

Kyoto University

**Subdimensional particles:
lattice models, field theories
and underlying mathematical structures**

Doctoral Thesis

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Abstract

Topologically ordered phases and their continuum descriptions provide a rich arena in which quantum many-body systems, algebraic topology, and quantum field theory meet. Recently, foliated fracton phases with subdimensional particles, which are excitations constrained to move only along lower-dimensional submanifolds, have emerged as a new class of quantum matter with potential applications to robust quantum information storage. These phases are naturally tied to foliation structures of the spatial manifold and exhibit unusual ground state degeneracies and subsystem conservation laws that go beyond conventional topological order.

In this thesis we investigate subdimensional particles from three complementary viewpoints: lattice stabilizer codes, mathematical structures on manifolds, and continuum gauge theories. As a warm-up, we revisit the toric code and show how its stabilizer structure, logical operators, and ground state degeneracy are encoded in cellular (co)homology. This “triangle of ideas” relating lattice models, BF-type topological quantum field theories, and topological invariants serves as a guiding paradigm.

We then extend this perspective to foliated fracton phases, focusing on the plaquette Ising model and the X-cube model as paradigmatic examples. We clarify how their subextensive ground state degeneracies and rigid line and membrane operators are organized by foliation-sensitive (co)homological data of the underlying manifold. Motivated by these observations, we construct a Godbillon-Vey $U(1)$ BF-like gauge theory coupled to a codimension-one foliation. By promoting the redundancies of the foliation data to gauge symmetries, the resulting continuum theory depends only on the foliation class and naturally realizes Wilson and higher-form operators whose mobility is restricted to the leaves. In $(2+1)$ - and $(3+1)$ - dimensions we show that this theory reproduces, respectively, the ground state degeneracy and operator algebra of the plaquette Ising and X-cube models, providing a foliated analogue of the BF-toric code correspondence.

Finally, we introduce matter field theories coupled to the foliation background and to the BF-like gauge fields. For complex scalars, fermions, and compact scalars we construct conserved currents that obey leafwise conservation laws enforcing subdimensional kinematics. We close by outlining extensions to higher-codimension foliations, more general G -structures, and symmetry topological field theories, and by emphasizing a generalized triangle relating microscopic models, geometric invariants, and continuum field theories for subdimensional particle systems.

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1 Introduction

For a long time, research on the phases of matter has been pursued from many perspectives including condensed matter physics and formal quantum field theories. Among them, a rather new concept called topological order pioneered a new path. This notion captures the deep correspondence between mathematical structures and the organization of quantum phases, and has grown into a broad interdisciplinary enterprise bridging condensed matter physics, topology, and quantum field theory [1, 2, 3]. Moreover, topological order provides a natural platform for realizing robust quantum information processing, since its inherent non-local degrees of freedom can encode quantum information in a way that is intrinsically protected from local noise [2, 4]. For this reason, the pursuit of experimental realization of topologically ordered states ranging from fractional quantum Hall systems to spin-liquid candidates and synthetic quantum simulators has become an active and influential direction of research [5, 6, 7, 8].

However, despite their remarkable robustness, topological quantum error-correcting codes such as the toric or surface code require substantial spatial and hardware overhead to realize fault tolerance, and they remain thermally fragile in realistic settings due to the proliferation of mobile anyonic excitations at finite temperature [9, 10]. These challenges have motivated the exploration of new classes of quantum phases featuring subdimensional particles, excitations constrained to move along lower-dimensional manifolds within the bulk. Such fractonic phases exhibit restricted mobility and unusual entanglement structures, providing new paradigms of quantum order and concrete realizations of subsystem and fractal symmetries that go beyond conventional topological phases [11, 12].

Subdimensional particles are deeply connected with the foliation structure of the underlying manifold, just as conventional topological orders are tied to its topological structure. In fractonic phases such as the X-cube model, excitations are constrained to move along specific lower-dimensional layers, effectively the leaves of a spatial foliation. Each foliation direction supports an emergent two-dimensional topological order, and under suitable finite-depth local unitary transformations the global structure can be decomposed into stacked two-dimensional layers together with a residual intrinsically three-dimensional fracton sector that cannot be trivialized by adding or removing such layers [13]. The ground state degeneracy of these models scales not with the topology of the full three-manifold but with the Betti numbers of the foliation leaves, reflecting the subdimensional character of the excitations and the subsystem nature of the conservation laws [13, 14, 15].

There are many field theoretic attempts to describe foliated fracton phases [13]. Some approaches focus on the conservation of dipole moments, which naturally restricts the motion of charged excitations and provides a continuum realization of fracton behavior. In this framework, fracton phases are interpreted as systems with higher-moment conservation laws, leading to the emergence of tensor gauge fields¹ that generalize conventional $U(1)$

¹These theories involve gauge fields with one ordinary time component, and rank-2 spatial tensor component $A = (A_0, A_{ij})$. In such field theories, spacetime symmetry is explicitly broken, but spatial symmetry

gauge structures [16, 17, 18, 19, 20, 21, 22, 23, 24, 25]. Such theories exhibit gauge-invariant charge and dipole conservation. Because of the dipole conservation, a charge cannot move alone, which means that the charge represents the fractonic excitation. From this approach, many new concepts have been actively explored in field theories, such as subsystem symmetries and UV/IR mixing that go beyond conventional topological quantum field theories. Subsystem symmetries (more generally, spatially modulated symmetries), which act on lower-dimensional manifolds embedded in space rather than on the whole system, capture the essence of restricted mobility and fractonic conservation laws [12, 26, 27, 28]. In these theories, local operators are often constrained to obey higher-moment conservation, and the associated tensor gauge structures share a close analogy with linearized gravity, where the symmetric rank-2 gauge field A_{ij} plays a role similar to the metric perturbation h_{ij} [29, 30, 31, 32]. This analogy provides a geometric intuition for the immobility of fractonic charges as a consequence of gauge redundancy and local constraints. Furthermore, such higher-moment constraints lead to exotic infrared behaviors that are sensitive to ultraviolet structures, a phenomenon known as UV/IR mixing [33, 34, 35]. This mixing implies that long-distance physics cannot be separated from microscopic data, a feature that sharply contrasts with ordinary renormalizable quantum field theories.

Also, there have been many approaches to extending conventional field theoretic methods to tensor gauge theories, including the incorporation of fermionic matter [25, 36, 37, 38, 39], topological terms [40, 41, 42], anomaly inflow [43, 44], and formulations on curved spacetime [45, 46, 47, 48].

There is another line of approach that focuses on the connection between field theories and the underlying foliation structures of spacetime. In this framework, one of the most extensively studied formulations is the foliated BF theory [49, 50, 51], where background one-forms specifying the foliation encode the subsystem structure and restricted mobility of fractonic excitations. This formulation allows the action to be written compactly in differential-form notation, making the theory explicit and facilitating a covariant treatment on general manifolds. In this theory, the restricted mobility of excitations is reflected in the existence of subdimensional Wilson operators, loop or surface operators that can only move within specific foliated submanifolds, representing fractonic particles with constrained motion [52, 53, 54, 55].

Furthermore, in this framework, several attempts have been made to generalize the notions like anomaly inflow [56, 57] or symmetry topological field theory (symTFT) to systems with foliated symmetries [58, 59, 60, 61]. These efforts aim to capture the nontrivial boundary-bulk correspondence and the generalized 't Hooft anomalies associated with foliated BF theories, extending the conventional ideas of topological response and anomaly cancellation to fractonic field theories.

However, the foliated BF theory does not fully capture the structure of a general foliation. In particular, it cannot incorporate arbitrary foliation geometries or account for the

is preserved.

redundancy associated with the defining normal one-forms. Moreover, due to the presence of UV/IR mixing, the theory faces intrinsic difficulties as a well-defined continuum field theory [52, 53].

To overcome these limitations, our group has introduced a new gauge theoretic formulation that embeds the redundancy of foliated geometries through the Godbillon-Vey characteristic class [62]. This approach not only removes the pathological UV/IR mixing inherent in previous foliated BF theories but also defines a consistent field theory of subdimensional particles on arbitrary foliated backgrounds. Furthermore, in our ongoing work, we are investigating non-invertible symmetries in the matter field theories of subdimensional particles, as well as the correspondence between symmetry operators in the (3+1)-dimensional theory and those in the X-cube model. These results suggest a deeper unification between continuum foliated field theories and lattice models of fracton order.

This thesis is organized as follows. Section 2 reviews known results on topological phases and topological field theories, with particular emphasis on the correspondence between topological orders, TQFTs, and their underlying mathematical structures. This viewpoint provides a unifying narrative that will be extended and generalized in the subsequent chapters. In section 3, we discuss characteristic features of subdimensional particles in lattice models. We then introduce our field-theoretic description based on background foliation structures and show how it reproduces and clarifies the properties of these lattice realizations. Section 4 develops the broader picture and outlines future directions for understanding the interplay between lattice models, field theories, and the mathematical structures that organize them.

Several appendices collect background material and technical details. Appendix A introduces basic notions of mathematical structures from categorical and geometric points of view, and makes explicit the assumptions that are often left implicit when one formulates field theories on manifolds endowed with additional structure. Appendix B contains remarks on higher-codimension foliations. Although this subject is not yet fully developed, we indicate possible generalizations of the main discussion in that direction. In Appendix C, we present the canonical quantization of our theory in detail; this computation supports the algebraic relations among gauge-invariant operators discussed in the main text.

2 Preliminary: toric code, topological BF theory, and topological structure

In this section, we review the relationships between topological order, topological quantum field theory, and the topological structure of a manifold. As shown in the following subsections, these notions represent different aspects of a common underlying framework. This relationship can be summarized as a “triangle of ideas,” as illustrated in Fig. 1. The toric code, as a representative lattice realization, can be understood through cellular (co)homology, which provides the mathematical backbone connecting it to the continuum BF-type TQFT, in which integral cohomology plays an essential role.

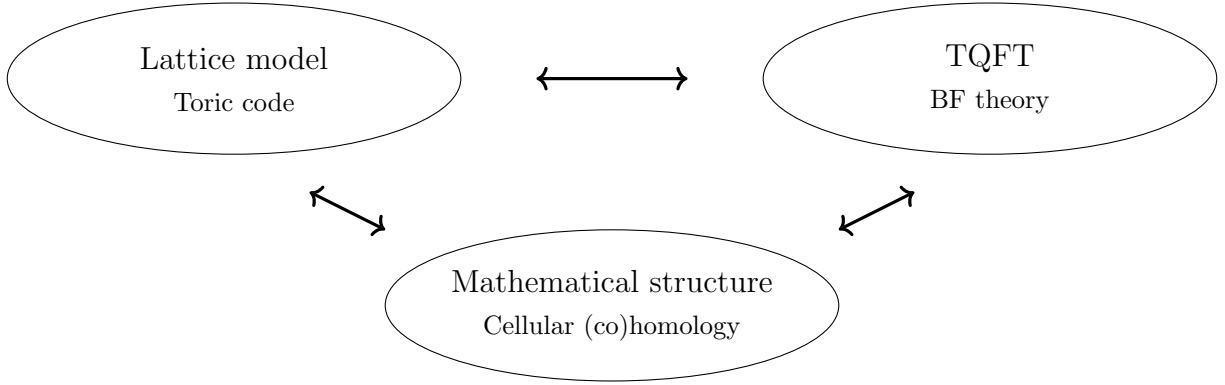


Figure 1: Triangle of ideas relating topologically ordered lattice models, BF-type TQFTs, and (co)homology, viewed as complementary descriptions of the same underlying topological structure of the manifold.

Moreover, this perspective can be extended to other mathematical structures as well. In section 3, we will apply this viewpoint to the case of foliation structures, and in appendix A we summarize several additional mathematical structures that are implicitly used in physics, many of which remain under active investigation.

2.1 Quantum error correction and stabilizer code

The toric code can be most clearly understood within the framework of the *stabilizer formalism*, which provides an algebraic description of quantum error correcting codes. Before analyzing the Hamiltonian and topological properties of the toric code in the subsequent subsections, we therefore review the basic concepts of quantum error correction and the stabilizer construction.

Quantum information encoded in many-body systems is inevitably exposed to local perturbations and decoherence. The central idea of *quantum error correction* [63] is to embed logical information into a subspace that is insensitive to a restricted set of local errors. Among the various constructions, the stabilizer formalism provides a particularly elegant and systematic framework for describing quantum error correcting codes [64, 65, 66, 67]. For comprehensive introductions to quantum error correction and the stabilizer formalism, we refer the reader to the standard textbooks and reviews [67, 68, 69, 70]. These references provide systematic expositions of the stabilizer framework, including its applications to topological codes.

Let us briefly recall the basic structure. For an n -qubit system, the Pauli group $\mathcal{P}_n$ is generated by tensor products of the single-qubit Pauli matrices² $\{I, X, Y, Z\}$ with overall phase factors $\{\pm 1, \pm i\}$. Namely, the Pauli group is generated by the Pauli matrices with

²We use the quantum information notation X, Y, Z for the Pauli matrices, instead of the conventional $\sigma_x, \sigma_y, \sigma_z$.

coefficients $\{\pm 1, \pm i\}$. The Pauli matrices have the following properties:

- Hermitian and unitary.
- Any pair of these matrices commute or anti-commute with each other.
- They become identity when squared, *i.e.* $X^2 = Y^2 = Z^2 = I$.
- Any 2×2 matrix M can be written as

$$M = \alpha X + \beta Y + \gamma Z + \delta I , \quad (2.1)$$

where α, β, γ and δ are some complex coefficients.

Thanks to the last one, any quantum operation is generated by the Pauli group. Moreover, because $Y = iX \cdot Z$, the generator of the Pauli group can be reduced to $\{I, X, Z\}$. These properties play a critical role in investigating the algebraic structure of stabilizer codes.

A stabilizer code is defined by choosing an Abelian subgroup $S \subset \mathcal{P}_n$ (excluding the element $-I^{\otimes n}$). The *code space* $\mathcal{H}_{\text{code}}$ is the common $+1$ eigenspace of all stabilizer operators $P \in S$:

$$\mathcal{H}_{\text{code}} = \{ |\psi\rangle \in \mathcal{H}_n \mid P|\psi\rangle = |\psi\rangle, \forall P \in S \} . \quad (2.2)$$

The dimension of this subspace determines the number of logical qubits. If S is generated by $n - k$ independent stabilizers, then the code encodes k logical qubits.

Logical operators are defined as elements of the normalizer $N(S)$ of S within $\mathcal{P}_n$ that are not contained in S itself:

$$N(S) = \{ P \in \mathcal{P}_n \mid PS = SP \} , \quad \text{logical operators: } N(S)/S . \quad (2.3)$$

These operators act nontrivially within the code space while preserving it. Pauli errors $E \in \mathcal{P}_n$ that anticommute with at least one stabilizer $P \in S$ map code states to orthogonal subspaces, enabling their detection and correction.

This construction can be viewed from another perspective. First, we choose a certain n -qubit Hilbert subspace, called the *code space* and spanned by the basis states $|\psi_j\rangle$, which generically represent the logical qubit state Hilbert space, the stabilizer group³ is defined as the subgroup of n -qubit Pauli operators P satisfying

$$P|\psi_j\rangle = |\psi_j\rangle , \quad (2.4)$$

for all j . This condition implies that the stabilizer group is Abelian, because all states $|\psi_j\rangle$ are simultaneous $+1$ eigenstates of the operators belonging to the stabilizer group.

³Strictly speaking, it would be more appropriate to regard this set as a kernel.

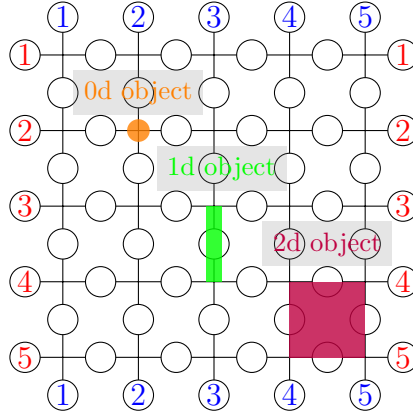


Figure 2: A two-dimensional lattice with 0-, 1-, and two-dimensional objects (vertices, edges, and plaquettes). Circles on the links indicate the locations of physical qubits. Red and blue numbers label pairs of links that are identified under periodic boundary conditions.

Let us consider the three-qubit repetition code as an illustrative example. In this model, one logical qubit is encoded into three physical qubits. The code space is spanned by two logical basis states,

$$|0\rangle_L := |000\rangle, \quad |1\rangle_L := |111\rangle. \quad (2.5)$$

The corresponding stabilizer group is generated by

$$ZZI, \quad ZIZ. \quad (2.6)$$

Each stabilizer operator measures the parity of two neighboring qubits, and the code space is the simultaneous $+1$ eigenspace of these operators. Within this framework, single-qubit bit-flip errors can be detected and corrected by identifying which stabilizer eigenvalue has changed sign.

The stabilizer formalism thus includes this simple repetition code. This algebraic formulation makes the stabilizer code a powerful and unifying language that encompasses a wide variety of quantum error correcting schemes, including concatenated codes, color codes, and topological codes. Among them, the *toric code* [2] is particularly remarkable because it realizes the stabilizer formalism on a two-dimensional lattice in a geometrically natural way. Its stabilizers correspond to local plaquette and vertex operators associated with the lattice geometry, and its logical operators are supported on noncontractible cycles of the manifold.

The next subsection 2.2 will present the Hamiltonian realization of the toric code and analyze its properties as a stabilizer code defined on a lattice.

2.2 Toric code

Before defining the toric code Hamiltonian, let us first describe the structure of the underlying square lattice with periodic boundary condition. As illustrated in Fig. 2, the lattice consists

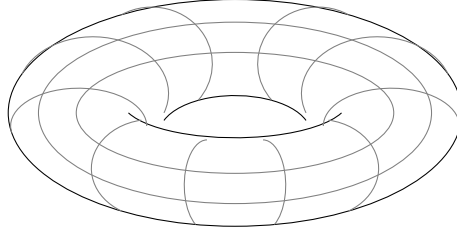


Figure 3: Square lattice with periodic boundary conditions, embedded in a torus.

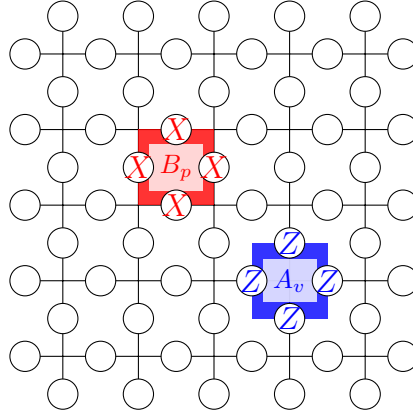


Figure 4: Local operators in the toric code. The vertex operator is given by the product of X operators on the edges adjacent to a vertex, and the plaquette operator is given by the product of Z operators on the edges around a plaquette.

of zero-dimensional objects (vertices), one-dimensional objects (edges), and two-dimensional objects (plaquettes). In the toric code, physical qubits are placed on the edges, while the vertices and plaquettes serve as the supports of stabilizer operators.

To avoid boundary effects and to reveal the topological nature of the model, we impose periodic boundary conditions (PBCs) along both spatial directions. With these identifications, the square lattice is naturally embedded on a two-dimensional torus, as depicted in Fig. 3. The nontrivial topology of the torus will play a crucial role in determining the ground state degeneracy and the structure of logical operators.

The toric code is a lattice spin model with intrinsic quantum error correcting properties, originally proposed by Kitaev [2]. It serves as one of the most fundamental realizations of topological quantum order in condensed matter and quantum information theory. This model is a topological phase of matter whose essential features such as the ground state degeneracy and the existence of nontrivial loop and string operators are determined by the topological structure of the manifold on which the lattice is defined.

We consider a square lattice with periodic boundary conditions, so that the system is defined on a torus. Physical qubits reside on the edges of the lattice. For each vertex v and

each plaquette p , we define the stabilizer operators

$$A_v := \prod_{e \in \delta v} X_e , \quad B_p := \prod_{e \in \partial p} Z_e , \quad (2.7)$$

where the product runs over the four edges incident to v for A_v , and over the four boundary edges of the plaquette p for B_p as shown in Fig. 4. These operators mutually commute and satisfy $A_v^2 = B_p^2 = I$. The Hamiltonian of the toric code is given by

$$H = - \sum_v A_v - \sum_p B_p . \quad (2.8)$$

The operators A_v and B_p are mutually commuting projectors⁴ and thus constitute a set of stabilizer generators in the sense of the stabilizer formalism reviewed in the previous subsection. Indeed, the Hamiltonian (2.8) is simply the negative sum of these stabilizers and therefore defines a stabilizer code whose code space coincides with the ground state subspace of H .

Because $A_v^2 = B_p^2 = I$ and all stabilizers commute, each term in the Hamiltonian can be diagonalized simultaneously. The ground states are those states that satisfy

$$A_v |\psi\rangle = |\psi\rangle , \quad B_p |\psi\rangle = |\psi\rangle , \quad (2.10)$$

for all vertices v and plaquettes p . Thus the ground state Hilbert space is the common +1 eigenspace of all stabilizer operators, in precise analogy with the general stabilizer code construction.

The excited states correspond to violations of these stabilizer constraints. A state for which $A_v = -1$ at a vertex v carries an *electric* excitation (often called an *e*-anyon), while a state for which $B_p = -1$ at a plaquette p carries a *magnetic* excitation (an *m*-anyon). Since a single Pauli Z operator anticommutes with the A_v 's on the two vertices adjacent to the acted edge, applying Z_e creates a pair of electric excitations on neighboring vertices. Similarly, applying X_e creates a pair of magnetic excitations on the two plaquettes adjacent to that edge. Individual *e* or *m* excitations cannot be created alone; they must appear in pairs, reflecting the underlying stabilizer constraints and the conservation of $\mathbf{Z}_2$ charge.

