{4,5} = 0
\end{cases}$$

Applying Fatou-index theorem to these fixed points;

$$\frac{1}{1-\mu_1} + \frac{1}{1-\mu_2} + \frac{1}{1-\mu_3} + \frac{1}{1-\mu_4} + \frac{1}{1-0} = 1, \quad (4)$$

where $\mu_i \neq 1$ ($1 < i < n$). Arranging this equation for the form of elementary symmetric functions;

$$\begin{aligned}
4 - 3(\mu_1 + \mu_2 + \mu_3 + \mu_4) + 2(\mu_1\mu_2 + \mu_1\mu_3 + \mu_1\mu_4 + \mu_2\mu_3 + \mu_2\mu_4 + \mu_3\mu_4) \\
- (\mu_1\mu_2\mu_3 + \mu_1\mu_2\mu_4 + \mu_1\mu_3\mu_4 + \mu_2\mu_3\mu_4) = 0.
\end{aligned}$$

Hence we have

$$4 - 3\sigma_{4,1} + 2\sigma_{4,2} - \sigma_{4,3} = 0. \quad (5)$$

For the equation (5), the cases $\mu_i = 1$ are also allowable.

Now we consider a polynomial $p(z) = a_4z^4 + a_3z^3 + a_2z^2 + a_1z + a_0 \in \text{Poly}_4(\mathbb{C})$ that has at least two fixed points. After affine conjugation, we can assume they are 0 and 1. Then, we will solve the following question: "Do the four multipliers

$$\mu_0 = p'(0), \mu_1 = p'(1), \mu_2 = p'(z_2), \mu_3 = p'(z_3),$$

where z_1, z_2 are fixed points of $p(z)$, determine the five coefficients a_4, a_3, a_2, a_1, a_0 of $p(z)$?"

In fact, the following equations hold;

$$\begin{aligned} a_0 &= 0 && \text{because of } f(0) = 0, \\ a_1 &= \mu_0 && \text{because of } f'(0) = \mu_0, \\ a_2 &= a_4 + 3 - 2\mu_0 - \mu_1 && \text{because of } f'(1) = \mu_1, \\ a_3 &= 1 - a_4 - a_2 - \mu_0 && \text{because of } f(1) = 1, \end{aligned}$$

and a_4 is a common root of the following two equations;

$$\begin{aligned} A_1 &= (\mu_2^2 - 2\mu_3\mu_2 + \mu_3^2 - \mu_0^2 + 2\mu_1\mu_0 - \mu_1^2)a_4^4 + (-4\mu_0^3 + (4\mu_1 + 8)\mu_0^2 + (-4\mu_1^2 - 8)\mu_0 + 4\mu_1^3 - 8\mu_1^2 + 8\mu_1)a_4^3 + (-6\mu_0^4 + (-4\mu_1 + 28)\mu_0^3 + (4\mu_1^2 + 4\mu_1 - 44)\mu_0^2 + (-4\mu_1^3 + 4\mu_1^2 - 8\mu_1 + 32)\mu_0 - 6\mu_1^4 + 28\mu_1^3 - 44\mu_1^2 + 32\mu_1 - 16)a_4^2 + (-4\mu_0^5 + (-12\mu_1 + 32)\mu_0^4 + (-8\mu_1^2 + 64\mu_1 - 96)\mu_0^3 + (8\mu_1^3 - 96\mu_1^2 + 128)\mu_0^2 + (12\mu_1^4 - 64\mu_1^3 + 96\mu_1^2 - 64)\mu_0 + 4\mu_1^5 - 32\mu_1^4 + 96\mu_1^3 - 128\mu_1^2 + 64\mu_1)a_4 - \mu_0^6 + (-6\mu_1 + 12)\mu_0^5 + (-15\mu_1^2 + 60\mu_1 - 60)\mu_0^4 + (-20\mu_1^3 + 120\mu_1^2 - 240\mu_1 + 160)\mu_0^3 + (-15\mu_1^4 + 120\mu_1^3 - 360\mu_1^2 + 480\mu_1 - 240)\mu_0^2 + (-6\mu_1^5 + 60\mu_1^4 - 240\mu_1^3 + 480\mu_1^2 - 480\mu_1 + 192)\mu_0 - \mu_1^6 + 12\mu_1^5 - 60\mu_1^4 + 160\mu_1^3 - 240\mu_1^2 + 192\mu_1 - 64 = 0, \\ A_2 &= (\mu_2 + \mu_3 + \mu_0 + \mu_1 - 4)a_4^2 + (2\mu_0^2 - 4\mu_0 - 2\mu_1^2 + 4\mu_1)a_4 + \mu_0^3 + (3\mu_1 - 6)\mu_0^2 + (3\mu_1^2 - 12\mu_1 + 12)\mu_0 + \mu_1^3 - 6\mu_1^2 + 12\mu_1 - 8 = 0. \end{aligned}$$