The ground state degeneracy of the toric code can be derived directly from the stabilizer structure. On a square lattice with N_v vertices, N_p plaquettes, and N_e edges, there are $N_v + N_p$ stabilizer generators. However, these generators are not all independent: the product of all vertex operators satisfies

$$\prod_v A_v = I , \quad (2.11)$$

⁴Rigorously, to interpret this Hamiltonian as a sum of projectors, it is easier to rewrite it as

$$H = - \sum_v \frac{1 + A_v}{2} - \sum_p \frac{1 + B_p}{2} + (\text{const}) , \quad (2.9)$$

where operators inside the sum satisfy the idempotent condition.

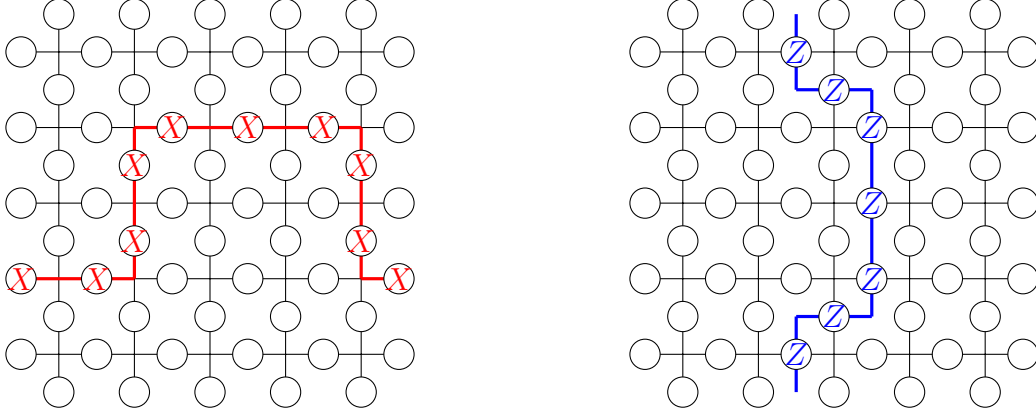


Figure 5: Logical operators in the toric code, given by products of X and Z along noncontractible loops on the torus. These loop operators implement the logical $\bar{X}$ and $\bar{Z}$ operations and generate the algebra of logical qubits.

and similarly, the product of all plaquette operators satisfies

$$\prod_p B_p = I . \quad (2.12)$$

Thus there are exactly two independent relations among the stabilizers. Since the Hilbert space has dimension 2^{N_e} and the number of independent stabilizer constraints is $(N_v + N_p - 2)$, the dimension of the ground state subspace is

$$2^{N_e - (N_v + N_p - 2)} = 2^2 = 4 . \quad (2.13)$$

On a square lattice embedded in a torus, the numbers of vertices, edges, and plaquettes satisfy

$$N_v - N_e + N_p = 0 , \quad (2.14)$$

which is precisely Euler's formula for a genus-one closed surface, $\chi = N_v - N_e + N_p = 2 - 2g$ with $g = 1$. Thus $N_e = N_v + N_p$, and the exponent reduces to

$$N_e - (N_v + N_p - 2) = 2 , \quad (2.15)$$

yielding the fourfold degeneracy above.

This fourfold degeneracy is topologically protected: it depends only on the genus of the underlying manifold, not on the microscopic details of the lattice. More generally, for a closed surface of genus g , Euler's formula

$$\chi = N_v - N_e + N_p = 2 - 2g \quad (2.16)$$

implies

$$N_e - (N_v + N_p - 2) = 2g , \quad (2.17)$$

so the toric code exhibits a ground state degeneracy

$$2^{N_e - (N_v + N_p - 2)} = 2^{2g} = 4^g , \quad (2.18)$$

which reflects the number of independent noncontractible cycles on the manifold.

The logical operators introduced earlier correspond precisely to noncontractible loop operators wrapping these cycles, as shown in Fig. 5. A product of Z operators along a noncontractible cycle defines one logical operator, while a product of X operators along the dual cycle defines another. Restricted to the four-dimensional ground state manifold, each intersecting pair of such Z - and X -type loops acts exactly as a logical Pauli pair $\bar{Z}$ and $\bar{X}$ on an encoded qubit (the same identification also works for the second logical qubit on the torus). These loop operators commute with all stabilizers but cannot be expressed as stabilizer products, thereby acting nontrivially within the four-dimensional ground state manifold. Their algebra encodes the essential topological structure of the toric code.

In the next subsection, we develop a more mathematical viewpoint on these properties. We will reinterpret the toric code in terms of cellular (co)homology of the underlying lattice, where the stabilizers, logical operators, and ground state degeneracy find natural expressions in terms of cycles, boundaries, and cohomology groups. This homological formulation provides a unifying and geometrically transparent understanding of the topological nature of the model.

2.3 Topological nature of the toric code

As shown in the previous subsection, the ground state degeneracy of the toric code is determined by the number of independent noncontractible loops on the underlying manifold. In this subsection, we clarify the mathematical origin of this topological nature by describing the toric code in terms of $\mathbf{Z}_2$ cellular cohomology [9, 71]. This formulation reveals that stabilizers, logical operators, and ground state degeneracy arise naturally from the homological structure of the lattice. The presentation here follows the treatment in [70], with some extensions and additional explanations for completeness.

Generally, (co)homology is built from a collection of (co)chains together with (co)boundary operators acting on them. To make these notions concrete in the present context, we explicitly construct the (co)chains and the corresponding (co)boundary operators on a lattice.

Let us first clarify what we mean by k -dimensional objects in a d -dimensional lattice. For example, in a two-dimensional square lattice such as the one shown in Fig. 2, the fundamental components are zero-cells (vertices), one-cells (edges), and two-cells (plaquettes). More generally, in a d -dimensional lattice, the basic building blocks are k -cells for all $0 \leq k \leq d$, each representing k -dimensional geometric objects from which the space is assembled.

With these k -cells in hand, we can define k -chains of the $\mathbf{Z}_2$ coefficient homology by assigning either 0 or 1 to each k -cell. Such an assignment itself is called an k -chain. Equivalently, an k -chain may be regarded as a function

$$c : \{k\text{-cells}\} \longrightarrow \{0, 1\} , \quad (2.19)$$

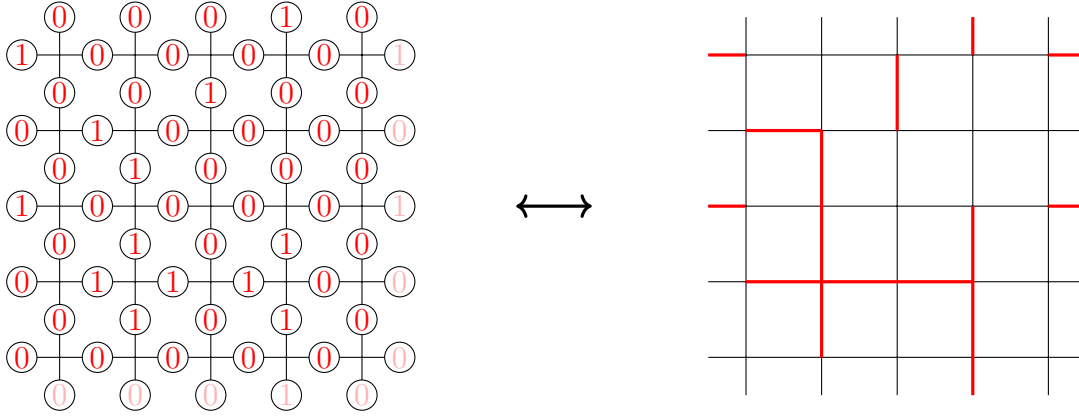


Figure 6: A 0-chain assigns a value 0 or 1 to each edge (one-dimensional cell) of the lattice.

where c specifies whether a given k -cell is included (1) or not (0).

In the case of the toric code, we are primarily interested in 1-chains on a two-dimensional lattice, since qubits reside on edges. Therefore, in what follows we focus on 1-chains in a two-dimensional lattice. Interpreting the value 0 or 1 on each edge as an indicator for the presence of that edge in the chain, a 1-chain can be identified with a subset of the set of all edges. An example of a 1-chain is illustrated in Fig. 6.

Having introduced chains, we are now ready to define homology; for this purpose we next introduce the boundary operators $\partial_k : C_k \rightarrow C_{k-1}$, which map k -chains to $(n-1)$ -chains. Let

$$\partial_2 : C_2 \longrightarrow C_1 \quad (2.20)$$

be the boundary map that sends each plaquette (a two-cell) to the sum of its boundary edges, and let

$$\partial_1 : C_1 \longrightarrow C_0 \quad (2.21)$$

be the boundary map that sends each edge (a one-cell) to its two endpoint vertices. Throughout, we work with $\mathbf{Z}_2$ coefficients, so boundaries are computed modulo 2.

Given the chain groups C_k and boundary operators $\partial_k : C_k \rightarrow C_{k-1}$, we define the group of k -cycles and k -boundaries by

$$Z_k := \ker \partial_k, \quad B_k := \text{im} \partial_{k+1}. \quad (2.22)$$

The fundamental relation $\partial_k \circ \partial_{k+1} = 0$ (*boundary has no boundary*) implies $B_k \subset Z_k$, and the k -th homology group is defined as the quotient

$$H_k := Z_k / B_k. \quad (2.23)$$

Having introduced chains and boundary operators, we now recall the dual notion of cochains and cohomology. For each k , the group of k -cochains is defined by

$$C^k := \text{Hom}(C_k, \mathbf{Z}_2). \quad (2.24)$$

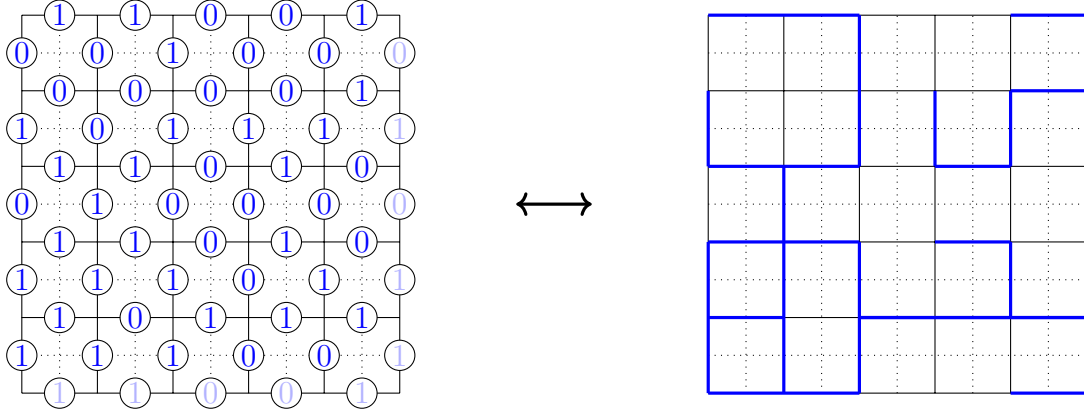


Figure 7: A 1-cochain assigns a value 0 or 1 to each edge (one-dimensional cell) of the dual lattice. The dotted graph represents the primary (original) lattice.

An element $\varphi \in C^k$ assigns to every k -cell a value in $\mathbf{Z}_2$, and there is a natural evaluation pairing

$$\langle \varphi, c \rangle := \varphi(c) \in \mathbf{Z}_2 \quad (\varphi \in C^k, c \in C_k) . \quad (2.25)$$

The coboundary operator

$$\delta_k : C^k \longrightarrow C^{k+1} \quad (2.26)$$

is defined as the dual of the boundary operator. Equivalently, it is characterized by the relation

$$\langle \delta_k \varphi^k, c_{k+1} \rangle = \langle \varphi^k, \partial_{k+1} c_{k+1} \rangle \quad (\varphi^k \in C^k, c_{k+1} \in C_{k+1}) . \quad (2.27)$$

One readily checks that $\delta_{k+1} \circ \delta_k = 0$, and the k -th cohomology group is defined by

$$H^k := \ker \delta_k / \text{im } \delta_{k-1} . \quad (2.28)$$

We can reinterpret this cohomological description in terms of homology on the dual lattice. The dual lattice is obtained by associating to each k -dimensional cell of the original d -dimensional lattice a $(d-k)$ -dimensional cell. In a two-dimensional square lattice with periodic boundary conditions, the dual lattice is the half-shifted lattice shown in Fig. 7.

With $\mathbf{Z}_2$ coefficients, k -cochains, coboundaries, and k -th cohomology groups on the original lattice are naturally identified with $(d-k)$ -chains, boundaries, and $(d-k)$ -th homology groups of the dual lattice, respectively (see Fig. 7). Under this identification, the evaluation pairing (2.25) is realized as the mod-2 intersection number between primary and dual cells, as illustrated in Fig. 8.

We are now ready to describe the toric code in the language of (co)homology. We fix once and for all a reference ground state $|\Omega\rangle$, which is a simultaneous $+1$ eigenstate of all vertex and plaquette stabilizers A_v and B_p .

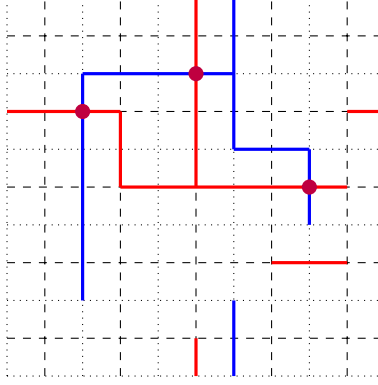


Figure 8: The evaluation pairing can be calculated by counting intersection points of chains and cochains.

In the operator-support viewpoint, a $\mathbf{Z}_2$ -valued 1-chain $c \in C_1$ specifies the support of an X -type operator, while a 1-cochain $\varphi \in C^1$ specifies the support of a Z -type operator. The evaluation pairing $\langle \varphi, c \rangle \in \mathbf{Z}_2$ then measures, modulo 2, the number of intersections between the supports of these X - and Z -strings.

In particular, this pairing encodes the phase arising from the anticommutation of X and Z . If we denote by $X(c)$ the product of X operators along the 1-chain c and by $Z(\varphi)$ the product of Z operators along the 1-cochain φ , their commutation relation is

$$X(c) Z(\varphi) = (-1)^{\langle \varphi, c \rangle} Z(\varphi) X(c) . \quad (2.29)$$

A general Pauli operator can be factorized into its X and Z parts as

$$\mathcal{O} = \mathcal{O}_X(c) \mathcal{O}_Z(\varphi) , \quad (2.30)$$

where $\mathcal{O}_X(c)$ (resp. $\mathcal{O}_Z(\varphi)$) is supported on the 1-chain c (resp. 1-cochain φ). Let $v^0 \in C^0$ denote a 0-cochain associated with a vertex and $p_2 \in C_2$ a 2-chain associated with a plaquette. If the supports satisfy

$$\langle \delta v^0, c \rangle = 0 , \quad \langle \varphi, \partial p_2 \rangle = 0 \quad \forall v^0 \in C^0, \forall p_2 \in C_2 , \quad (2.31)$$

then $\mathcal{O}$ maps ground states to ground states. Indeed, these conditions are equivalent to

$$\mathcal{O} A_v = A_v \mathcal{O} , \quad \mathcal{O} B_p = B_p \mathcal{O} \quad \forall v, \forall p , \quad (2.32)$$

so $\mathcal{O}$ commutes with all stabilizers and hence preserves the ground state subspace.

Using the defining relation between boundary and coboundary,

$$\langle \delta v^0, c \rangle = \langle v^0, \partial c \rangle , \quad \langle \varphi, \partial p_2 \rangle = \langle \delta \varphi, p_2 \rangle , \quad (2.33)$$

the above conditions can be rewritten as

$$\langle v^0, \partial c \rangle = 0 , \quad \langle \delta \varphi, p_2 \rangle = 0 \quad \forall v^0 \in C^0, \forall p_2 \in C_2 , \quad (2.34)$$

which implies $\partial c = 0$ and $\delta\varphi = 0$. In other words, $c \in Z_1$ and $\varphi \in Z^1$: the X -support is a 1-cycle and the Z -support is a 1-cocycle.

However, operators built purely from stabilizers act trivially on the ground states. If the supports satisfy

$$c = \partial q, \quad \varphi = \delta\chi, \quad (2.35)$$

for some 2-chain q and 0-cochain χ , then the corresponding operator $\mathcal{O}_X(c)\mathcal{O}_Z(\varphi)$ can be written as a product of vertex and plaquette stabilizers.

Nontrivial logical operators are therefore classified by the homology and cohomology classes of their supports. On the torus T^2 , these are the groups $H_1(T^2; \mathbf{Z}_2)$ and $H^1(T^2; \mathbf{Z}_2)$, and the corresponding nontrivial classes give rise to logical $\bar{X}$ - and $\bar{Z}$ -type operators, respectively.

The resulting ground state degeneracy is obtained by counting the number of independent logical qubits, or equivalently the number of independent pairs of logical $\bar{X}$ and $\bar{Z}$ operators. For the torus, one finds

$$\text{GSD} = 2^{\dim H_1(T^2; \mathbf{Z}_2)} = 2^{\beta_1}, \quad (2.36)$$

where β_1 is the first Betti number, defined as the dimension of the first homology group.

More generally, let M be a closed d -dimensional manifold and work with coefficients in $\mathbf{Z}_2$. Poincaré duality provides canonical isomorphisms

$$H_k(M; \mathbf{Z}_2) \cong H^{d-k}(M; \mathbf{Z}_2), \quad (2.37)$$

relating homology in degree k to cohomology in degree $d - k$. On the other hand, since $\mathbf{Z}_2$ is a field, the universal coefficient theorem implies that

$$\dim H^k(M; \mathbf{Z}_2) = \dim H_k(M; \mathbf{Z}_2) \quad (2.38)$$

for all k . Combining these two facts, the Betti numbers satisfy

$$\dim H_k(M; \mathbf{Z}_2) = \dim H^k(M; \mathbf{Z}_2) = \dim H_{d-k}(M; \mathbf{Z}_2). \quad (2.39)$$

In particular, for a closed surface ($d = 2$) one has

$$\beta_1 := \dim H_1(M; \mathbf{Z}_2) = \dim H^1(M; \mathbf{Z}_2) =: \beta^1, \quad (2.40)$$

so the ground state degeneracy can equally well be expressed in terms of the dimension of the first cohomology group.

For a closed two-dimensional manifold M , the Betti numbers satisfy the Euler-Poincaré relation

$$\chi(M) = \beta_0(M) - \beta_1(M) + \beta_2(M), \quad (2.41)$$

where $\chi(M)$ is the Euler characteristic. In a cellular decomposition adapted to the lattice, this can also be written as

$$\chi(M) = N_v - N_e + N_p, \quad (2.42)$$

with N_v , N_e , and N_p the numbers of vertices, edges, and plaquettes, respectively. For a connected closed orientable surface of genus g one has $\beta_0(M) = \beta_2(M) = 1$ and $\chi(M) = 2 - 2g$, so

$$\beta_1(M) = 2g. \quad (2.43)$$

Hence, for the toric code defined on such a surface, the ground state degeneracy can be written in terms of the genus as

$$\text{GSD} = 2^{\beta_1(M)} = 2^{2g}. \quad (2.44)$$

As a related remark, the same chain-cochain framework also encompasses simpler models such as the ferromagnetic quantum Ising model with periodic boundary condition. In this case, the relevant objects are 0-chains, and the Hamiltonian may be written as

$$H = -J \sum_e X_e, \quad (2.45)$$

where the sum runs over all generators of 0-boundaries. Here X_e denotes the two-qubit Pauli X operator acting on the spin located at two ends of each edge e (nearest neighbor interaction); the set of 0-coboundaries is empty since there is no negative-dimensional cell. From this viewpoint, the model is sensitive only to the zero-th Betti number β_0 , which counts the connected components of the lattice.

Although the Ising model does not realize a long-range entangled phase in the sense usually associated with topological order, it nevertheless admits a description within the same homological language. The distinction lies in the level of topology that the model probes: the Ising model is governed solely by β_0 , whereas the toric code detects higher Betti numbers such as $\beta_1(T^2) = 2$, that encode nontrivial global topology and give rise to a robust degeneracy structure.

The (co)homological viewpoint is not restricted to the square lattice with periodic boundary conditions in two-dimensions or periodic spin chain in 1 dimension. In general, given a d -dimensional cell complex Σ and a choice of cell dimension k on which qubits live, one can define a CSS-type Hamiltonian with X -stabilizers supported on k -boundaries and Z -stabilizers supported on k -coboundaries in the dual complex. The resulting ground state degeneracy is then controlled by the k -th Betti number,

$$\text{GSD} = 2^{\dim H_k(\Sigma; \mathbb{Z}_2)} = 2^{\beta_k}, \quad (2.46)$$

and is therefore a purely topological invariant of Σ .

From this perspective, models whose degeneracy depends on Betti numbers β_k with $0 < k < d$ realize genuinely topologically ordered phases, as in the toric code where $\text{GSD} = 2^{2g}$ is governed by β_1 of a genus- g surface. By contrast, systems whose degeneracy is controlled only by β_0 or β_d (such as the Ising ferromagnet, where β_0 counts connected components) are naturally viewed as conventional symmetry-breaking or trivial phases rather than exhibiting intrinsic topological order.

As a final remark, although this subsection has focused on the topological and (co)homological structure underlying the toric code, a complete understanding of its practical role as a quantum error correcting code requires additional tools: syndrome extraction, decoding procedures, fault-tolerant gate constructions, and threshold analyses. For readers interested in the operational aspects of the toric code as a quantum error correcting code, we refer to the following reviews and textbooks [67, 68, 69, 72].

2.4 Topological BF theory

Although a fully rigorous correspondence is still lacking, it is widely believed that the structure of the ground-state Hilbert space of topologically ordered phases is captured by topological quantum field theories (TQFTs). Indeed, in a number of exactly solvable lattice models, an explicit mapping between ground states and continuum TQFTs has been constructed [73, 74, 75]. In particular, the ground states of the toric code model are described by the topological BF theory [76]. Let us consider the following action on a three dimensional manifold M_3 (two-dimensional space with time direction):

$$S = \frac{k}{2\pi} \int_{M_3} b \wedge dc , \quad (2.47)$$

where b and c are $U(1)$ one-form gauge fields and k is a nonzero integer level.

The gauge redundancies of the BF theory are given by

$$\begin{aligned} b &\rightarrow b + d\alpha , \\ c &\rightarrow c + d\tilde{\alpha} , \end{aligned} \quad (2.48)$$

where α and $\tilde{\alpha}$ are $U(1)$ -valued functions. To ensure gauge invariance of the action, it is convenient to introduce a manifold N such that $\partial N = M$ and rewrite the action as

$$S = \frac{k}{2\pi} \int_M b \wedge dc = \frac{k}{2\pi} \int_N db \wedge dc . \quad (2.49)$$

Since $db \wedge dc$ is manifestly invariant under the gauge transformations, the BF action is gauge invariant as well [77].⁵

⁵In $2 + 1$ dimensions this construction always works, because every closed three-dimensional manifold is null-cobordant. In higher dimensions or for more general manifolds, one must instead use the framework developed by [78, 79] to define the action in a fully gauge invariant manner.

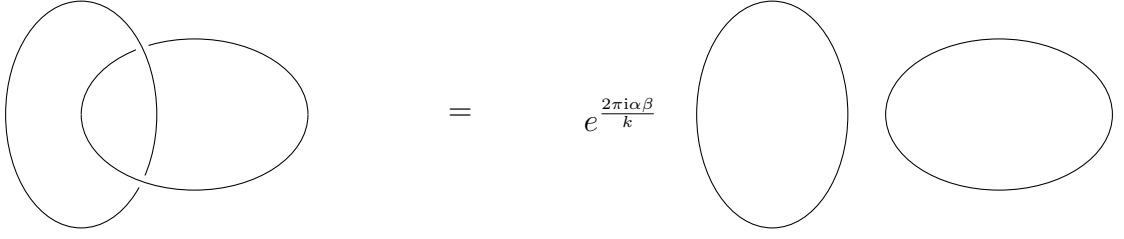


Figure 9: Linking of Wilson loops in BF theory. The b - and c -type Wilson lines $W_b(C)$ and $W_c(C')$ are linked once, and their nontrivial linking number $\text{Link}(C, C')$ produces the mutual braiding phase $\exp(2\pi i \text{Link}(C, C')/k)$ characteristic of the $\mathbb{Z}_k$ BF topological order.

The equations of motion are

$$db = 0, \quad dc = 0, \quad (2.50)$$

which can be interpreted as the conservation laws of $U(1)$ one-form symmetries [80]. These constraints imply that the Wilson loop operators constructed from b and c ,

$$W_b(C) = \exp\left(i\alpha \int_C b\right), \quad W_c(C') = \exp\left(i\beta \int_{C'} c\right), \quad (2.51)$$

are topological line operators: their expectation values depend only on the homotopy classes of the closed curves C and C' in the spacetime manifold M , and are therefore invariant under smooth deformations of the paths within M .

A characteristic feature of the BF theory is that these two types of Wilson loops exhibit a nontrivial braiding phase (Fig. 9). If we insert both $W_b(C)$ and $W_c(C')$ in the path integral and evaluate the action on configurations where the supports of these loops have a nontrivial linking number $\text{Link}(C, C')$, the correlation functions obey

$$W_b(C) W_c(C') = \exp\left(\frac{2\pi i}{k} \text{Link}(C, C')\right) W_c(C') W_b(C), \quad (2.52)$$

to be understood inside expectation values. This relation realizes, in the continuum description, the mutual statistics of the e - and m -type excitations of the toric code.

To make contact with the operator formulation familiar from lattice models, it is useful to reinterpret these topological line operators in the Hamiltonian picture. In canonical quantization on a spatial surface Σ , the BF action implies the fundamental commutation relation

$$[b_i(x), c_j(y)] = \frac{2\pi}{k} \varepsilon_{ij} \delta^{(2)}(x - y), \quad (2.53)$$

where i, j are spatial indices [81]. Upon exponentiation, this yields the exchange algebra for Wilson loops: if C and C' are closed curves on Σ with intersection number $\text{Int}(C, C')$, then

$$W_b(C) W_c(C') = \exp\left(\frac{2\pi i}{k} \text{Int}(C, C')\right) W_c(C') W_b(C). \quad (2.54)$$

Thus the continuum operator algebra reproduces the same intersection-number-controlled commutation relations as in lattice models: for instance, when $k = 2$ the factor above becomes $(-1)^{\text{Int}(C, C')}$, giving precisely the mutual anticommutation between e - and m -string operators of the toric code.

A characteristic feature of topological field theories, such as the BF theory in 2+1 dimensions, is that the Hamiltonian vanishes identically,

$$H = 0 \ , \tag{2.55}$$

so the theory has no local propagating degrees of freedom. The Hilbert space is instead determined purely by the pattern of spontaneous breaking of the two $U(1)$ one-form symmetries down to their discrete subgroups. Consequently, physical states are labelled by the gauge-invariant Wilson loops modulo their fusion relations, and the dimension of the Hilbert space on a spatial surface Σ is fixed entirely by the topology of Σ .

For a spatial torus T^2 , the Hilbert space obtained by canonical quantization has dimension k^2 , generated for example by the b -type and c -type Wilson loops wrapping the two noncontractible cycles. In particular, for $k = 2$ the Hilbert space is four-dimensional. Although every state in this Hilbert space is a ground state because the Hamiltonian vanishes, it is convenient to write the ground state degeneracy in the suggestive form

$$\text{GSD} = 2^{\beta_1(T^2)} \ , \tag{2.56}$$

where $\beta_1(T^2) = 2$ is the first Betti number of the torus, exactly as in the toric code. The resulting degeneracy agrees with the dimension of the ground state Hilbert space of the toric code. More generally, for an arbitrary closed spatial manifold Σ , the level- k BF theory reproduces the ground state Hilbert space and the algebra of topological line operators of the $\mathbf{Z}_k$ homological code associated with the cellular chain complex of Σ . In particular, the $k = 2$ BF theory realizes the operator algebra and topological degeneracy characteristic of $\mathbf{Z}_2$ topological order.

To summarize, the toric code provides an explicit realization of how topological order, cellular (co)homology, and topological BF theory represent complementary descriptions of the same underlying structure. The ground state Hilbert space, topological degeneracy, and algebra of nonlocal operators can all be understood either in terms of homology classes of the lattice or as topological line operators in the continuum BF theory. This correspondence exemplifies the “triangle of ideas” introduced at the beginning of this chapter.

In section 3, we will transplant this triangle of ideas to foliated geometries, leading to subsystem symmetries and subdimensional mobility that characterize foliated fracton phases.

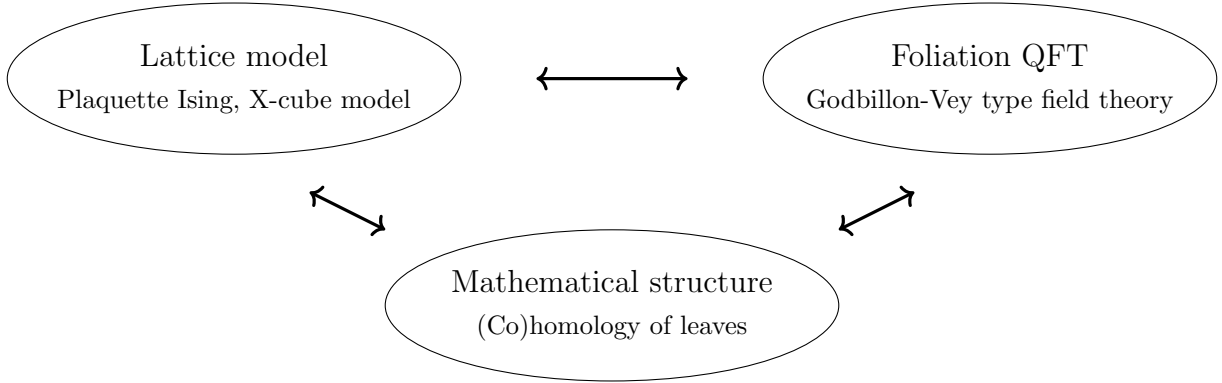


Figure 10: Triangle of ideas for foliated phases: the three corners represent foliation-based lattice models with subdimensional excitations, a tentative “foliation (co)homology” (a cellular (co)homology adapted to the foliation structure whose precise mathematical formulation remains to be developed), and BF-type (or Godbillon-Vey BF-type) topological field theories coupled to background foliation data.

3 Lattice theory and continuum field theory of subdimensional particles

In this section, we extend the discussion of section 2 to the case of foliation structures. We introduce lattice models that explicitly couple to an underlying foliation (foliated fracton orders) and examine how their ground state degeneracies are determined by the mathematical invariants associated with the foliation. We then formulate a continuum quantum field theory coupled to the foliation structure, following the framework developed in our work [62].

This section provides another realization of the “triangle of ideas” (Fig. 10) introduced at the beginning of section 2, connecting lattice models, mathematical invariants of foliations, and continuum field theories. In contrast to the purely topological case, however, the foliation setting remains far less developed, particularly regarding the mathematical understanding of lattice models coupled to foliation structures. Much of the necessary lattice-level mathematical formulation is still incomplete and requires further theoretical development.

3.1 Foliation structure

As we will discuss in later sections, subdimensional particles are deeply connected with the foliation structure of the underlying manifold [13], in much the same way that topological orders are tied to the topological structure of space. To prepare for the lattice and field-theoretic constructions developed later in this section, this subsection provides a basic introduction to foliation structures and their associated characteristic classes. In particular, we review the definition of a foliation, the geometry encoded by an integrable hypersurface distribution, and the Godbillon-Vey characteristic class. For comprehensive textbook, we

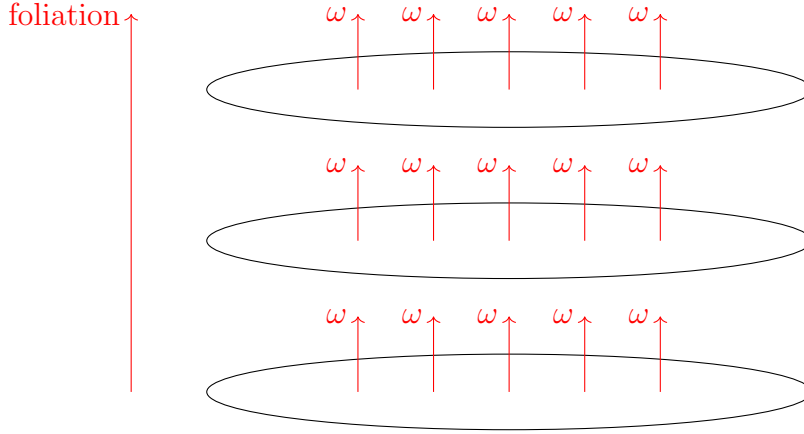


Figure 11: Schematic illustration of a foliated manifold. The manifold is decomposed into infinitely many leaves, shown as wavy lines, which are locally specified by the normal 1-form field ω .

refer to [82].

A foliation is one of the mathematical structures that can be defined on a manifold, on the same footing as topological, differentiable, or Riemannian structures. For a unified viewpoint on such structures, see appendix A. Intuitively, a foliation decomposes a manifold into a collection of lower-dimensional submanifolds, called leaves. Stacking these leaves recovers the full higher-dimensional manifold, much like strata in a geological formation or the layered sheets of a millefeuille.

We can regard this stacking of lower-dimensional manifolds as an intrinsic geometric structure of the ambient manifold. Such a structure is called a foliation. In the case of a codimension-one foliation, the structure can be described locally by a nowhere-vanishing one-form ω , which specifies the direction normal to the leaves, that is, the direction transverse to the stacked lower-dimensional manifolds (see Fig. 11).

The normal nowhere-vanishing one-form ω must satisfy the Frobenius integrability condition

$$\omega \wedge d\omega = 0 , \quad (3.1)$$

so that its kernel defines an integrable distribution and hence a codimension-one foliation. Moreover, two normal one-forms ω and ω' determine the same foliation structure whenever they are related by a nowhere-vanishing function, $\omega' = \rho\omega$ with smooth $\rho \neq 0$ everywhere (on a co-oriented foliation one may further restrict to $\rho > 0$). This means that only the direction of the normal form is physically and geometrically relevant, while its magnitude carries no information. Consequently, any quantity or field-theoretic construction associated with the foliation structure must depend solely on the direction of ω and remain insensitive to its overall normalization.⁶

⁶In general, a codimension- q foliation on M may be encoded by a rank- q subbundle $N^*\mathcal{F} \subset T^*M$ (the

We now introduce the Godbillon-Vey invariant, a characteristic class associated with codimension-one foliations of a manifold [83]. Comments on possible extensions to higher codimension are collected in appendix B. The construction of the invariant involves auxiliary data with nontrivial redundancies, which reflect the underlying foliation structure itself. We begin with a globally defined normal one-form ω whose kernel describes a codimension-one integrable distribution; equivalently, ω obeys the Frobenius integrability condition (3.1), so that it defines a foliation of the manifold. This condition implies the existence of a 1-form η satisfying

$$d\omega = \omega \wedge \eta . \quad (3.2)$$

There is a certain freedom in the choice of the 1-form η satisfying (3.2). First, if η is a solution, then so is $\eta + u\omega$ for any smooth function u . Namely, we have a redundancy of

$$\eta \longrightarrow \eta + u\omega . \quad (3.3)$$

This implies that the component of η in the ω -direction is irrelevant. Second, when the normal one-form is rescaled by a positive function, $\omega' = \rho\omega$, the corresponding η must transform accordingly. Indeed, one finds

$$d(\rho\omega) = d\rho \wedge \omega + \rho d\omega = d\rho \wedge \omega + \rho\omega \wedge \eta = \omega \wedge (\eta - d(\log \rho)) , \quad (3.4)$$

and therefore the appropriate transformation is

$$\eta \longrightarrow \eta - d(\log \rho) . \quad (3.5)$$

Using the 1-form η , we can construct a characteristic class of a codimension-one foliation, known as the Godbillon-Vey class, defined by

$$[\eta \wedge d\eta] , \quad (3.6)$$

where the square brackets denote the de Rham cohomology class represented by the enclosed differential form. This class contains the redundancies associated with the choice of η

conormal bundle to the foliation $\mathcal{F}$), described locally by q linearly independent one-forms $\{\omega_a^i\}_{i=1}^q$ on patches U_a with transition functions $\omega_a^i = (g_{ab})^i_j \omega_b^j$, where $g_{ab} : U_a \cap U_b \rightarrow \text{GL}(q, \mathbf{R})$ satisfies the cocycle condition on triple overlaps. This cocycle condition is precisely the global consistency requirement ensuring that the local frames glue to a globally defined subbundle $N^*\mathcal{F}$. Integrability is imposed by the Frobenius condition $d\omega_a^i \in (\omega_a^1, \dots, \omega_a^q)$ (equivalently, $d\omega_a^i = \sum_j \eta_a^{ij} \wedge \omega_a^j$), which is a *local* differential condition (and hence can be checked patchwise once $N^*\mathcal{F}$ is globally defined). If the foliation is *co-oriented* (i.e. $N^*\mathcal{F}$ is oriented), the structure group reduces to $\text{GL}^+(q, \mathbf{R})$, so the transition functions may be chosen with $\det g_{ab} > 0$. In the codimension-one case ($q = 1$), this means $\omega_a = \rho_{ab}\omega_b$ with $\rho_{ab} > 0$, so the structure group is $\mathbf{R}_{>0}$, which is contractible; hence $N^*\mathcal{F}$ is trivial and one may choose a globally defined nowhere-vanishing one-form ω with $T\mathcal{F} = \ker \omega$. Accordingly, in this setting the global Frobenius condition $\omega \wedge d\omega = 0$ already captures integrability. In the main text, we use this fact and treat ω as a globally defined representative (understood up to $\omega \rightarrow \rho\omega$ with $\rho > 0$).

which we referred to above. Indeed, under the transformation $\eta \rightarrow \eta + u\omega$ for any smooth function u , one has

$$\begin{aligned} (\eta + u\omega) \wedge d(\eta + u\omega) &= \eta \wedge d\eta + u\omega \wedge d\eta + \eta \wedge d(u\omega) \\ &= \eta \wedge d\eta + 2u\omega \wedge d\eta - d(\eta \wedge u\omega) , \end{aligned} \quad (3.7)$$

up to terms proportional to $\omega \wedge \omega$, or $\omega \wedge d\omega$, which are vanished because of standard differential form calculation or integrability condition (3.1). The term proportional to $\omega \wedge d\eta$ also vanishes. Indeed, the Frobenius condition (3.1) implies

$$0 = d(d\omega) = d(\omega \wedge \eta) = d\omega \wedge \eta - \omega \wedge d\eta = \omega \wedge \eta \wedge \eta - \omega \wedge d\eta \quad \Rightarrow \quad \omega \wedge d\eta = 0 . \quad (3.8)$$

Hence all the extra contributions generated by the shift $\eta \rightarrow \eta + u\omega$ vanish in cohomology, and the class $[\eta \wedge d\eta]$ is well defined.

When the manifold M is a closed oriented three-dimensional manifold, the Godbillon-Vey class of a codimension-one foliation $\mathcal{F}$, which we denote by $\langle \omega \rangle$, defines an element $\text{GV}(\langle \omega \rangle) \in H^3(M; \mathbf{R})$. By pairing this class with the fundamental class $[M] \in H_3(M; \mathbf{Z})$, namely by integrating over the entire manifold, one obtains a numerical invariant of the foliated manifold,

$$\text{GV}(M, \langle \omega \rangle) := \langle \text{GV}(\langle \omega \rangle), [M] \rangle = \int_M \eta \wedge d\eta . \quad (3.9)$$

To summarize, a codimension-one foliation on a manifold is specified by a nowhere-vanishing normal one-form ω obeying the Frobenius integrability condition, together with the associated one-form η defined by $d\omega = \omega \wedge \eta$. The Godbillon-Vey class $[\eta \wedge d\eta]$ and its integrated invariant provide a powerful way to characterize global features of the foliation, while simultaneously exhibiting nontrivial redundancies under the transformations of (ω, η) . These redundancies will play a central role in the field-theoretic framework we develop in later sections. We again refer the reader to appendix A, where this viewpoint is phrased in terms of G -structures, and to appendix B, which contains additional remarks on foliation structures of higher codimensions.

3.2 Lattice models

In this section, we discuss lattice spin models that host subdimensional particles in two or three spatial dimensions. As we will see, the structure of such lattice theories is intimately connected to the foliation structure of the underlying spatial manifold.

In this subsection, we focus primarily on examples whose subdimensional excitations can be directly related to an underlying foliation structure. It should be emphasized, however, that the landscape of lattice models hosting subdimensional particles is much broader. Significant progress has been made on systems whose mobility constraints arise not from foliations, but from more intricate structures such as fractal structure [11, 84] or other generalized structures [85, 86]. These models exhibit distinct types of restricted mobility that cannot be captured by foliation data alone.

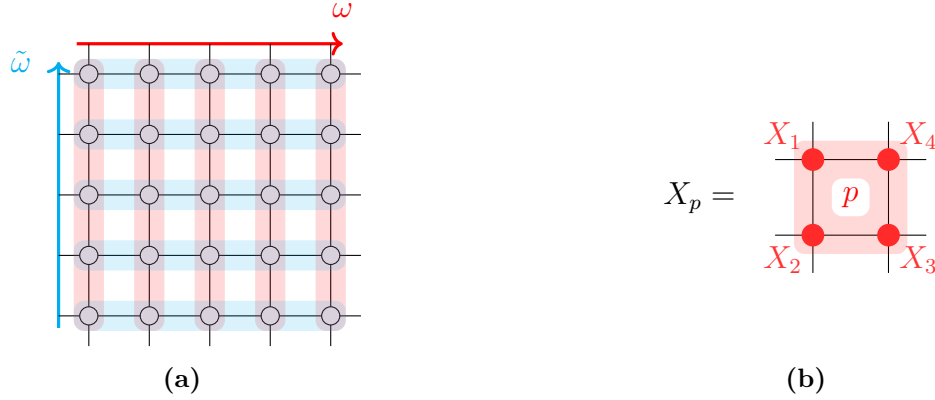


Figure 12: (a) Lattice spins with two independent foliation structures in the model (3.10). The physical degrees of freedom are qubits, indicated by circles (o). There are two independent normal 1-forms, ω and $\tilde{\omega}$, pointing in the horizontal and vertical directions, respectively. The red and blue rectangles represent leaves associated with the normal 1-forms ω and $\tilde{\omega}$. (b) The operator X_p in the (2+1)-dimensional plaquette Ising model (3.10), written as a product $X_p = X_1 X_2 X_3 X_4$, where each X_i is a Pauli X operator acting on one of the four spins at the corners of the plaquette p .

3.2.1 Plaquette Ising model

In this subsection, we consider the (2+1)-dimensional plaquette Ising model [62, 87, 88, 89, 90, 91] on the two torus T^2 equipped with two trivial foliations, as a simple example of a lattice model exhibiting subdimensional excitations.

In the plaquette Ising model, the physical degrees of freedom are qubits located on the vertices of a square lattice equipped with two linearly independent trivial foliations, as illustrated in Fig. 12a. The Hamiltonian is

$$H = - \sum_{p : \text{plaquettes}} X_p, \quad (3.10)$$

where X_p denotes the product of Pauli- X operators acting on the four vertices at the corners of a plaquette (see Fig. 12b).

As in the case of the toric code discussed in section 2.2, the ground state degeneracy of this model can be computed by analyzing the constraints imposed by the plaquette interactions. All local terms in the Hamiltonian mutually commute, so the model can again be regarded as a stabilizer code. Since each plaquette operator X_p is self-inverse, the structure of the constraints is closely analogous to the toric code: ground states are the simultaneous +1 eigenstates of all X_p .