Above two equations have common roots if and only if $\mu_0, \mu_1, \mu_2, \mu_3$ satisfy the equation (5). Since $\mu_0, \mu_1, \mu_2, \mu_3$ are the four multipliers of $p(z)$ and they should satisfy the equation (5), the two equations always have common roots. Hence five coefficients of $p(z)$ are calculated by its four multipliers, however, this calculation is not decisive when they have distinct two common roots.

For the case of $\text{Poly}_n(\mathbb{C})$, it is clear from (4) that the equation corresponds to (5) cannot have the term of $\sigma_{n,n}$. Hence we can put

$$c_0 + c_1\sigma_{n,1} + c_2\sigma_{n,2} + \cdots + c_{n-1}\sigma_{n,n-1} = 0$$

where c_k ($0 \leq k \leq n-1$) are functions of n variable.

Paying attention to the form of elementary symmetric functions, we obtain the following equation;

$$c_k = (-1)^k \binom{n-1}{k} n / \binom{n}{k} = n - k.$$

where $\binom{n}{k}$ means binomial coefficient. For convenience, put $\sigma_{n,0} = 1$. we have

$$\sum_{k=0}^{n-1} (-1)^k (n-k) \sigma_{n,k} = 0. \quad (6)$$

Question Is the moduli space $M_n(\mathbb{C})$ for polynomials of degree n canonically isomorphic to $\mathbb{C}^{n-1}$ with coordinates $\sigma_1, \sigma_2, \dots, \sigma_{n-2}$, and σ_n ?

3.2. Symmetry locus

Proposition 3 *A polynomial of degree four has a non-trivial automorphism if and only if it is conjugate to a map in the unique normal form*

$$\{z^4 + az\}, \quad a \in \mathbb{C}.$$

For a map $p(z)$ in this normal form, $\text{Aut}(p)$ is a cyclic group of order three.

Outline of proof. Let $p(z) \in \text{Poly}_4(\mathbb{C})$.

1. In the case of a map $p(z)$ with multiple fixed points.
 - (a) The case of $p(z)$ with a fixed point of order four: $\text{Aut}(p)$ is non-trivial.
 - (b) The case of $p(z)$ with a fixed point of order three: $\text{Aut}(p)$ is trivial.
 - (c) The case of $p(z)$ with two fixed points of order two: there is not such $p(z)$.
 - (d) The case of $p(z)$ with a fixed point of order two: $\text{Aut}(p)$ is trivial.
2. In the case of a map $p(z)$ with four distinct fixed points.
 - (a) The case of four distinct multipliers: $\text{Aut}(p)$ is trivial.
 - (b) The case that only two of multipliers are coincide: $\text{Aut}(p)$ is trivial.
 - (c) The case of two pair of same multipliers: there is not such $p(z)$.
 - (d) The case of three same multipliers: By an affine conjugation, if three fixed points (whose multipliers are same) are mapped on the vertices of a regular triangle whose barycenter is the origin and the other fixed point on the origin, then $\text{Aut}(p)$ is non-trivial. Otherwise $\text{Aut}(p)$ is trivial.
 - (e) The case of four same multipliers: there is not such $p(z)$.

Therefore a map $p(z)$ has non-trivial automorphisms if and only if $p(z)$ is in the case 1-(a) and the first part of 2-(d). We can check easily that these maps coincide with the normal form $\{z^4 + az\}$. ■

Conjecture A polynomial of degree n has a non-trivial automorphism if and only if it is conjugate to a map in the unique normal form

$$\left\{ z^n + \sum_{k|(n-1), k \neq n-1} A(k) z^k \right\}$$

where $A(k)$ are parameters in $\mathbb{C}$.

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