On a square lattice of size $L_\omega \times L_{\tilde{\omega}}$ with periodic boundary conditions and one qubit per vertex, the total Hilbert space has dimension $2^{L_\omega L_{\tilde{\omega}}}$. There is one plaquette operator per plaquette, so the total number of stabilizer generators is $N_p = L_\omega L_{\tilde{\omega}}$. However, these

generators are not all independent. For each leaf of the foliation defined by ω , the product of plaquette operators along that leaf is the identity,

$$\prod_{p \in \omega^T} X_p = I , \quad (3.11)$$

and similarly, for each leaf of the foliation defined by $\tilde{\omega}$,

$$\prod_{p \in \tilde{\omega}^T} X_p = I . \quad (3.12)$$

There is, however, one redundancy between these relations, because

$$\prod_{\text{all leaves } p \in \omega^T} X_p = \prod_{\text{all leaves } p \in \tilde{\omega}^T} X_p . \quad (3.13)$$

Thus the number of independent relations is $L_\omega + L_{\tilde{\omega}} - 1$, and the number of independent stabilizer generators is

$$N_{\text{stab}}^{(\text{ind})} = N_p - (L_\omega + L_{\tilde{\omega}} - 1) = L_\omega L_{\tilde{\omega}} - (L_\omega + L_{\tilde{\omega}} - 1) . \quad (3.14)$$

Since the number of qubits is $N_q = L_\omega L_{\tilde{\omega}}$, the number of encoded qubits is

$$k = N_q - N_{\text{stab}}^{(\text{ind})} = L_\omega + L_{\tilde{\omega}} - 1 , \quad (3.15)$$

and the resulting ground-state degeneracy is

$$\text{GSD} = 2^k = 2^{L_\omega + L_{\tilde{\omega}} - 1} . \quad (3.16)$$

In the present geometry each leaf is topologically a circle, so its first Betti number is $\beta_1 = 1$. More suggestively, we may write

$$\text{GSD} = 2^{\beta_1^\omega L_\omega + \beta_1^{\tilde{\omega}} L_{\tilde{\omega}} - 1} , \quad (3.17)$$

where β_1^ω and $\beta_1^{\tilde{\omega}}$ denote the first Betti numbers of the leaves of the foliations specified by ω and $\tilde{\omega}$, respectively (for the square lattice with periodic boundaries one has $\beta_1^\omega = \beta_1^{\tilde{\omega}} = 1$), and L_ω and $L_{\tilde{\omega}}$ are the numbers of lattice periods along the ω - and $\tilde{\omega}$ -directions. Thus the ground state degeneracy grows linearly with the number of layers in each foliation, reflecting the subsystem-symmetric character of the model in contrast to the purely topological degeneracy of the toric code. In the toric code, the stabilizer constraints select closed 1-cycles modulo boundaries; here, the plaquette operators impose layer-by-layer constraints along each foliation direction, leading to independent degrees of freedom associated with each leaf of the foliations.

The logical operators in this model take the form of rigid, subsystem-like strings. All products of Pauli- X operators along straight lines act as logical $\bar{X}$ operators, reflecting the fact that the plaquette constraints do not restrict such rigid line operators. In contrast, only

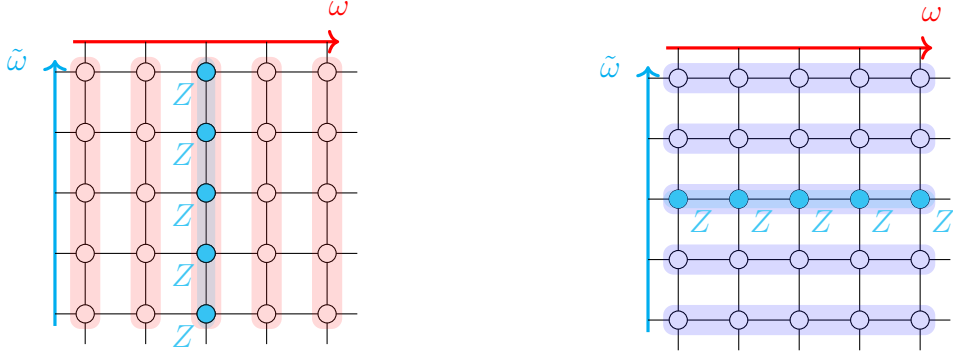


Figure 13: Two types of logical operators in the Ising plaquette theory (3.10). The logical operators are depicted as green dots in the left and right panels, which are embedded in leaves associated to the normal 1-forms ω and $\tilde{\omega}$, respectively.

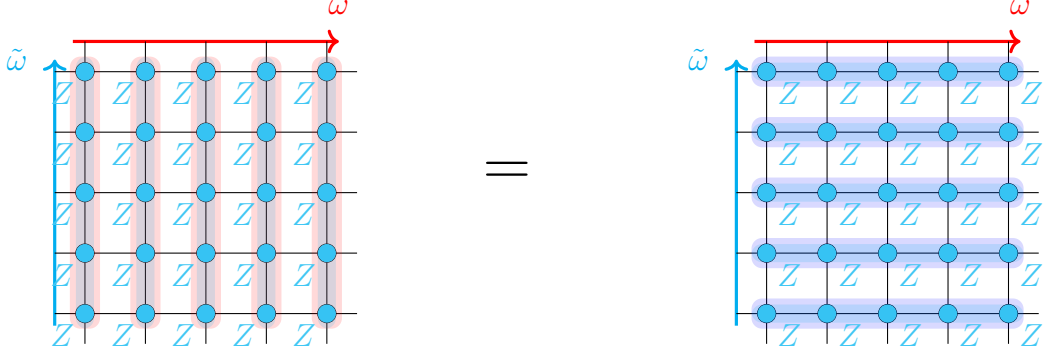


Figure 14: The left panel depicts a horizontal arrangement of vertical logical loop operators, whereas the right panel illustrates a vertical arrangement of horizontal loop operators. These two operator distributions are identical.

rigid Z -loop operators extending along the directions transverse to the foliations defined by ω and $\tilde{\omega}$ become logical $\bar{Z}$ operators. Examples of such operators are shown in Fig. 13. As illustrated in Fig. 14, the product of all logical $\bar{Z}$ operators transverse to ω is equivalent to the product of all logical $\bar{Z}$ operators transverse to $\tilde{\omega}$. This global constraint reduces the number of independent logical qubits by one, resulting in the term -1 in the exponent of (3.13). In the continuum limit, where L_ω and $L_{\tilde{\omega}}$ are sent to infinity, this offset becomes negligible and the degeneracy is governed solely by the foliation-dependent subsystem structure.

The fractionalized quasiparticles in this theory arise as the endpoints of rigid Z -string operators oriented transversely to either the ω - or $\tilde{\omega}$ -foliation. Such excitations behave as lineons: they can move freely only along the direction transverse to the foliation on which the corresponding Z -string is supported, and remain immobile in any other direction. Thus the restricted mobility of the excitations directly reflects the subsystem structure imposed by the pair of trivial foliations.

3.2.2 X-cube model

The X-cube model is one of the pioneering lattice realizations of subdimensional particles, originally proposed in Ref. [12]. It is defined on a three-dimensional cubic lattice with periodic boundary conditions. In three spatial dimensions, the lattice consists not only of vertices, edges, and plaquettes, but also of elementary cubes. With periodic boundary conditions, the cubic lattice is naturally regarded as being embedded in the three-dimensional torus T^3 .

The physical degrees of freedom of the X-cube model are qubits placed on the edges of the lattice. These edge qubits serve as the physical qubits in the sense of the stabilizer formalism introduced in section 2.1.

The Hamiltonian of the X-cube model is

$$H = - \sum_{c \text{ cubes}} B_c - \sum_{v \text{ vertices}} (A_v^x + A_v^y + A_v^z) , \quad (3.18)$$

where the stabilizer operators are defined as follows. The cube operator B_c is the product of Pauli- X operators over the twelve edges of the cube c (see Fig. 15):

$$B_c = \prod_{e \text{ consisting } c} X_e. \quad (3.19)$$

At each vertex v , there are three “cross” operators A_v^x , A_v^y , and A_v^z , each supported on the four edges contained in a coordinate plane through v . Explicitly,

$$A_v^x = Z_{e_1^{(x)}} Z_{e_2^{(x)}} Z_{e_3^{(x)}} Z_{e_4^{(x)}} , \quad (3.20)$$

$$A_v^y = Z_{e_1^{(y)}} Z_{e_2^{(y)}} Z_{e_3^{(y)}} Z_{e_4^{(y)}} , \quad (3.21)$$

$$A_v^z = Z_{e_1^{(z)}} Z_{e_2^{(z)}} Z_{e_3^{(z)}} Z_{e_4^{(z)}} , \quad (3.22)$$

where the sets of four edges $\{e_i^{(\mu)}\}_{i=1}^4$ ($\mu = x, y, z$) correspond to the edges lying in the plane normal to the μ -direction. These operators are illustrated in Figs. 16-18. All terms in the Hamiltonian are Hermitian and satisfy $O^2 = I$, and they mutually commute. Thus, as in the toric code, the X-cube model is a commuting-projector stabilizer code, with its ground state Hilbert space given by the common +1 eigenspace of all cube and vertex operators.

Let us now count the dimension of the ground-state Hilbert space. The full Hilbert space has dimension

$$2^{N_e}, \quad N_e = 3 L_x L_y L_z , \quad (3.23)$$

since there are three edges per unit cell on a cubic lattice with L_x, L_y, L_z sites in the three directions. The ground states are those states that satisfy all stabilizer constraints imposed by the cube operators B_c and the vertex operators A_v^x, A_v^y, A_v^z . To determine the number of independent constraints, we must count the number of independent stabilizer generators.

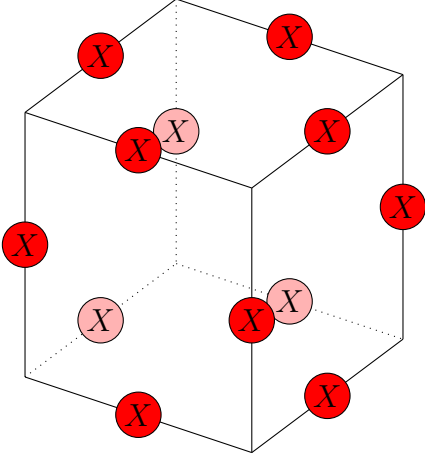


Figure 15: Cube X operator in the X-cube model. Each such operator is supported on a single cube of the three-dimensional lattice, acting on the qubits located on its edges.

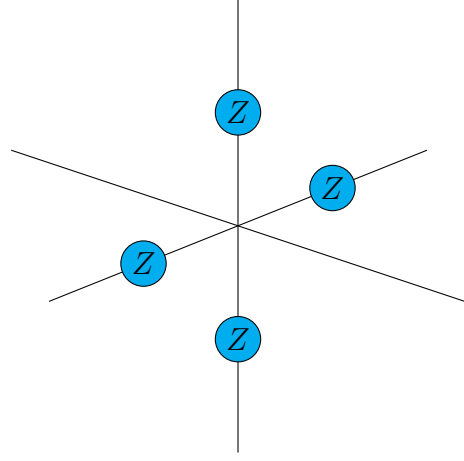


Figure 16: Vertex operator in the X-cube model supported on the y - z plane. It acts on the four edges adjacent to a given vertex within that plane.

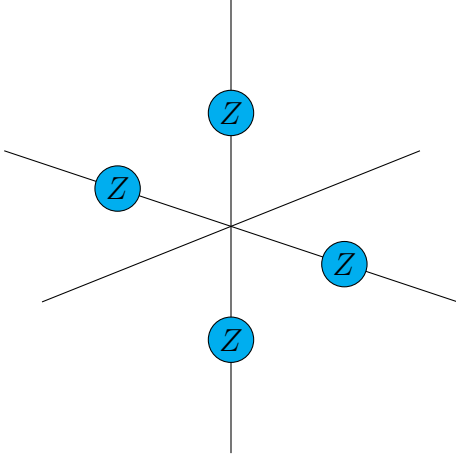


Figure 17: Vertex operator in the X-cube model supported on the z - x plane. It acts on the four edges adjacent to a given vertex within that plane.

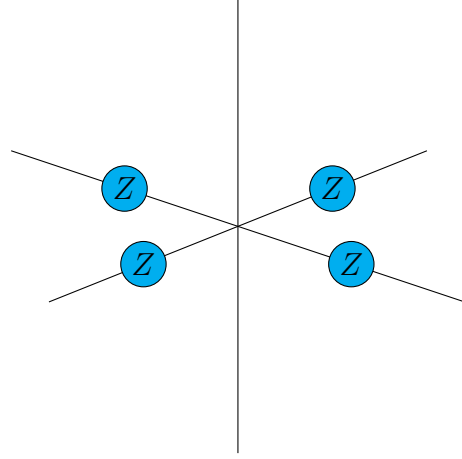


Figure 18: Vertex operator in the X-cube model supported on the x - y plane. It acts on the four edges adjacent to a given vertex within that plane.

There are

$$N_c = L_x L_y L_z \quad (3.24)$$

cube operators B_c , and

$$N_v = L_x L_y L_z \quad (3.25)$$

vertices. At each vertex v there are three cross operators A_v^x, A_v^y, A_v^z , but they satisfy the

local relation

$$A_v^x A_v^y A_v^z = 1 , \quad (3.26)$$

so only two of them are independent. In what follows, we take A_v^x and A_v^y to be the independent generators at each vertex. Thus the total number of independent vertex stabilizers is

$$2N_v = 2L_x L_y L_z . \quad (3.27)$$

However, not all of these stabilizers are independent. There exist additional global relations among both the cube operators and the vertex operators, which further reduce the number of independent constraints.

First consider the cube stabilizers. For each coordinate plane, the product of all cube operators in that plane is the identity:

$$\prod_{x,y} B_c = I , \quad \prod_{y,z} B_c = I , \quad \prod_{z,x} B_c = I . \quad (3.28)$$

These equations are not all independent, because the product over the entire three-dimensional lattice can be viewed either as a stack of all xy -plane relations, or as a stack of all yz - or zx -plane relations. As a result, there are only $(L_x + L_y + L_z - 2)$ independent global relations among the $N_c = L_x L_y L_z$ cube operators, so the number of independent cube stabilizers is

$$N_c^{(\text{ind})} = N_c - (L_x + L_y + L_z - 2) . \quad (3.29)$$

Next consider the vertex stabilizers. As discussed above, we take A_v^x and A_v^y to be the two independent local generators at each vertex, giving $2N_v = 2L_x L_y L_z$ vertex stabilizers before imposing global relations. There are additional plane-wise constraints: for each yz -plane the product of all A_v^x in that plane is the identity, and for each zx -plane the product of all A_v^y in that plane is the identity. Equivalently, one can also express an xy -plane relation in terms of $A_v^z = A_v^x A_v^y$:

$$\prod_{y,z} A_v^x = I , \quad \prod_{z,x} A_v^y = I , \quad \prod_{x,y} A_v^z = \prod_{x,y} A_v^x A_v^y = I . \quad (3.30)$$

One of these plane-wise relations is redundant, since

$$\prod_{x,y,z} A_v^z = \left(\prod_{x,y,z} A_v^x \right) \left(\prod_{x,y,z} A_v^y \right) . \quad (3.31)$$

Thus there are $(L_x + L_y + L_z - 1)$ independent global relations among the vertex stabilizers, and the number of independent vertex stabilizers is

$$N_v^{(\text{ind})} = 2N_v - (L_x + L_y + L_z - 1) = 2L_x L_y L_z - (L_x + L_y + L_z - 1) . \quad (3.32)$$

Putting these together, the total number of independent stabilizers is

$$N_{\text{stab}}^{(\text{ind})} = N_c^{(\text{ind})} + N_v^{(\text{ind})} \quad (3.33)$$

$$= [N_c - (L_x + L_y + L_z - 2)] + [2N_v - (L_x + L_y + L_z - 1)] \quad (3.34)$$

$$= 3L_x L_y L_z - 2(L_x + L_y + L_z) + 3. \quad (3.35)$$

Since the dimension of the ground-state Hilbert space is

$$2^{N_e - N_{\text{stab}}^{(\text{ind})}}, \quad (3.36)$$

we obtain

$$\text{GSD} = 2^{3L_x L_y L_z - [3L_x L_y L_z - 2(L_x + L_y + L_z) + 3]} \quad (3.37)$$

$$= 2^{2(L_x + L_y + L_z) - 3}. \quad (3.38)$$

This reproduces the well-known result that the X-cube model has a subextensive ground state degeneracy

$$\text{GSD} = 2^{2L_x + 2L_y + 2L_z - 3}, \quad (3.39)$$

which grows linearly with the linear system sizes L_x, L_y, L_z (measured in units of cubic cells), rather than with the total volume.

Having characterized the ground state degeneracy, let us now describe the structure of logical operators in the X-cube model. As a stabilizer code, its logical operators are those operators that commute with all cube and vertex stabilizers but are not themselves products of stabilizers. In contrast to the toric code, where logical operators are topologically non-trivial loops, the X-cube model admits logical operators supported on rigid one-dimensional lines (Fig. 19a) and two-dimensional membranes (strips) (Fig. 19b).

Rigid line operators—products of Pauli- X operators along straight, noncontractible lines—serve as logical $\bar{X}$ operators. A striking feature of these operators, in contrast to the toric code, is that these lines cannot be smoothly deformed or bent without leaving the ground state subspace. Conversely, membrane (or strip) operators are constructed as products of Pauli- Z operators over an entire two-dimensional sheet orthogonal to one of the coordinate axes. To define a nontrivial logical operator, the membrane must wrap noncontractibly around at least one of the two directions along this sheet. Such membrane operators commute with all stabilizers but cannot be generated by them, and therefore act as logical $\bar{Z}$ operators. These rigid line and membrane (strip) operators account for the subextensive scaling of the ground state degeneracy, reflecting the presence of three independent foliation directions in the X-cube model.

Excitations above the ground state arise from violations of stabilizer constraints. A single Pauli- X operator acting on an edge anticommutes with the four vertex operators that touch that edge (two at each endpoint), creating four violated A_v terms. These excitations can be moved without additional energy cost by extending the X string straight along the direction

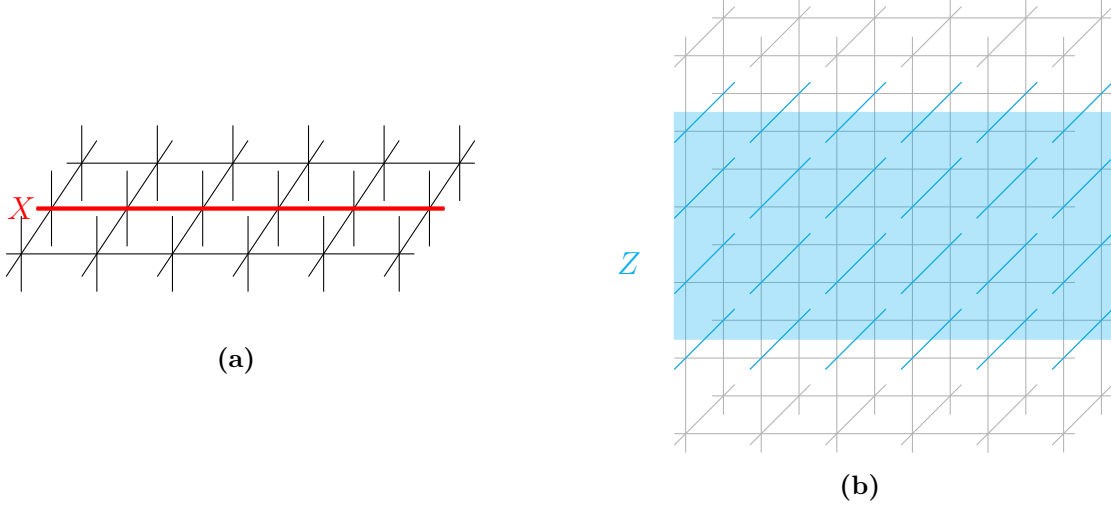


Figure 19: Logical operators in the X-cube model. (a) A rigid logical $\bar{X}$ line operator, given by the product of Pauli- X on a straight, noncontractible line of edges (shown along the x direction). Unlike in the toric code, the support cannot be smoothly deformed or bent without leaving the ground-state subspace. (b) A logical $\bar{Z}$ membrane (strip) operator, given by the product of Pauli- Z over a two-dimensional sheet orthogonal to one of the coordinate axes (shown for a fixed- z sheet); to be nontrivial, the membrane must wrap noncontractibly around at least one direction within the sheet.

of that edge; any attempt to bend the string would create extra vertex violations. In this way one obtains lineon excitations that are mobile only along a single coordinate axis.

Conversely, acting with Pauli- Z operators along the boundary of an elementary membrane (strip) anticommutes with the cube operators B_c and creates four fracton excitations at the corners of the membrane. Each individual fracton is strictly immobile, but suitable bound states of fractons can acquire enhanced mobility: pairs of fractons separated along a line behave as lineons, while appropriate four-fracton composites can move within a two-dimensional plane and are interpreted as planons. Thus the excitation structure of the X-cube model exhibits a characteristic hierarchy of fractons, lineons, and planons, directly reflecting its subsystem-symmetric and foliated nature.

The subextensive ground state degeneracy obtained above for the X-cube model can be reformulated in a way that highlights its geometric and foliated structure. Let $\omega_1, \omega_2, \omega_3$ denote the three foliation directions of the cubic X-cube model. In this picture, the logical operators naturally fall into two classes:

- rigid “line” operators supported on noncontractible 1-cycles, defined by the intersections of leaves from two of the foliation structures and running parallel to the remaining direction,
- “membrane (or strip)” operators supported on noncontractible two-dimensional sheets

(strips) orthogonal to one of the coordinate axes, which intersect the rigid lines and thus measure their eigenvalues.

Thus the counting of logical operators reduces to the counting of independent 1-cycles on the leaves transverse to the three foliations. Let $\beta_1^{(\omega_i)}$ denote the first Betti number of a leaf transverse to ω_i (for T^3 one has $\beta_1^{(\omega_i)} = 2$ for each i). Let L_{ω_i} denote the number of parallel leaves along the foliation direction ω_i . Then the ground state degeneracy takes the form

$$\text{GSD} = 2^{\sum_{i=x,y,z} (\beta_1^{(\omega_i)} L_{\omega_i})} - \Delta, \quad (3.40)$$

where Δ accounts for relations among stabilizers (such as $A_v^x A_v^y A_v^z = 1$ and the global product constraints) depending on information of the underlying manifold [13]. In the cubic lattice realization on T^3 we have $\beta_1^{(\omega_i)} = 2$ and $L_{\omega_i} = L_i$, giving

$$\text{GSD} = 2^{2(L_1+L_2+L_3)-3}, \quad (3.41)$$

in agreement with the explicit stabilizer counting. This geometric interpretation naturally anticipates the foliated field-theoretic description developed in the next sections.

3.3 Gauge field theory coupled to foliation structure

In section 2.4 we saw that the toric code on a closed surface admits a clean continuum description in terms of a topological BF theory, where the logical loop operators are realized as Wilson loop operators of a one-form gauge fields. By contrast, the lattice models with subdimensional excitations discussed in the preceding subsections, such as the plaquette Ising model and the X-cube model, are intimately tied to foliation structures of the spatial manifold. This contrast naturally leads to the question of whether there exists a continuum field-theoretic framework that plays, for foliated fracton phases, the same role that BF theory plays for the toric code, simultaneously capturing the geometric data of the foliation and the restricted mobility of the corresponding excitations.

In this subsection we construct such a continuum description in the form of a “topological” gauge theory coupled to the foliation geometry through the Godbillon-Vey like gauge redundancy introduced in section 3.1. The theory, proposed in Ref. [62], is designed so that its ground state Hilbert space on a foliated manifold reproduces that of lattice models with subdimensional excitations, playing for foliated fracton phases a role analogous to that of BF theory for the toric code. Couplings to dynamical matter fields and the resulting subdimensional particles will be discussed in later subsections.

In this subsection we analyze the continuum action

$$\begin{aligned} S[b^{(d-p)}, c^{(p)}, \lambda^{(p)}, \zeta^{(d-p-1)}; \langle \omega^{(1)} \rangle] \\ = \frac{k}{2\pi} \int_{M_{d+1}} \left[b^{(d-p)} \wedge dc^{(p)} - \lambda^{(p)} \wedge (d\zeta^{(d-p-1)} \wedge \omega^{(1)} - \omega^{(1)} \wedge b^{(d-p)}) \right], \quad k \in \mathbf{Z}. \end{aligned} \quad (3.42)$$

Here M_{d+1} is a $(d+1)$ -dimensional spacetime and $b^{(d-p)}$, $c^{(p)}$, $\lambda^{(p)}$, and $\zeta^{(d-p-1)}$ are differential form fields defined on M_{d+1} . The fields $c^{(p)}$, $b^{(d-p)}$, and $\zeta^{(d-p-1)}$ are treated as $U(1)$ higher-form gauge fields of degree p , $d-p$, and $d-p-1$, respectively. The one-form $\omega^{(1)}$ is the normal form specifying the codimension-one foliation $\langle \omega^{(1)} \rangle$ introduced in section 3.1, and k is an integer-valued constant, analogous to the level in Chern-Simons theory.

Importantly, this theory is *not* a mere variant of foliated BF theory. Instead, it incorporates the intrinsic redundancies of the foliation structure through a novel gauge principle acting on the normal 1-forms $\omega^{(1)}$ and $\eta^{(1)}$. Concretely, the gauge redundancies are

$$c^{(p)} \rightarrow c^{(p)} + \chi^{(p-1)} \wedge \omega^{(1)}, \quad \lambda^{(p)} \rightarrow \lambda^{(p)} + (-1)^{p-1} \chi^{(p-1)} \wedge \eta^{(1)} - d\chi^{(p-1)}, \quad (3.43)$$

$$c^{(p)} \rightarrow c^{(p)} + d\gamma^{(p-1)}, \quad (3.44)$$

$$b^{(d-p)} \rightarrow b^{(d-p)} - d\beta^{(d-p-1)}, \quad \zeta^{(d-p-1)} \rightarrow \zeta^{(d-p-1)} + (-1)^{d-p} \beta^{(d-p-1)}, \quad (3.45)$$

$$\zeta^{(d-p-1)} \rightarrow \zeta^{(d-p-1)} + d\tilde{\beta}^{(d-p-2)}, \quad (3.46)$$

$$\lambda^{(p)} \rightarrow \lambda^{(p)} + \delta^{(p-1)} \wedge \omega^{(1)}, \quad (3.47)$$

$$\lambda^{(p)} \rightarrow \rho^{-1} \lambda^{(p)}, \quad \omega^{(1)} \rightarrow \rho \omega^{(1)}, \quad (3.48)$$

where $\eta^{(1)}$ is a background 1-form.⁷ The first two lines encode the analog of (3.3) and (3.5), which are redundancies of the Godbillon-Vey class. The last transformation ensures that the action depends only on the underlying foliation structure and not on a particular representative of the normal 1-form.

The equations of motion following from (3.42) are

$$dc^{(p)} + (-1)^{(d-p)(p+1)} \lambda^{(p)} \wedge \omega^{(1)} = 0, \quad (3.50)$$

$$db^{(d-p)} = 0, \quad (3.51)$$

$$d\zeta^{(d-p-1)} \wedge \omega^{(1)} - \omega^{(1)} \wedge b^{(d-p)} = 0, \quad (3.52)$$

$$d(\lambda^{(p)} \wedge \omega^{(1)}) = 0. \quad (3.53)$$

These relations encode the constrained, subdimensional nature of gauge invariant operators in the theory. In particular, the first transformation plays a crucial role. Because $\lambda^{(p)}$ can be shifted by the gauge transformations (3.43) and (3.47), it does not fix an absolute value of $\lambda^{(p)}$; instead, it constrains the field strength of $c^{(p)}$. It is natural to reinterpret the resulting equation of motion as a condition on the component of $dc^{(p)}$ tangent to the leaves, namely as a “subsystem higher-form symmetry” conservation:

$$(dc^{(p)})|_{\omega^T} = 0, \quad (3.54)$$

⁷Here $\eta^{(1)}$ is the background field defined in (3.2). Under the rescaling (3.48), it transforms as

$$\eta^{(1)} \rightarrow \eta^{(1)} - d(\log \rho), \quad (3.49)$$

so that the full set of gauge transformations remains consistent.

where the subscript ω^T denotes the projection to directions tangent to the foliation (i.e. the component annihilated by contraction with the normal one-form $\omega^{(1)}$).

We now turn to gauge invariant extended operators in this theory. A first class of such operators is given by Wilson operators of the p -form field $c^{(p)}$:

$$U_\beta[\Sigma_p] := \exp \left(i\beta \int_{\Sigma_p} c^{(p)} \right), \quad (3.55)$$

where Σ_p is a closed, oriented p -dimensional submanifold lying entirely within a single d -dimensional leaf of the foliation, and β is an integer valued parameter. These operators are invariant under the p -form gauge symmetry of $c^{(p)}$ and, due to the foliation constraint on the field strength, naturally realize subsystem-like symmetries supported on individual leaves. The conservation of subsystem higher-form symmetry implies that the expectation value of the Wilson operator (3.55) depends only on the homotopy class of the foliation leaf.

A second class of operators is constructed from the $(d-p)$ -form field $b^{(d-p)}$ and the $(d-p-1)$ -form field $\zeta^{(d-p-1)}$. For a $(d-p)$ -dimensional manifold $\mathcal{N}_{d-p}$, whose connected components of boundary lie within a leaf, and may have boundary or be closed, we define

$$W_\gamma[\mathcal{N}_{d-p}] := \exp \left(i\gamma \int_{\partial\mathcal{N}_{d-p}} \zeta^{(d-p-1)} \right) \exp \left(i\gamma \int_{\mathcal{N}_{d-p}} b^{(d-p)} \right), \quad (3.56)$$

where γ is an integer valued parameter. The particular combination of the bulk integral of $b^{(d-p)}$ and the boundary integral of $\zeta^{(d-p-1)}$ is chosen so that $W_\gamma[\mathcal{N}_{d-p}]$ is invariant under the higher-form gauge transformations of $b^{(d-p)}$ and $\zeta^{(d-p-1)}$ (3.45).

A key property of these operators is that $W_\gamma[\mathcal{N}_{d-p}]$ becomes trivial whenever every connected component of $\mathcal{N}_{d-p}$ lies entirely within a single leaf of the foliation. This follows directly from the equation of motion (3.52). Nontrivial operators therefore arise only when $\mathcal{N}_{d-p}$ has components that interpolate between different leaves. Moreover, $W_\gamma[\mathcal{N}_{d-p}]$ is invariant under smooth deformations of $\mathcal{N}_{d-p}$ as long as the boundary $\partial\mathcal{N}_{d-p}$ remains confined to the corresponding leaves because (3.51) and (3.52) represent higher-form symmetry-like conservation law. In this sense, $W_\gamma[\mathcal{N}_{d-p}]$ depends only on the foliation-sensitive homology class of $\mathcal{N}_{d-p}$ with boundary.

Let us first comment on the role of the gauge field $c^{(p)}$. Although $c^{(p)}$ does not couple explicitly to the normal 1-form $\omega^{(1)}$ in the action, it still retains information about the foliation through the shift symmetry (3.43). This symmetry is the field-theoretic manifestation of the redundancy (3.3) of the Godbillon-Vey data, and thus encodes the way in which the foliation structure is imprinted on the dynamics of $c^{(p)}$.

We now analyze the ground state degeneracy of the U(1) BF-like theory (3.42) using canonical quantization. Because the theory contains several constraints and gauge redundancies, the quantization procedure is somewhat subtle. A complete and systematic treatment in 2+1 dimensions is presented in appendix C; in this subsection we simply summarize the key ingredients required to determine the ground state degeneracy.

Since the Hamiltonian of this theory vanishes identically, the structure of the ground state Hilbert space is governed entirely by the algebra of gauge-invariant operators. A straightforward computation shows that the Wilson-type operator $U_\beta[\Sigma_p]$ and the $W_{\gamma'}[\mathcal{N}_{d-p}]$ obey

$$U_\beta[\Sigma_p] W_{\gamma'}[\mathcal{N}_{d-p}] = \exp\left(\frac{2\pi i \beta \gamma'}{k} n(\Sigma_p, \mathcal{N}_{d-p})\right) W_{\gamma'}[\mathcal{N}_{d-p}] U_\beta[\Sigma_p], \quad (3.57)$$

where $n(\Sigma_p, \mathcal{N}_{d-p})$ denotes the intersection number of the closed p -dimensional submanifold Σ_p with the $(d-p)$ -dimensional submanifold with boundary $\mathcal{N}_{d-p}$. This nontrivial commutation relation encodes the spontaneous breaking of the subsystem p -form symmetry generated by (3.55).

As in the familiar case of Chern-Simons theory or BF theory, the ground state Hilbert space is obtained by constructing irreducible representations of this operator algebra. The resulting degeneracy is

$$\text{GSD} \sim k^{\beta_p^\omega L_\omega} = k^{\beta_{d-p}^\omega L_\omega} (= \infty), \quad (3.58)$$

where β_p^ω and β_{d-p}^ω are the p -th and $(d-p)$ -th Betti numbers of a leaf, and L_ω is the system size along the direction specified by the normal 1-form ω (in continuum, it becomes infinity). This yields a subextensive ground state degeneracy characteristic of foliated fracton-like phases, which becomes formally infinite in the continuum field theory $L_\omega \rightarrow \infty$.

3.3.1 2+1 dimensions: plaquette Ising field theory

We now illustrate the general construction by specializing the action (3.42) to the (2+1)-dimensional case, and compare the resulting continuum theory with the plaquette Ising model introduced in section 3.2.1. For $d = 2$ and $p = 1$ the action takes the form

$$S[b^{(1)}, c^{(1)}, \lambda^{(1)}, \phi; \langle \omega^{(1)} \rangle] = \frac{k}{2\pi} \int_{M_3} \left[b^{(1)} \wedge dc^{(1)} - \lambda^{(1)} \wedge (d\phi \wedge \omega^{(1)} - \omega^{(1)} \wedge b^{(1)}) \right], \quad k \in \mathbf{Z}. \quad (3.59)$$

Here $b^{(1)}$ and $c^{(1)}$ are U(1) one-form gauge fields, $\lambda^{(1)}$ is another one-form field, and ϕ is a compact scalar. More precisely, ϕ is nothing but the 0-form $\zeta^{(0)}$ appearing in the general action (3.42), but we have relabeled it for notational clarity. For the sake of simplicity, in the remainder of this subsection we will omit the superscripts indicating the form degree and simply write b , c , λ , ϕ , and ω for the corresponding differential forms.

The normal one-form ω encodes the codimension-one foliation of the spatial manifold; for instance, on $M_3 = S^1 \times T^2$ with a trivial foliation by straight lines one may take $\omega^{(1)} = dy$ so that the leaves are the x -circles at fixed y .

Specializing the equations of motion (3.50)-(3.53) of section 3.3 to the present case yields

$$dc + \lambda \wedge \omega = 0, \quad (3.60)$$

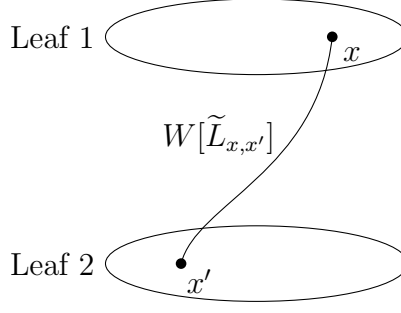


Figure 20: Nontrivial configuration of the line operator $W[\tilde{L}_{x,x'}]$. In this figure the endpoints x and x' lie on different leaves, so $\tilde{L}_{x,x'}$ cannot be deformed into a single leaf and the corresponding operator cannot be gauged away.

$$db = 0 , \quad (3.61)$$

$$d\phi \wedge \omega - \omega \wedge b = 0 , \quad (3.62)$$

$$d(\lambda \wedge \omega) = 0 . \quad (3.63)$$

In a coordinate system adapted to the foliation, these equations imply that the curvature of c vanishes along directions tangent to the leaves, while any nontrivial flux is tied to variation in the transverse ω -direction. In other words, the allowed field configurations are “layered” along the foliation, mirroring the way the plaquette Ising model organizes its degrees of freedom leaf by leaf.

Let us now introduce the gauge invariant operators of the (2+1)-dimensional theory. The first type is a Wilson operator built from c :

$$U_\beta[\mathcal{C}] := \exp \left(i\beta \int_{\mathcal{C}} c \right) , \quad (3.64)$$

where $\mathcal{C}$ is a closed curve lying entirely within a single two-dimensional leaf, and β is an integer-valued parameter. Thanks to the equations of motion and gauge redundancies, the expectation value of $U_\beta[\mathcal{C}]$ depends only on the homotopy (or homology) class of $\mathcal{C}$ within the leaf: the operator is topological under smooth deformations of the curve that stay on the same leaf, but cannot be deformed across leaves. These operators generate a subsystem 1-form symmetry acting independently on each leaf, and are the continuum counterparts of the rigid products of Pauli- Z operators along straight lines in the plaquette Ising model.

A second family of gauge invariant operators involves the fields b and ϕ . For an open path $L_{x,x'}$ connecting x' to x , which is not required to remain within a single leaf, we define

$$W_\gamma[L_{x,x'}] := \exp(-i\gamma \phi(x)) \exp \left(i\gamma \int_{L_{x,x'}} b \right) \exp(i\gamma \phi(x')) , \quad (3.65)$$

as shown in Fig. 20. The combination of endpoint operators built from ϕ and the line integral of b ensures that $W_\gamma[L_{x,x'}]$ is invariant under the gauge transformations (3.43)-(3.48) discussed in section 3.3.

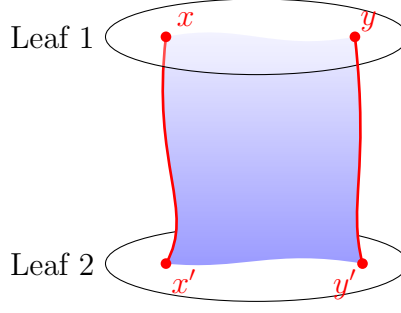


Figure 21: Schematic picture of a surface interpolating between two line segments $\tilde{L}_{x,x'}$ and $\tilde{L}_{y,y'}$. The two red lines represent the topological line operators $W[\tilde{L}_{x,x'}]$ and $W[\tilde{L}_{y,y'}]$. In this configuration, the points x and y lie on the same leaf, while x' and y' lie on another leaf, so the two line operators are related by sliding their endpoints along the leaves across the interpolating surface.

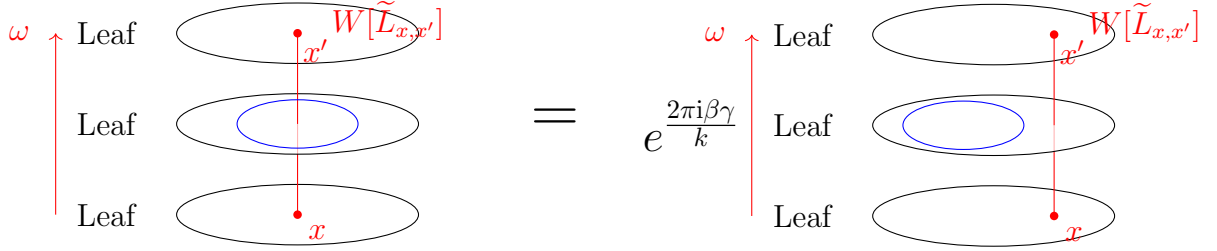


Figure 22: Schematic illustration of the linking between a Wilson loop $U_\beta[C] = \exp(i\beta \int_C c)$ lying on a leaf and an open line operator $W_\gamma[L_{x,x'}]$ built from b and ϕ . In expectation values constrained by the Ward-Takahashi identities, sliding $W_\gamma[L_{x,x'}]$ across $U_\beta[C]$ to undo their linking produces a phase $\exp(2\pi i \beta \gamma / k)$ determined by their intersection (linking) number.

In particular, the equations of motion for b and ϕ (see (3.61) and (3.62)) imply that, up to gauge equivalence, the bulk of the path $L_{x,x'}$ can be smoothly deformed anywhere in the ambient space, while the endpoints x and x' can slide freely along their respective leaves without changing $W_\gamma[L_{x,x'}]$ as shown in Fig. 21. Consequently, $W_\gamma[L_{x,x'}]$ becomes trivial whenever the entire path $L_{x,x'}$, including its endpoints, can be smoothly deformed so as to lie within a single leaf. In other words, if x and x' lie on the same leaf and $L_{x,x'}$ can be continuously pushed into that leaf, the corresponding operator is equivalent to the identity. Nontrivial operators therefore correspond to paths whose endpoints cannot be simultaneously embedded into a single leaf, i.e. paths that genuinely cross between different leaves, providing the continuum analogue of Pauli-X operators that measure the eigenvalues of the rigid Pauli-Z loop operators on each leaf.

This structure can be probed by combining a Wilson loop $U_\beta[C] = \exp(i\beta \oint_C c)$ supported on a leaf with an open line operator $W_\gamma[L_{x,x'}]$ in the bulk constructed from b and ϕ . The

Ward-Takahashi identities associated with the b - and ϕ -equations of motion then ensure that, when we continuously deform $L_{x,x'}$ so as to unwind any linking between C and $L_{x,x'}$, the correlator $\langle U_\beta[C] W_\gamma[L_{x,x'}] \rangle$ changes only by a universal topological phase, $\exp(2\pi i \beta \gamma / k)$ for a single linking:

$$U_\beta[C] W_\gamma[L_{x,x'}] = \exp\left(\frac{2\pi i \beta \gamma}{k} \text{Link}(C, L_{x,x'})\right) W_\gamma[L_{x,x'}] U_\beta[C] , \quad (3.66)$$

while any further dependence on the detailed shapes of the operators is trivial (see Fig. 22).

In Hamiltonian formalism, the canonical commutation relations derived in appendix C imply that $U_\beta[C]$ and $W_\gamma[L_{x,x'}]$ obey an intersection-number-controlled algebra. If we denote by $n(C, L_{x,x'})$ the intersection number between the closed curve C and the open path $L_{x,x'}$, we obtain

$$U_\beta[C] W_\gamma[L_{x,x'}] = \exp\left(\frac{2\pi i \beta \gamma}{k} n(C, L_{x,x'})\right) W_\gamma[L_{x,x'}] U_\beta[C] . \quad (3.67)$$

As in the BF theory discussed in section 2.4, the ground state Hilbert space is determined by the irreducible representations of this operator algebra. For a foliation with leaves of first Betti number β_1^ω and with L_ω parallel leaves along the ω -direction, the resulting ground state degeneracy is

$$\text{GSD} \sim k^{\beta_1^\omega L_\omega} (= \infty) . \quad (3.68)$$

In the case of a trivial foliation of T^2 by circles one has $\beta_1^\omega = 1$, so that $\text{GSD} = k^{L_\omega}$.

As an important remark, to reproduce the plaquette Ising model of section 3.2.1, which involves two independent trivial foliations ω and $\tilde{\omega}$, we simply take two copies of the above (2+1)-dimensional theory, one for each foliation direction. For $k = 2$ this yields

$$\text{GSD} \sim 2^{\beta_1^\omega L_\omega + \beta_1^{\tilde{\omega}} L_{\tilde{\omega}}} (= \infty) , \quad (3.69)$$

in agreement with the lattice result (3.17) up to the constant finite-size shift. Thus the (2+1)-dimensional foliation-coupled gauge theory (3.59) provides a continuum field-theoretic realization of the plaquette Ising ground state structure and subsystem symmetries, in direct analogy with how BF theory captures the toric code in section 2.4.

3.3.2 3+1 dimensions: X-cube-like field theory

We now turn to the (3+1)-dimensional case of the Godbillon-Vey type gauge theory (3.42) and show how it reproduces the Hilbert space structure of the X-cube model discussed in section 3.2.2. In (3+1) dimensions with $p = 1$, the basic fields in (3.42) consist of a two-form gauge field $b^{(2)}$, a one-form gauge field $c^{(1)}$, a one-form field $\lambda^{(1)}$, and a one-form field $\zeta^{(1)}$:

$$\begin{aligned} & S[b^{(2)}, c^{(1)}, \lambda^{(1)}, \zeta^{(1)}; \langle \omega^{(1)} \rangle] \\ &= \frac{k}{2\pi} \int_{M_{3+1}} \left[b^{(2)} \wedge dc^{(1)} - \lambda^{(1)} \wedge (d\zeta^{(1)} \wedge \omega^{(1)} - \omega^{(1)} \wedge b^{(2)}) \right] , \quad k \in \mathbf{Z} . \end{aligned} \quad (3.70)$$

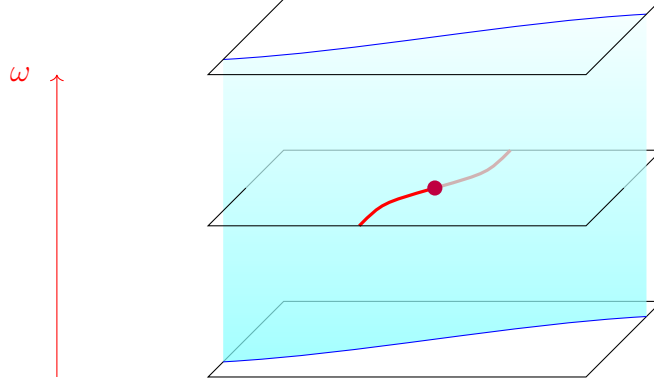


Figure 23: Schematic picture of the intersection between a leafwise loop operator $U_\beta[C]$ and a transverse membrane (strip) operator $W_\gamma[\mathcal{N}_2]$ in the (3+1)-dimensional Godbillon-Vey BF theory. The closed loop C lies inside a single leaf, while the infinitely thin strip $\mathcal{N}_2$ runs transversely across the foliation, so that on a time slice they intersect at a single point. This intersection contributes a phase $\exp(2\pi i \beta \gamma / k)$ to the commutation relation between $U_\beta[C]$ and $W_\gamma[\mathcal{N}_2]$, illustrating how the intersection number controls the algebra of gauge invariant operators.

In contrast to the (2+1)-dimensional case, in which the field $\zeta^{(0)}$ could be rewritten as a compact scalar ϕ , here all dynamical fields naturally appear as differential forms. As in the previous subsection, for notational simplicity we omit the superscripts indicating form degree in what follows.

The gauge invariant operators are loop operator embedded in a leaf:

$$U_\beta[\Sigma] := \exp \left(i\beta \int_C c \right) , \quad (3.71)$$

and membrane operator with boundary:

$$W_\gamma[\mathcal{N}_2] := \exp \left(i\gamma \int_{\partial \mathcal{N}_2} \zeta \right) \exp \left(i\gamma \int_{\mathcal{N}_2} b \right) . \quad (3.72)$$

As in the 2+1-dimensional case, these two operators exhibit a nontrivial mutual linking: the correlator $\langle U_\beta[\mathcal{C}] W_\gamma[\mathcal{N}_2] \rangle$ is invariant under smooth deformations of $\mathcal{C}$ and $\mathcal{N}_2$ that preserve their mutual linking, while a deformation that changes the linking number between the loop $\mathcal{C}$ and the membrane $\mathcal{N}_2$ induces a universal topological phase in the expectation value, determined solely by this linking number and independent of the detailed shapes of the operators.

In Hamiltonian formalism, c loop operators behave as planon loop operators, and b operators behave as strip operators. On a fixed time slice, the gauge invariant loop operator built from c is

$$U_\beta[C] = \exp \left(i\beta \oint_C c \right) , \quad (3.73)$$

where C is a closed curve lying entirely inside a single leaf. Within the gauge invariant (physical) Hilbert space, this loop acts as a topological planon operator: its action depends only on the homology class of C in the leaf of the codimension-one foliation, and any deformation of C within the same leaf is implemented by a product of local gauge transformations.

Similarly, the gauge invariant extended operator associated with b is a membrane (or strip) operator $W_\gamma[\mathcal{N}_2]$ whose support is a two-dimensional surface $\mathcal{N}_2$ stretching between neighboring leaves and whose boundary lines lie along the leaves. In canonical variables, this operator is constructed so that the bulk b -flux and boundary contributions combine to commute with the gauge generators, so $W_\gamma[\mathcal{N}_2]$ also preserves the physical Hilbert space. Its boundary acts as a pair of one-dimensional defects that can be freely slide along the leaves without changing any physical state, while displacing the strip so as to move its boundary across different leaves corresponds to creating or transporting subdimensional excitations.

The nontrivial structure arises when the loop C and the strip $\mathcal{N}_2$ intersect. Let $n(C, \mathcal{N}_2)$ denote the intersection number between C and the bulk membrane $\mathcal{N}_2$ on a time slice. Then the algebra of gauge-invariant operators takes the form

$$U_\beta[C] W_\gamma[\mathcal{N}_2] = \exp\left(\frac{2\pi i \beta \gamma}{k} n(C, \mathcal{N}_2)\right) W_\gamma[\mathcal{N}_2] U_\beta[C], \quad (3.74)$$

so the intersection pairing controls a central phase in their commutation relation (see Fig. 23). On a compact foliated spatial manifold, the ground state degeneracy is determined by the number of independent intersection pairs between nontrivial leafwise loops and transverse strips, and thus grows with the global foliated topology of the spatial manifold. In particular, for a regular stack of leaves with β_1^ω independent leafwise one-cycles per leaf and L_ω leaves, the degeneracy scales as

$$\text{GSD} \sim k^{\beta_1^\omega L_\omega}, \quad (3.75)$$

which diverges, $\text{GSD} \rightarrow \infty$, in the continuum limit where the number of leaves L_ω becomes infinite. The scaling of the ground state degeneracy agrees with that of the X-cube model (3.40).

To model the X-cube phase, we consider, for concreteness, three mutually orthogonal foliations of the spatial three-manifold M_3 ,⁸ characterized by normal one-forms $\omega_{(1)}$, $\omega_{(2)}$, and $\omega_{(3)}$. On a three-torus T^3 with coordinates (x, y, z) , a convenient choice is

$$\omega_{(1)} = dx, \quad \omega_{(2)} = dy, \quad \omega_{(3)} = dz, \quad (3.76)$$

so that each foliation consists of parallel two-tori transverse to one of the coordinate directions. For each foliation, labelled by $j = 1, 2, 3$, we introduce an independent copy of the gauge fields

$$(b_{(j)}, c_{(j)}, \lambda_{(j)}, \zeta_{(j)}), \quad (3.77)$$

⁸Note that all orientable compact 3-manifolds are known to admit total foliations [92].

and define the (3+1)-dimensional action

$$S_{\text{X-cube}} := \sum_{j=1}^3 S[b_{(j)}, c_{(j)}, \lambda_{(j)}, \zeta_{(j)}; \langle \omega_{(j)} \rangle] , \quad (3.78)$$

where each sector is given by (3.70):

$$S[b_{(j)}, c_{(j)}, \lambda_{(j)}, \zeta_{(j)}; \langle \omega_{(j)} \rangle] = \frac{k}{2\pi} \int_{M_{3+1}} \left[b_{(j)} \wedge dc_{(j)} - \lambda_{(j)} \wedge (d\zeta_{(j)} \wedge \omega_{(j)} - \omega_{(j)} \wedge b_{(j)}) \right] , \quad k \in \mathbf{Z} . \quad (3.79)$$

Thus the continuum theory relevant for the X-cube phase is simply a sum of three copies of the (3+1)-dimensional Godbillon-Vey-type BF theory (3.70), one associated with each foliation direction.

Each sector labelled by j enjoys the gauge redundancies discussed in section 3.3 for the general theory, with the obvious replacement $(b, c, \lambda, \zeta, \omega) \rightarrow (b_{(j)}, c_{(j)}, \lambda_{(j)}, \zeta_{(j)}, \omega_{(j)})$. In particular, the subsystem-like shift symmetry of $c_{(j)}$

$$c_{(j)} \rightarrow c_{(j)} + u_{(j)} \omega_{(j)} \quad (3.80)$$

encodes the redundancy of the Godbillon-Vey data for the j -th foliation and is responsible for the restricted mobility of excitations in the corresponding direction.

Let us now discuss the ground state Hilbert space. Since the Hamiltonian of the theory again vanishes identically, the ground states are entirely characterized by the algebra of gauge invariant operators. For each foliation sector j , we can construct Wilson-type operators supported on closed curves lying within a single leaf,

$$U_{\beta, (j)}[\mathcal{C}_{(j)}] := \exp \left(i\beta \int_{\mathcal{C}_{(j)}} c_{(j)} \right) , \quad (3.81)$$

where $\mathcal{C}_{(j)}$ is a closed one-dimensional submanifold embedded in a leaf of the j -th foliation. Similarly, we can define operators built from $b_{(j)}$ and $\zeta_{(j)}$ supported on two-dimensional manifolds that connect different leaves:

$$W_{\gamma, (j)}[\mathcal{N}_{2, (j)}] := \exp \left(i\gamma \int_{\partial \mathcal{N}_{2, (j)}} \zeta_{(j)} \right) \exp \left(i\gamma \int_{\mathcal{N}_{2, (j)}} b_{(j)} \right) , \quad (3.82)$$

where $\mathcal{N}_{2, (j)}$ is a (not necessarily leaf-preserving) surface whose boundary lies within leaves of the j -th foliation. Upon discretization, $U_{\beta, (j)}$ and $W_{\gamma, (j)}$ are the continuum counterparts of the rigid line and strip logical operators illustrated in Fig. 19 of the X-cube model.

The commutation relation between $U_{\beta, (j)}$ and $W_{\gamma, (j)}$ is again governed by their intersection number as (3.74):

$$U_{\beta, (j)}[\mathcal{C}_{(j)}] W_{\gamma, (j)}[\mathcal{N}_{2, (j)}] = \exp \left(\frac{2\pi i \beta \gamma}{k} n(\mathcal{C}_{(j)}, \mathcal{N}_{2, (j)}) \right) W_{\gamma, (j)}[\mathcal{N}_{2, (j)}] U_{\beta, (j)}[\mathcal{C}_{(j)}] , \quad (3.83)$$

where $n(\mathcal{C}_{(j)}, \mathcal{N}_{2,(j)})$ denotes the intersection number within the spatial manifold M_3 . Each independent pair of operators with unit intersection behaves as a conjugate pair of “position” and “momentum” operators in the topological Hilbert space, and therefore contributes a factor of k to the ground state degeneracy.

On a three-torus T^3 with the trivial foliations specified above, each leaf of the j -th foliation is a two-torus T^2 , which has first Betti number $\beta_1(T^2) = 2$. If we discretize the foliation into L_μ leaves along the j -direction, then in each sector there are $2L_{(j)}$ independent Wilson-type operators $U^{\beta,(j)}$ and the same number of conjugate operators $W_{\gamma,(j)}$. The contribution of the j -th sector to the ground state degeneracy is therefore

$$\text{GSD}_{(j)} \sim k^{2L_{(j)}} (= \infty) . \quad (3.84)$$

Taking the tensor product of the three foliation sectors, we obtain the total degeneracy

$$\text{GSD} \sim k^{2(L_{(1)}+L_{(2)}+L_{(3)})} (= \infty) . \quad (3.85)$$

For $k = 2$ this reproduces the characteristic subextensive scaling of the X-cube model (3.39), up to a foliation-independent constant shift in the exponent (which arises from global stabilizer relations in the lattice model and is not captured by the continuum description). In this sense, the (3+1)-dimensional Godbillon-Vey type field theory (3.79) plays for the X-cube model the same role that the BF theory plays for the toric code: it reproduces, in the continuum, the operator algebra and subextensive ground state degeneracy of the underlying foliated fracton phase.

However, an important subtlety remains. In the X-cube lattice model the logical loop operators are rigid lineon loops, whereas in our continuum description above we have realized planon loop operators. There are at least two possible ways to bridge this gap.

One possibility is to work with composite operators in the field theory. For example, we may consider the loop operator

$$U_\beta[\mathcal{C}] := \exp \left(i\beta \int_{\mathcal{C}} (c_{(1)} + c_{(2)}) \right) . \quad (3.86)$$

To preserve gauge invariance, the loop $\mathcal{C}$ must be embedded along the codimension-two locus given by the intersection of the leaves defined by $\omega_{(1)}$ and $\omega_{(2)}$.

A second, more radical possibility is to formulate an intrinsically codimension-two continuum field theory equipped with a foliation structure from the outset. The precise construction of such a theory is still under active development, and we will not pursue it in detail here. Instead, we restrict ourselves to a brief discussion in appendix B.

3.4 Matter theory coupled to c

In this subsection we introduce matter theories coupled to the gauge field $c^{(p)}$ appearing in (3.42). We add to the action a coupling of the form

$$\int c^{(p)} \wedge \star J_{\text{cl}}^{(p)} , \quad (3.87)$$

where $J_{\text{cl}}^{(p)}$ is a classical p -form current. Gauge invariance of this term under (3.43) and (3.44) requires $J_{\text{cl}}^{(p)}$ to satisfy

$$d \star J_{\text{cl}}^{(p)} = 0, \quad \omega^{(1)} \wedge \star J_{\text{cl}}^{(p)} = 0. \quad (3.88)$$

The first condition is the usual conservation law associated with the 0-form generalized symmetry whose background gauge field is $c^{(p)}$. The second condition is specific to the foliated Godbillon-Vey-type BF structure: it states that the source is compatible with the foliation in the sense that the Hodge dual $\star J_{\text{cl}}^{(p)}$ has no component along the normal direction picked out by $\omega^{(1)}$. Equivalently, $\star J_{\text{cl}}^{(p)}$ takes values in the subbundle of forms tangent to the leaves, and the theory only admits background sources that respect this constraint.

For concreteness, let us focus on the case $p = 1$. In local coordinates, the first condition reduces to

$$\partial_\mu J_{\text{cl}}^\mu = 0, \quad (3.89)$$

which is the usual continuity equation for a conserved current. The second condition can be rewritten as

$$\omega_\mu J_{\text{cl}}^\mu = 0, \quad (3.90)$$

meaning that the current has no component along the normal direction specified by ω .

This is precisely the type of kinematic restriction that characterizes subdimensional particle theories. The foliation encoded by ω singles out preferred lower-dimensional submanifolds (the leaves), and the constraint $\omega_\mu J_{\text{cl}}^\mu = 0$ implements the statement that the worldlines of charged excitations must lie inside these leaves. In (2+1) dimensions, for example, J_{cl} can be interpreted as the worldline current of a lineon that is allowed to move only along a single spatial direction (the direction tangent to the leaves), and is immobile in the ω -direction. More generally, in $(d+1)$ dimensions and for higher-form currents $J^{(p)}$ one obtains extended subdimensional objects whose worldvolumes are confined to the leaves, so that their allowed motion is restricted to a $(d-1)$ -dimensional subspace.

From the viewpoint of generalized global symmetries, the coupling $\int c^{(p)} \wedge \star J_{\text{cl}}^{(p)}$ states that the theory admits a conserved p -form charge measured by integrating $\star J^{(p)}$ on codimension- p hypersurfaces. The foliation constraint implies that these hypersurfaces can be chosen leaf by leaf, and that charge cannot flow between different leaves. Equivalently, each leaf carries its own conserved charge, leading to an extensive (leafwise) set of conservation laws. In subsequent sections we will replace the classical source J_{cl} by dynamical matter fields whose Noether currents obey the same constraints, and we will see explicitly how the foliation structure enforces lineon- or fracton-like kinematics in the corresponding quantum field theories.

Below we present three examples of matter theories in which dynamical scalar or fermionic fields are coupled to the $U(1)$ gauge field $c^{(1)}$ so that the associated conserved current satisfies the above constraints. Throughout this subsection we work in Euclidean signature for convenience, but the discussion carries over straightforwardly to Lorentzian spacetime.

3.4.1 Complex scalar theory

We now construct an explicit example of a matter theory whose quantum current J satisfies the foliated conservation laws

$$d \star J = 0 \ , \quad \omega \wedge \star J = 0 \ . \quad (3.91)$$

A particularly simple realization is provided by a complex scalar field minimally coupled to the background one-form $\Lambda\omega$. We consider the action

$$S = \int_{M_{d+1}} \left[((d + i\Lambda\omega)\Phi^*) \wedge \star((d - i\Lambda\omega)\Phi) + V(|\Phi|^2) + F(\Lambda\omega) \right] , \quad (3.92)$$

where Φ is a complex scalar field with potential $V(|\Phi|^2)$, Λ is a real scalar field, and ω is a fixed background foliation normal one-form. The first term is the usual kinetic term for a charged scalar, with the background gauge connection replaced by the composite one-form $\Lambda\omega$, so that the associated $U(1)$ current will automatically obey the foliation constraint discussed in the previous subsection. The last term $F(\Lambda\omega)$ denotes a gauge-invariant functional of the one-form $\Lambda\omega$, which we require to satisfy the identity

$$\partial_\mu \left(\frac{\delta \int F(X)}{\delta X_\mu} \right) \Big|_{X=\Lambda\omega} = 0 \ . \quad (3.93)$$

This condition ensures that the $\Lambda\omega$ -sector does not introduce additional sources for the $U(1)$ current and is compatible with the constrained conservation law encoded by ω .

A representative choice is the kinetic term for $\Lambda\omega$,

$$F_{\text{kin}}(\Lambda\omega) = d(\Lambda\omega) \wedge \star d(\Lambda\omega) , \quad (3.94)$$

which indeed obeys (3.93). Viewing $F_{\text{kin}}(X)$ as a functional of a general one-form X and varying with respect to X_μ , one finds

$$\partial_\mu \left(\frac{\delta \int F_{\text{kin}}(X)}{\delta X_\mu} \right) \Big|_{X^\sigma = \Lambda\omega^\sigma} = -4 \partial_\mu \partial_\nu (\partial^\nu X^\mu - \partial^\mu X^\nu) \Big|_{X^\sigma = \Lambda\omega^\sigma} = 0 , \quad (3.95)$$

where the last equality follows from the symmetry of $\partial_\mu \partial_\nu$ and the antisymmetry of $(\partial^\nu X^\mu - \partial^\mu X^\nu)$. Thus F_{kin} provides a concrete example of a kinetic term for the composite gauge field $\Lambda\omega$ that preserves the desired constrained conservation law for the scalar current.

In a local coordinate patch, the corresponding Lagrangian density takes the form

$$\mathcal{L} = (\partial_\mu + i\Lambda\omega_\mu)\Phi^* (\partial^\mu - i\Lambda\omega^\mu)\Phi + V(|\Phi|^2) + F(\Lambda\omega) . \quad (3.96)$$

Varying Φ and Λ we obtain the equations of motion

$$(\partial_\mu - i\Lambda\omega_\mu)(\partial^\mu - i\Lambda\omega^\mu)\Phi - \frac{\delta V(X)}{\delta X} \Big|_{X=|\Phi|^2} \Phi = 0 , \quad (3.97)$$

$$\mathrm{i}(\Phi^* \partial_\mu \Phi - \partial_\mu \Phi^* \Phi) \omega^\mu + 2\Lambda \omega_\mu \omega^\mu \Phi^* \Phi + \left. \frac{\delta F(X)}{\delta X^\mu} \right|_{X^\nu = \Lambda \omega^\nu} \omega^\mu = 0 , \quad (3.98)$$

together with the complex conjugate of the first line. Here we abuse the symbol $\delta/\delta X_\mu$ to denote a functional derivative with respect to the function variable X . The first equation is a Klein-Gordon-type equation with additional interactions, minimally coupled to the composite background connection $\Lambda\omega$, while the second equation plays the role of an effective constraint that ties the scalar density $\Phi^*\Phi$ (and, more generally, the matter configuration) to the background foliation data through Λ .

The Lagrangian (3.96) is invariant under an ordinary global $U(1)$ phase symmetry

$$\Phi \rightarrow e^{\mathrm{i}\alpha} \Phi , \quad \Phi^* \rightarrow e^{-\mathrm{i}\alpha} \Phi^* \quad (\alpha : \text{constant}) . \quad (3.99)$$

A convenient representative of the associated Noether current (defined up to identically conserved improvements) is

$$J^\mu = \mathrm{i}(\Phi^* \partial^\mu \Phi - \partial^\mu \Phi^* \Phi) + 2\Lambda \omega^\mu \Phi^* \Phi + \left. \frac{\delta F(X)}{\delta X_\mu} \right|_{X_\nu = \Lambda \omega_\nu} , \quad (3.100)$$

which satisfies the usual conservation law

$$\partial_\mu J^\mu = 0 . \quad (3.101)$$

Here the first two terms are the familiar current of a charged scalar minimally coupled to the background connection $\Lambda\omega$, while the last term gives an identically conserved contribution thanks to the condition (3.93) and can be viewed as an “improvement” compatible with the foliation structure. In particular, this choice of representative will be useful below, where the Λ equation of motion allows us to show that J^μ is orthogonal to the normal one-form ω .

With this choice of representative for J^μ , the Λ equation of motion in (3.98) further implies that the current is orthogonal to the normal one-form ω ,

$$\omega_\mu J^\mu = 0 , \quad (3.102)$$

so that J^μ represents the flow of excitations constrained to move along codimension-one submanifolds. Geometrically, the condition (3.102) means that J^μ takes values in the tangent bundle of the leaves: the current has no component along the normal direction picked out by ω , and hence charge can be transported only within each leaf and cannot flow between different leaves. In this sense the complex scalar field provides a continuum realization of subdimensional excitations whose motion is restricted by the foliation structure.

Up to now, the discussion has been classical. The corresponding quantum statements can be formulated using Ward-Takahashi identities. The quantum version of the conservation law (3.101) is

$$\langle \partial_\mu J^\mu \rangle = 0 , \quad (3.103)$$

while the quantum analog of the mobility constraint (3.102) follows from varying Λ in the path integral:

$$\langle \omega_\mu J^\mu \rangle \propto \int \mathcal{D}\Phi^* \mathcal{D}\Phi \mathcal{D}\Lambda \frac{\delta}{\delta \Lambda} e^{-S} = 0 . \quad (3.104)$$

Finally, we couple the scalar matter theory to the U(1) Godbillon-Vey BF system (3.42) with $p = 1$. The full action is

$$\begin{aligned} S = \int_{M_{d+1}} & \left[((d + i\Lambda\omega + ic)\Phi^*) \wedge \star((d - i\Lambda\omega - ic)\Phi) + V(|\Phi|^2) + F(\Lambda\omega) \right] \\ & + \frac{ik}{2\pi} \int_{M_{d+1}} \left[b^{(d-1)} \wedge dc - \lambda \wedge (d\zeta^{(d-2)} \wedge \omega - \omega \wedge b^{(d-1)}) \right] . \end{aligned} \quad (3.105)$$

This theory is invariant under the following gauge transformations:

$$c \rightarrow c + u\omega , \quad \lambda \rightarrow \lambda + u\eta - du , \quad \Lambda \rightarrow \Lambda - u , \quad (3.106)$$

$$c \rightarrow c + d\alpha , \quad \Phi \rightarrow e^{i\alpha}\Phi , \quad \Phi^* \rightarrow e^{-i\alpha}\Phi^* , \quad (3.107)$$

$$b^{(d-1)} \rightarrow b^{(d-1)} - d\beta^{(d-2)} , \quad \zeta^{(d-2)} \rightarrow \zeta^{(d-2)} + \beta^{(d-2)} , \quad (3.108)$$

$$\lambda \rightarrow \rho^{-1}\lambda , \quad \omega \rightarrow \rho\omega , \quad (3.109)$$

$$\lambda \rightarrow \lambda + v\omega . \quad (3.110)$$

In particular, the combination $c + \Lambda\omega$ transforms as an ordinary U(1) gauge field. Consequently, we may add standard gauge invariant terms constructed from this combination without spoiling the foliated Godbillon-Vey structure. For example, a Maxwell term can be included as

$$\frac{1}{4} d(c + \Lambda\omega) \wedge \star d(c + \Lambda\omega) . \quad (3.111)$$

In this coupled system, $c + \Lambda\omega$ plays the role of the dynamical U(1) gauge field that minimally couples to the scalar Φ , while $b^{(d-1)}$, λ , and $\zeta^{(d-2)}$ enforce the foliated constraints characteristic of the Godbillon-Vey BF sector.

3.4.2 Fermion QED-like theory

We next introduce a fermionic counterpart of the scalar QED-like model discussed in the previous subsection and couple it to the Godbillon-Vey BF theory (3.42). In this subsection we work, for simplicity, at the level of the Lagrangian in a coordinate system. The basic fermionic Lagrangian is

$$\mathcal{L} = \bar{\psi} \gamma^\mu (\partial_\mu - i\Lambda\omega_\mu) \psi + V(\psi, \bar{\psi}) + F(\Lambda\omega) , \quad (3.112)$$

where ψ is a Dirac spinor, $V(\psi, \bar{\psi})$ denotes a local interaction built from fermion bilinears (for instance a mass term or a Thirring-type self-interaction), and $F(\Lambda\omega)$ is a gauge invariant

functional of the one-form $\Lambda\omega$ obeying the condition (3.93). The first term is the usual Dirac kinetic term with the background connection $\Lambda\omega$ playing the role of an effective $U(1)$ gauge field, so that the corresponding fermionic current will inherit the foliation constraint in a manner parallel to the scalar case.

Varying the action with respect to $\bar{\psi}$ and Λ yields the equations of motion

$$\gamma^\mu(\partial_\mu - i\Lambda\omega_\mu)\psi + \frac{\delta V(\psi, \bar{\psi})}{\delta \bar{\psi}} = 0 , \quad \omega_\mu \bar{\psi} \gamma^\mu \psi = 0 , \quad (3.113)$$

together with the Dirac-conjugate equation obtained by varying with respect to ψ . Thus, from the second equation, the fermion bilinear $\bar{\psi} \gamma^\mu \psi$ is forced to be tangent to the leaves of the foliation.

The Lagrangian (3.112) possesses a vector $U(1)$ global symmetry⁹

$$\psi \rightarrow e^{i\alpha} \psi , \quad \bar{\psi} \rightarrow \bar{\psi} e^{-i\alpha} \quad (\alpha : \text{constant}) , \quad (3.114)$$

with associated Noether current

$$J^\mu = \bar{\psi} \gamma^\mu \psi + \left. \frac{\delta F(X)}{\delta X_\mu} \right|_{X_\nu = \Lambda\omega_\nu} . \quad (3.115)$$

Using the equations of motion together with the property (3.93) of F , one finds that this current obeys

$$\partial_\mu J^\mu = 0 , \quad \omega_\mu J^\mu = 0 . \quad (3.116)$$

Hence the fermionic $U(1)$ charge is conserved and its flow is confined to directions tangent to the foliation, precisely mirroring the subdimensional mobility pattern obtained in the complex scalar case.

Finally, we couple the fermions to the $U(1)$ Godbillon-Vey BF theory (3.42) with $p = 1$. For definiteness, let us first specialize to $d = 2$, i.e. a $(2+1)$ -dimensional spacetime, where b_μ and λ_μ are one-forms and the $(d-2)$ -form ζ reduces to a scalar field ϕ . In this case the full Lagrangian takes the form

$$\begin{aligned} \mathcal{L} = & \bar{\psi} \gamma^\mu (\partial_\mu - i\Lambda\omega_\mu - ic_\mu) \psi + V(\psi, \bar{\psi}) + F(\Lambda\omega) \\ & + \frac{ik}{2\pi} \epsilon^{\mu\nu\rho} (b_\mu \partial_\nu c_\rho - \lambda_\mu (\partial_\nu \phi \omega_\rho - \omega_\nu b_\rho)) . \end{aligned} \quad (3.117)$$

This Lagrangian is invariant under the gauge redundancies

$$\begin{aligned} c_\mu &\rightarrow c_\mu + u \omega_\mu , & \lambda_\mu &\rightarrow \lambda_\mu + u \eta_\mu - \partial_\mu u , & \Lambda &\rightarrow \Lambda - u , \\ b_\mu &\rightarrow b_\mu - \partial_\mu \theta , & \phi &\rightarrow \phi + \theta , \\ c_\mu &\rightarrow c_\mu + \partial_\mu \tilde{\theta} , & \psi &\rightarrow e^{i\tilde{\theta}} \psi , & \bar{\psi} &\rightarrow \bar{\psi} e^{-i\tilde{\theta}} , \\ \lambda_\mu &\rightarrow \rho^{-1} \lambda_\mu , & \omega_\mu &\rightarrow \rho \omega_\mu , \\ \lambda_\mu &\rightarrow \lambda_\mu + v \omega_\mu . \end{aligned} \quad (3.118)$$

⁹If the interaction $V(\psi, \bar{\psi})$ is absent, an axial $U(1)$ symmetry is present as well.

The extension to general spacetime dimension $(d+1)$ is straightforward: one takes $b^{(d-1)}$ and $\zeta^{(d-2)}$ to be higher-form fields as in (3.42), while the fermionic sector and the composite gauge field $c + \Lambda\omega$ retain the same structure. In this way, the foliated Godbillon-Vey BF theory coupled to fermions provides a natural higher-dimensional generalization of the $(2+1)$ -dimensional model described above.

3.4.3 Compact scalar theory

The third, and simplest, model that provides an arena for subdimensional properties is the compact scalar theory. We consider the simplest compact boson in $d+1$ dimensions with subdimensional current flow, described by the action

$$S = \frac{R^2}{4\pi} \int_{M_{d+1}} (d\phi - \Lambda\omega) \wedge \star(d\phi - \Lambda\omega). \quad (3.119)$$

Here ϕ is a compact scalar with 2π -periodicity, $\phi \sim \phi + 2\pi$, Λ is a real scalar field, and ω is the background foliation one-form introduced in the previous subsections. The combination $(d\phi - \Lambda\omega)$ plays the role of an effective covariant derivative, and, as we will see, its structure enforces a subdimensional kinematics for the corresponding conserved current. In what follows, we use this simplest model as a concrete playground to explore the characteristic properties of subdimensional particles. A more detailed analysis of this compact scalar theory and its subdimensional properties can be found in Refs. [93].

The equations of motion are

$$d \star (d\phi - \Lambda\omega) = 0, \quad (3.120)$$

$$\omega \wedge \star(d\phi - \Lambda\omega) = 0. \quad (3.121)$$

The action (3.119) is invariant under a $U(1)_{\text{shift}}$ (“electric”) shift symmetry

$$U(1)_{\text{shift}} : \quad \phi \mapsto \phi + \alpha, \quad \alpha : \text{const.} \quad (3.122)$$

The associated Noether current is

$$J_{\text{shift}} = d\phi - \Lambda\omega, \quad (3.123)$$

which obeys the conservation law

$$d \star J_{\text{shift}} = 0. \quad (3.124)$$

Moreover, the Λ -equation of motion (3.121) implies the additional constraint

$$\omega \wedge \star J_{\text{shift}} = 0, \quad (3.125)$$

so that the flow of J_{shift} is confined to the leaves of the foliation. In components, (3.125) can be written as

$$\omega_\mu J_{\text{shift}}^\mu = 0, \quad (3.126)$$

which means that the current has no component along the normal direction specified by ω and is everywhere tangent to the leaves. Thus the $U(1)_{\text{shift}}$ charge realizes a genuinely subdimensional symmetry: excitations carrying this charge can propagate only within each leaf and cannot move between different leaves.

There is also a second conserved quantity in this theory, associated with a “winding” (magnetic) symmetry. Since ϕ is compact, its winding is measured by the current

$$J_{\text{wind}} = \star d\phi, \quad (3.127)$$

which satisfies a conservation law that follows from the Bianchi identity $d^2 = 0$:

$$d \star J_{\text{wind}} = 0. \quad (3.128)$$

Because J_{shift} and J_{wind} are built from the same underlying field ϕ , the mobility constraint on the shift current (3.125) also induces a nontrivial constraint on the winding current. Using $J_{\text{shift}} = d\phi - \Lambda\omega$ and the definition $J_{\text{wind}} = \star d\phi$, one finds, after a short piece of algebra, that

$$\omega \wedge \star J_{\text{shift}} \sim \star \omega \wedge J_{\text{shift}} = \star \omega \wedge \star J_{\text{wind}} - \Lambda(\star \omega) \wedge \omega = 0, \quad (3.129)$$

or equivalently

$$\star \omega \wedge \star J_{\text{wind}} = \Lambda(\star \omega) \wedge \omega. \quad (3.130)$$

This relation shows that the flow of the winding current is constrained by the “dual foliation” encoded in the d -form $\star \omega$: the allowed winding configurations cannot be arbitrary, but must be compatible with the foliation structure and with the profile of the auxiliary field Λ . For this reason, we will refer to the magnetic current flow as “quasi-subdimensional”: its charge is conserved, but its motion is restricted only indirectly through the geometric constraint involving $\star \omega$, rather than being strictly confined to the leaves as in the electric shift current.

To summarize, the compact scalar theory (3.119) supports two intertwined conserved quantities. The shift current J_{shift} realizes a strictly subdimensional symmetry, whose charge can move only along the leaves. The winding current J_{wind} is also conserved but subject to an additional geometric constraint involving $\star \omega$, so that its flow is quasi-subdimensional, restricted by the dual foliation rather than being fully subdimensional in the same sense. In this sense the model hosts a subdimensional shift current and a quasi-subdimensional winding current, providing a simple continuum example of the rich kinematics familiar from lattice models with fracton-like excitations.

Exchanging shift symmetry and winding symmetry So far we have focused on a compact scalar theory in which the shift current is subdimensional while the winding current is only quasi-subdimensional. However, one can equally well consider a “dual” compact scalar theory in which the shift current is only quasi-subdimensional whereas the winding current exhibits genuinely subdimensional kinematics. Let us consider the following action

$$S = \frac{R^2}{4\pi} \int ((d\phi - \Lambda\omega) \wedge \star(d\phi - \Lambda\omega) - (\Lambda\omega) \wedge \star(\Lambda\omega)). \quad (3.131)$$

The equations of motion are

$$d \star (d\phi - \Lambda\omega) = 0 \quad (3.132)$$

$$\omega \wedge \star d\phi = 0 . \quad (3.133)$$

In this theory we define the $U(1)_{\text{shift}}$ current as

$$J_{\text{shift}} := d\phi - \Lambda\omega , \quad (3.134)$$

which by the equations of motion satisfies the conservation law

$$d \star J_{\text{shift}} = 0 , \quad (3.135)$$

together with the foliation constraint

$$\omega \wedge \star J_{\text{shift}} = \Lambda\omega \wedge \star\omega . \quad (3.136)$$

In other words, J_{shift} is a conserved current whose flow is confined to the leaves: it has no component along the normal direction specified by ω , and thus implements a genuinely subdimensional $U(1)$ shift symmetry.

Thus, in this case, the shift symmetry current does not describe a flow of subdimensional particles. On the other hand, the $U(1)_{\text{wind}}$ winding symmetry current

$$J_{\text{wind}} = \star d\phi , \quad (3.137)$$

obeys

$$d \star J_{\text{wind}} = 0 , \quad (3.138)$$

and

$$(\star\omega) \wedge \star J_{\text{wind}} = 0 . \quad (3.139)$$

Therefore, in 1+1 dimensions the flow of the shift-symmetry current is only quasi-subdimensional, whereas the flow of the winding-symmetry current is truly subdimensional with respect to the foliation structure defined by $\star\omega$.¹⁰

¹⁰In 1+1 dimensions, the one-form $\star\omega$ automatically defines a foliation structure, since the would-be obstruction

$$\star\omega \wedge d(\star\omega) \quad (3.140)$$

has degree three and hence vanishes identically in two-dimensional spacetime.

Physical meaning of scalar theories In this subsection we clarify the physical content of (3.119) and its dual formulation (3.131). In both cases the scalar field Λ plays the role of an auxiliary variable that imposes constraint conditions on the scalar configuration.

In the case of (3.119), Λ appears quadratically and without derivatives, so it behaves as a Gaussian auxiliary field. Integrating out Λ we can rewrite the action as

$$\begin{aligned} S &= \frac{R^2}{4\pi} \int \left(d\phi - \frac{\langle \omega, d\phi \rangle}{\langle \omega, \omega \rangle} \omega \right) \wedge \star \left(d\phi - \frac{\langle \omega, d\phi \rangle}{\langle \omega, \omega \rangle} \omega \right) \\ &= \frac{R^2}{4\pi} \int ((1 - P_\omega) d\phi) \wedge \star ((1 - P_\omega) d\phi) , \end{aligned} \quad (3.141)$$

where P_ω denotes the projection onto the direction of ω ,

$$P_\omega(\alpha) := \frac{\langle \omega, \alpha \rangle}{\langle \omega, \omega \rangle} \omega , \quad (3.142)$$

and $\langle , \rangle$ is the inner product induced by the spacetime metric on one-forms. Thus the dynamics only involves the components of $d\phi$ transverse to ω .

To make this more explicit, choose a local coordinate system and (in a locally orthonormal frame) align ω along a single spatial direction, say $\omega \propto dx^j$. Then the action reduces schematically to

$$S = \frac{R^2}{4\pi} \int \sum_{\mu \neq j} \partial_\mu \phi \partial^\mu \phi , \quad (3.143)$$

showing that the kinetic term contains derivatives of ϕ only in the directions transverse to x^j . In other words, fluctuations of ϕ along the j -th direction do not cost action at this quadratic level: the dependence of ϕ on x^j is not fixed by (3.119) alone, and is instead left as a flat direction to be lifted (if at all) by boundary conditions or additional terms in the effective action.

By contrast, the dual action (3.131) can be written, in the same adapted coordinate system, in the form

$$S = \frac{R^2}{4\pi} \int \sum_{\mu \neq j} \partial_\mu \phi \partial^\mu \phi \quad (\partial_j \phi = 0) . \quad (3.144)$$

Formally the quadratic functional of ϕ looks identical, but now the equation of motion for the auxiliary field imposes the constraint $\partial_j \phi = 0$ as a genuine condition on the allowed configurations. In the first formulation (3.119), the x^j -dependence of ϕ is completely unconstrained by the leading dynamics (a flat direction), whereas in the dual description (3.131) it is completely frozen by the constraint. This distinction will be important when we discuss how the compact scalar realizes subdimensional excitations and their dual descriptions.

Comments on further analyses The subdimensional compact scalar theory provides a simple but versatile arena for non-perturbative analyses, such as T-duality, bosonization, and leafwise conformal symmetries, as discussed in [93]. These developments indicate that even this minimal model captures many of the characteristic features of subdimensional phases, and it may serve as a useful continuum starting point for further explorations.

3.5 Matter theory coupling to b and ζ

We can also consider matter theories coupled to the higher-form gauge fields b and ζ appearing in (3.42). In what follows, we focus on the case with a two-form $b^{(2)}$ and a one-form $\zeta^{(1)}$, namely $p = d - 2$. For notational simplicity we will omit the superscripts indicating the form degree. As discussed in section 3.3.2, the endpoint excitation of a line operator built from ζ is identified with the fracton excitation living at the corner of the strip operator in the X-cube model described in section 3.2.2.

The gauge symmetry (3.45) acts simultaneously on b and ζ . Therefore, the conservation laws of the matter currents coupled to these fields are intertwined and are no longer independent.

We introduce classical currents $J_{1,\text{cl}}^{(1)}$ and $J_{2,\text{cl}}^{(2)}$ by adding the couplings

$$\zeta^{(1)} \wedge \star J_{1,\text{cl}}^{(1)}, \quad b^{(2)} \wedge \star J_{2,\text{cl}}^{(2)}, \quad (3.145)$$

to the action. The equations of motion then imply

$$d \star J_{1,\text{cl}}^{(1)} = 0, \quad (3.146)$$

$$\omega \wedge \star J_{1,\text{cl}}^{(1)} = 0, \quad (3.147)$$

and

$$d \star J_{2,\text{cl}}^{(2)} + \star J_{1,\text{cl}}^{(1)} = 0. \quad (3.148)$$

These equations show that the current $J_{1,\text{cl}}^{(1)}$ is conserved, has no component along the normal one-form of the foliation, and at the same time acts as a source for the two-form current $J_{2,\text{cl}}^{(2)}$. In other words, the conserved one-form current cannot flow independently: any flow of $J_{1,\text{cl}}^{(1)}$ necessarily induces a correlated flow of $J_{2,\text{cl}}^{(2)}$. This intertwined structure reproduces the characteristic kinematic constraint of fracton systems, where isolated charges are effectively immobile and motion is possible only in conjunction with higher-form (dipolar or fractal) degrees of freedom.

At present, however, an explicit quantum field theory in which both $J_{1,\text{cl}}^{(1)}$ and $J_{2,\text{cl}}^{(2)}$ arise as dynamical quantum currents is still unknown. Our discussion in this subsection should therefore be viewed as a semi-classical, constraint-based description of the desired fracton-like sector. Constructing a fully-fledged continuum quantum field theory that realizes these coupled currents, and clarifying its relation to lattice models of fracton order, remains an important problem for future work.

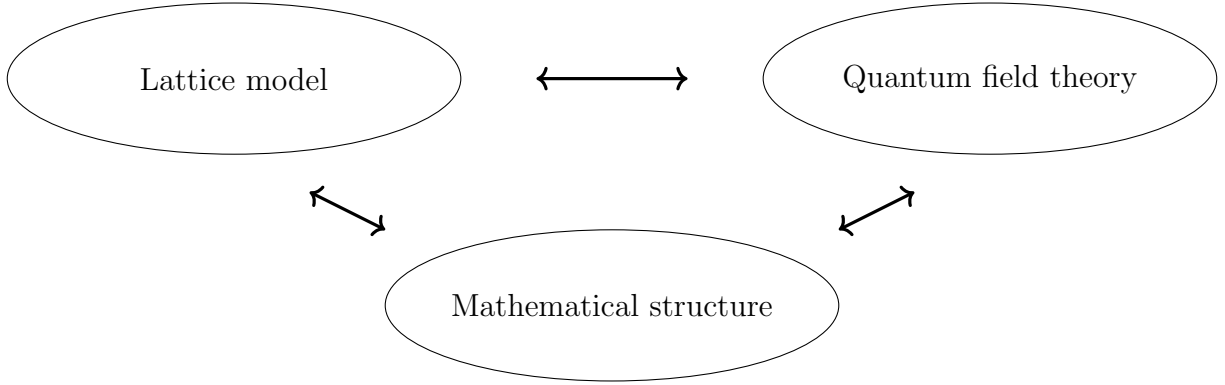


Figure 24: Schematic “triangle of ideas” relating lattice models, mathematical structures, quantum field theories. Each vertex represents, respectively, a microscopic Hamiltonian description on a lattice, its reformulation in terms of some mathematical data on a cell complex embedded in the underlying manifold, and a continuum field theory capturing the same mathematical structure. The arrows indicate correspondences between ground state degeneracies, excitation content, and mathematical invariants, emphasizing how algebraic topology mediates between discrete lattice realizations and continuum field-theoretic descriptions.

4 Discussion

In this thesis we have explored the interplay between lattice models with topological or foliated fracton order, the underlying mathematical structures of the spatial manifold, and continuum gauge theories. In section 2 we revisited the toric code as a paradigmatic example: its ground-state Hilbert space and logical operators can be expressed in terms of cellular (co)homology, and this combinatorial structure is in turn reproduced by a topological BF theory whose de Rham cohomology data encode the same information. This relationship was summarized as a “triangle of ideas” (Fig. 1). One of the main messages of the present work is that a closely analogous triangle can be constructed for foliated fracton phases.

In section 3, we introduced a Godbillon-Vey type $U(1)$ BF-like theory coupled to a codimension-one foliation $\langle\omega\rangle$. The key idea was to regard the redundancies of the foliation data (ω, η) as gauge symmetries and to build a continuum action whose gauge structure faithfully reflects these redundancies. Once this is done, the equations of motion and the gauge invariant operators automatically exhibit the restricted mobility characteristic of subdimensional excitations. In $(2+1)$ dimensions, the resulting theory reproduces the ground state degeneracy and subsystem one-form symmetries of the plaquette Ising model, while in $(3+1)$ dimensions the multi-foliated version captures the subextensive ground state degeneracy and operator algebra characteristic of X-cube-type models. In both cases, the continuum description provides a foliated analogue of the BF-toric-code correspondence (see Fig. 10).

A second message is that the foliation-coupled theory highlights a natural way in which geometry organizes the operator content of subdimensional particle theories. By elevating

the foliation redundancies to gauge symmetries, the continuum theory depends only on the underlying foliation $\langle\omega\rangle$, rather than on a specific choice of normal 1-form. Gauge invariant Wilson operators of the p -form field c are confined to live on individual leaves, while operators built from b and ζ interpolate between leaves in a controlled manner. The resulting algebra of extended operators is governed by leaf-wise intersection numbers, and the ground state degeneracy is expressed in terms of “foliation Betti numbers” and the number of leaves. In this sense, the foliation plays an organizing role for foliated fracton phases analogous to that of cellular (co)homology for the toric code, even though a fully systematic notion of “foliated (co)homology” has not yet been established.

We further coupled dynamical matter fields to the gauge field c , and showed that gauge invariance forces the associated conserved currents to flow entirely within the leaves of the foliation. This provides a continuum realization of codimension-one excitations and clarifies how subdimensional mobility emerges from the interplay between higher-form gauge symmetry and foliation geometry. In the compact boson example we also observed how subdimensional properties are implemented at the level of both the electric shift and magnetic winding currents, with the former realizing a genuinely subdimensional symmetry and the latter exhibiting only quasi-subdimensional mobility controlled by the dual foliation encoded in $\star\omega$ and the auxiliary field Λ .

Beyond these concrete models, the broader perspective developed here is that the “triangle of ideas” (Fig. 24) is not specific to cellular (co)homology or to foliation structures. As emphasized in section 2 and appendix A, many field theories implicitly rely on background geometric data that can be packaged as a G -structure or a suitable generalization. Foliations, Riemannian metrics, spin structures, almost complex structures, and contact structures are all examples of such background geometries.

The results of this thesis suggest that whenever a phase of matter or a family of field theories is naturally associated with a given mathematical structure, one should expect an analogous triangle relating

- lattice or other microscopic realizations,
- continuum (topological or anomaly) field theories encoding the same data,
- and invariants of the underlying geometric structure (such as (co)homology groups, characteristic classes, or cobordism groups).

In the foliated case, the role of “invariants” is played by foliation-sensitive (co)homology (even though a fully systematic formulation remains to be developed), and the corresponding continuum description is provided by the Godbillon-Vey BF action constructed in this work.

It is natural to expect that this framework can be extended to much more general settings. One direction is to replace codimension-one foliations by more general Haefliger structures or singular foliations, and to investigate whether suitable continuum gauge theories can be constructed whose operator algebras are controlled by the corresponding characteristic

classes. Another direction is to apply the same logic to other G -structures: for instance, to explore how contact structures might organize subdimensional excitations in systems with chiral edge modes, or how complex and Kähler structures might appear in generalized symmetry-protected phases. In the familiar BF–toric-code correspondence, the matching of ground-state sectors can be understood precisely as the canonical isomorphism between the integral cohomology of the spatial manifold and the cellular cohomology of a CW decomposition; in other words, what is really being identified are the characteristic cohomology classes associated with the underlying geometric structure. This suggests that, in many other situations, one should likewise be able to organize the relation between microscopic models and continuum descriptions in terms of the characteristic classes attached to the relevant geometric or topological structures. In all these cases, one may hope to realize a triangle relating microscopic models, geometric invariants, and continuum field theories in a way analogous to the toric-code/BF and foliation/Godbillon-Vey examples studied here.

Finally, it would be interesting to combine the present approach with the modern viewpoint on generalized and non-invertible symmetries and their symmetry topological field theories. The Godbillon-Vey BF-like theories constructed here naturally host subsystem higher-form symmetries whose operators are constrained by the foliation. Understanding their symmetry TFT descriptions, anomaly structures, and possible holographic interpretations is an important open problem for future work. We expect that pursuing these directions will further clarify the unifying role of geometric structures in organizing phases of matter and their continuum descriptions, and will extend the “triangle of ideas” to a broader class of mathematical and physical contexts.

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A Mathematical structure and spacetime symmetry

The modern axiomatic notion of a *mathematical structure* was systematically developed by Bourbaki [94]. Roughly speaking, a structure consists of an underlying set together with additional data (such as operations, relations, or topologies) and a list of axioms that this data must satisfy. Many familiar objects in mathematics, such as groups, topological spaces, manifolds, and so on, can be viewed in this unified way.

Category theory provides a convenient language to organize such structures and the maps that preserve them. A *category* consists of a collection of *objects* and *morphisms* between them, together with a composition law and identity morphisms. In practice, one often considers a category whose objects are mathematical structures of a given type (for example, all groups, or all manifolds endowed with a fixed kind of additional structure) and whose morphisms are structure-preserving maps between them. For a comprehensive introduction to this point of view, see [95].

Although we will not attempt a full systematic development here, let us record a few standard examples to build some intuition for what a mathematical structure and its associated category look like:

- **Top**: the category of topological spaces. The objects are sets equipped with a topology, and the morphisms are continuous maps.
- **Grp**: the category of groups. The objects are groups, and the morphisms are group homomorphisms.
- **Vect_k**: the category of vector spaces over a field k . The objects are k -vector spaces, and the morphisms are k -linear maps.

In each case, the “structure” is precisely the additional data on the underlying set (topology, group law, vector space structure) that the morphisms are required to preserve.

Among such examples, structures defined on smooth manifolds play a central role in field theories formulated via an action functional. These geometric data are conveniently described in terms of *G-structures* [96, 97], which may be viewed as reductions of the frame bundle from $\mathrm{GL}(d+1, \mathbf{R})$ to a subgroup G . A choice of G -structure encodes the background spacetime geometry and specifies which local frame transformations are regarded as spacetime symmetries of the theory.

A.1 G-structure and spacetime symmetry

In this subsection, we elaborate on the relationship between spacetime symmetries in field theories and background geometrical data in terms of G -structures and their generalizations.

The notion of a G -structure provides a unifying framework for a wide class of geometric structures on manifolds, including, for example, Riemannian and foliated structures. We will give a precise definition of G -structures later in this section and illustrate it with several basic examples relevant to the main text. These examples will clarify how specifying a background structure selects the allowed local frame transformations and thereby encodes the spacetime symmetries of a given field theory.

We begin by recalling what we mean by spacetime symmetries in a field theory. Consider a $(d+1)$ -dimensional spacetime manifold M_{d+1} . A spacetime symmetry is a diffeomorphism

$$f : M_{d+1} \rightarrow M_{d+1} , \quad (\text{A.1})$$

that preserves the chosen background geometric data on M_{d+1} . Equivalently, the background fields encoding the mathematical structure (G -structure, such as metric, foliation and so on) are taken to be invariant under the pullback by f . In the remainder of this appendix, we make this statement precise and explain how such symmetries are naturally phrased in terms of automorphisms of a given G -structure.

Under a general diffeomorphism f , we require the action of the field theory to be invariant:

$$S[a, b, \dots; X, Y, \dots] = S[f^*a, f^*b, \dots; f^*X, f^*Y, \dots] , \quad (\text{A.2})$$

where $a, b, \dots$ denote dynamical fields (such as a scalar field ϕ or a gauge field A), and $X, Y, \dots$ denote background fields encoding the chosen geometric structure (such as a foliation $\langle \omega \rangle$ or a metric g). Here f^* denotes the pullback induced by the diffeomorphism f acting on each field. In particular, if f preserves the background fields $X, Y, \dots$ in the sense that

$$f^*X = X , \quad f^*Y = Y , \dots , \quad (\text{A.3})$$

then the invariance condition (A.2) reduces to

$$S[a, b, \dots; X, Y, \dots] = S[f^*a, f^*b, \dots; X, Y, \dots] . \quad (\text{A.4})$$

This is precisely what is usually meant by a spacetime symmetry in the physics literature: the background fields are held fixed and only the dynamical fields are transformed. In this sense the backgrounds $X, Y, \dots$ play the role of external parameters, transforming trivially under those diffeomorphisms that are regarded as symmetries. The set of such diffeomorphisms forms the automorphism group of the chosen geometric structure (for example, of a given G -structure)

A.2 Examples of G -structures in field theories

To make this more concrete, let us now present several examples of field theories and their spacetime symmetry groups.

Example 1. Topological field theories

As a first illustration, let us consider ordinary topological field theories such as Chern-Simons theory [98] or topological BF theory [76]. From the present viewpoint, and ignoring internal gauge degrees of freedom, the only nontrivial background geometric datum on spacetime is an orientation μ on a smooth $(d+1)$ -dimensional manifold M_{d+1} . Thus (M_{d+1}, μ) is an object in the category of oriented manifolds, and its automorphism group is the group of orientation-preserving diffeomorphisms,

$$\text{Aut}(M_{d+1}; \mu) := \{ f \in \text{Diff}(M_{d+1}) \mid f^* \mu = \mu \}. \quad (\text{A.5})$$

Globally, the structure of $\text{Aut}(M_{d+1}; \mu)$ depends in a complicated way on the topology of M_{d+1} , and it is an infinite-dimensional group.

For many field-theoretic purposes, however, only the *local* form of the spacetime symmetry is relevant. Linearizing an orientation-preserving diffeomorphism in a neighborhood of a point, one obtains an element of the affine group acting on $\mathbf{R}^{d+1}$. Infinitesimally, this is described by the Lie algebra

$$\mathfrak{gl}(d+1, \mathbf{R}) \oplus \mathfrak{t}^{d+1}, \quad (\text{A.6})$$

where $\mathfrak{gl}(d+1, \mathbf{R})$ is the Lie algebra of the general linear group and $\mathfrak{t}^{d+1} \cong \mathbf{R}^{d+1}$ is the Lie algebra of translations. Exponentiating this algebra yields, in a local coordinate patch, the familiar orientation-preserving affine group,

$$\text{GL}^+(d+1, \mathbf{R}) \ltimes \mathbf{R}_{\text{translation}}^{d+1}, \quad (\text{A.7})$$

where $\text{GL}^+(d+1, \mathbf{R})$ denotes the subgroup of $\text{GL}(d+1, \mathbf{R})$ with positive determinant and $\mathbf{R}_{\text{translation}}^{d+1}$ is the additive group of translations. In this sense, an oriented topological field theory can be regarded as a field theory whose background spacetime structure is encoded by the oriented frame bundle with structure group $\text{GL}^+(d+1, \mathbf{R})$.

Example 2. Field theories with Lorentz symmetry

As a second example, consider a relativistic field theory with Lorentz invariance but without time-reversal symmetry. The standard model of particle physics with CP violation (and hence T violation by the CPT theorem) is a familiar instance of this situation for high-energy physicists.

In this case the relevant background geometric data on a smooth $(d+1)$ -dimensional spacetime M_{d+1} consist of a Lorentzian pseudo-Riemannian metric g , together with a choice of time orientation ξ and, when needed, a spacetime orientation μ . In the concrete setting of the standard model, it is natural to assume a fixed time orientation (to define causal structure and spinor representations) and a fixed spacetime orientation (relevant for parity-violating interactions such as the weak force). The triple (g, ξ, μ) specifies a time-oriented, oriented

Lorentzian manifold, and the associated automorphism group is the group of diffeomorphisms preserving these structures:

$$\text{Aut}(M_{d+1}; g, \xi, \mu) := \{f \in \text{Diff}(M_{d+1}) \mid f^*g = g, f^*\xi = \xi, f^*\mu = \mu\} . \quad (\text{A.8})$$

Globally, this is an infinite-dimensional group whose detailed structure depends on M_{d+1} , but its local form is easily understood. Linearizing around a point, one obtains the affine group generated by Lorentz transformations and translations. Infinitesimally this is captured by the Lie algebra

$$\mathfrak{so}(d, 1) \oplus \mathfrak{t}^{d+1} , \quad (\text{A.9})$$

where $\mathfrak{so}(d, 1)$ is the Lorentz Lie algebra and $\mathfrak{t}^{d+1} \cong \mathbf{R}^{d+1}$ denotes translations. Exponentiating, one arrives in a local coordinate patch at the usual time-orientation- and orientation-preserving Poincaré group

$$\text{SO}^+(d, 1) \ltimes \mathbf{R}_{\text{translation}}^{d+1} , \quad (\text{A.10})$$

where $\text{SO}^+(d, 1)$ denotes the connected component of the Lorentz group that preserves both the spacetime orientation and the chosen time orientation. In this sense, a relativistic field theory with Lorentz but not time-reversal symmetry can be regarded as a field theory whose background spacetime structure is encoded by a time-oriented, oriented Lorentzian G -structure with structure group $\text{SO}^+(d, 1)$.

Example 3. Conformal field theories

As a third example, let us consider (Euclidean) conformal field theories (CFTs). In addition to an orientation μ on a $(d+1)$ -dimensional manifold M_{d+1} , the relevant background data consist of a *conformal structure*, that is, an equivalence class $[g]$ of Riemannian metrics modulo Weyl rescalings:

$$g \sim g' \stackrel{\text{def}}{\iff} g' = e^\rho g , \quad \rho \in \mathbf{R} . \quad (\text{A.11})$$

Thus a Euclidean CFT is naturally viewed as a field theory defined on a conformal manifold $(M_{d+1}, [g], \mu)$.

The automorphism group of this background is the group of orientation-preserving diffeomorphisms that preserve the conformal class:

$$\text{Aut}(M_{d+1}; [g], \mu) := \{f \in \text{Diff}(M_{d+1}) \mid f^*[g] = [g], f^*\mu = \mu\} . \quad (\text{A.12})$$

In general, the global structure of this group depends sensitively on the choice of $(M_{d+1}, [g])$. However, in the special case where M_{d+1} is (conformally) flat, the infinitesimal conformal transformations form the familiar finite-dimensional conformal Lie algebra

$$\mathfrak{so}(d+2, 1) \cong (\mathbf{M} \oplus \mathbf{D} \oplus \mathbf{K}) \oplus \mathbf{P} , \quad (\text{A.13})$$

where $\mathbf{M}$, $\mathbf{D}$, $\mathbf{K}$ and $\mathbf{P} \cong \mathfrak{t}^{d+1}$ denote, respectively, the generators of rotations, dilatations, special conformal transformations, and translations.

Exponentiating this algebra, one obtains in a local flat coordinate patch the usual Euclidean conformal group

$$\mathrm{SO}(d+2, 1) . \quad (\text{A.14})$$

From the present viewpoint, a Euclidean CFT is therefore a field theory whose background spacetime data are encoded by a conformal G -structure with structure group $\mathrm{SO}(d+2, 1)$ acting on a conformal manifold $(M_{d+1}, [g], \mu)$.

Example 4. Field theories with foliation structure

As a final example, let us turn to field theories built on a foliation structure, such as the Godbillon-Vey-type field theories studied in this thesis. On a smooth $(d+1)$ -dimensional manifold M_{d+1} , we consider a codimension- q foliation specified (locally) by a collection of linearly independent one-forms $\langle \omega_i \rangle$, $i = 1, \dots, q$, whose common kernel defines the tangent distribution to the leaves.¹¹ We also assume an orientation μ on M_{d+1} . The background geometric data are thus $(\langle \omega_i \rangle, \mu)$.

The corresponding automorphism group is the group of orientation-preserving diffeomorphisms that map the foliation to itself:

$$\mathrm{Aut}(M_{d+1}; \langle \omega_i \rangle, \mu) := \{ f \in \mathrm{Diff}(M_{d+1}) \mid f^* \langle \omega_i \rangle = \langle \omega_i \rangle, f^* \mu = \mu \} , \quad (\text{A.15})$$

where $f^* \langle \omega_i \rangle = \langle \omega_i \rangle$ means that the pulled-back system of normal one-forms spans the same subbundle as the original one. As before, the global structure depends on the detailed topology of the foliation, but its local linear structure can be described more explicitly.

In a local frame adapted to the foliation, the tangent bundle splits into a direct sum of a $(d+1-q)$ -dimensional subbundle tangent to the leaves and an q -dimensional normal subbundle. The linear part of a foliation-preserving diffeomorphism at a point is then represented by a block upper-triangular matrix of the form

$$\begin{aligned} & \mathrm{GL}(d+1-q, q)^+ \\ & := \left\{ \begin{pmatrix} A & 0 \\ B & C \end{pmatrix} \in \mathrm{GL}^+(d+1, \mathbf{R}) \mid A \in \mathrm{GL}(d+1-q, \mathbf{R}), C \in \mathrm{GL}(q, \mathbf{R}), B \in \mathrm{Mat}_{q \times (d+1-q)}(\mathbf{R}) \right\} , \end{aligned} \quad (\text{A.16})$$

where the superscript $+$ again denotes the connected component of the identity (e.g. preserving the chosen orientation). The corresponding Lie algebra is

$$\mathfrak{gl}(d+1-q, q) := \left\{ \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \mid a \in \mathfrak{gl}(d+1-q, \mathbf{R}), c \in \mathfrak{gl}(q, \mathbf{R}), b \in \mathrm{Mat}_{q \times (d+1-q)}(\mathbf{R}) \right\} , \quad (\text{A.17})$$

¹¹Globally one should regard $\langle \omega_i \rangle$ as defining the normal bundle to the foliation, subject to the Frobenius integrability conditions, but the local description in terms of one-forms suffices for our purposes.

and the local spacetime symmetry algebra once again includes translations,

$$\mathfrak{gl}(d+1-q, q) \oplus \mathfrak{t}^{d+1}, \quad (\text{A.18})$$

so that in a local coordinate patch the foliation-preserving spacetime symmetry group takes the form

$$\text{GL}(d+1-q, q)^+ \ltimes \mathbf{R}_{\text{translation}}^{d+1}. \quad (\text{A.19})$$

From these examples we see that, for each choice of background mathematical structure on spacetime (orientation, metric, conformal class, foliation, and so on), the local spacetime symmetry can be decomposed into a semi-direct product of a matrix group and the translation group. The matrix-group part can be understood as the structure group of an appropriate G -structure, obtained by reducing the frame bundle from $\text{GL}(d+1, \mathbf{R})$ to a subgroup G that preserves the given geometric data. In this way, the notion of a G -structure provides a unified language for describing spacetime symmetries compatible with a specified background mathematical structure.

A.3 Reduction of translational symmetries in field theories with defects

So far we have focused on the relation between the *matrix part* of the spacetime symmetry group and the corresponding G -structure. In many physically relevant situations, however, not only the linear (matrix) part but also the translational symmetries are reduced to subgroups. This happens naturally when lower-dimensional defects, boundaries, or interfaces are present in spacetime. For example, in conformal field theories, extended defects break the full conformal group down to the subgroup that preserves the defect configuration [99, 100].

In the remainder of this appendix we illustrate how such symmetry reductions work in the presence of defects and how they can be incorporated into the G -structure viewpoint.

Example 1. Field theories on a punctured spacetime

As a first example of reduced translational symmetry, let us consider a $(d+1)$ -dimensional spacetime manifold M_{d+1} equipped with a distinguished point $x_0 \in M_{d+1}$ (a *puncture*, so that we have a pointed space). In the absence of any additional geometric data, the natural spacetime symmetry group that preserves the puncture is

$$\text{Aut}(M_{d+1}; x_0) := \{f \in \text{Diff}(M_{d+1}) \mid f(x_0) = x_0\}. \quad (\text{A.20})$$

The local form of this symmetry depends on whether one looks at the puncture itself or at a generic point. At the fixed point x_0 , any infinitesimal symmetry must leave x_0 fixed, so the translational part of (A.6) is completely broken. The remaining linear part is described by the Lie algebra

$$\mathfrak{gl}(d+1, \mathbf{R}), \quad (\text{A.21})$$

and the corresponding local symmetry group around x_0 is

$$\mathrm{GL}(d+1, \mathbf{R}) . \quad (\text{A.22})$$

In other words, near the puncture the spacetime symmetry is purely linear: all affine translations would move the marked point and are therefore not allowed.

By contrast, in a neighborhood of any point $x \neq x_0$ the presence of the puncture is invisible to the local geometry, and the spacetime symmetry reduces to the usual affine form discussed in (A.6) and (A.7). Thus this simple example already shows how specifying additional structures (such as a marked point) can partially break the translational part of the spacetime symmetry while leaving the linear part intact.

Example 2. Field theories in the presence of defects

As a second example, let us consider a $(d+1)$ -dimensional spacetime M_{d+1} containing a q -dimensional embedded defect $N_q \subset M_{d+1}$. In the absence of any additional geometric data, the natural notion of spacetime symmetry is the group of diffeomorphisms that preserve the defect as a set:

$$\mathrm{Aut}(M_{d+1}; N_q) := \{f \in \mathrm{Diff}(M_{d+1}) \mid f(N_q) = N_q\} . \quad (\text{A.23})$$

To understand the local structure of this symmetry, choose coordinates adapted to the defect so that points are labeled by $(x_{\parallel}, x_{\perp})$, with $x_{\parallel} \in \mathbf{R}^q$ along N_q and $x_{\perp} \in \mathbf{R}^{d+1-q}$ transverse to it. At a point on the defect, the linear part of a defect-preserving diffeomorphism cannot translate in the transverse directions (otherwise the defect would be moved), but can translate along the defect and act linearly on both tangent and normal directions. Infinitesimally, the corresponding Lie algebra takes the form

$$\mathfrak{gl}(q, \mathbf{R}) \oplus \mathfrak{gl}(d+1-q, \mathbf{R}) \oplus \mathfrak{t}^q , \quad (\text{A.24})$$

where $\mathfrak{gl}(q, \mathbf{R})$ and $\mathfrak{gl}(d+1-q, \mathbf{R})$ act on the tangent and normal directions, respectively, and $\mathfrak{t}^q \cong \mathbf{R}^q$ denotes translations along the defect worldvolume.

Exponentiating this algebra, the local symmetry group near a point on N_q can be represented as

$$(\mathrm{GL}(q, \mathbf{R}) \times \mathrm{GL}(d+1-q, \mathbf{R})) \ltimes \mathbf{R}_{\text{translation}}^q , \quad (\text{A.25})$$

where the matrix part corresponds to block-diagonal linear transformations preserving the splitting of the tangent space into directions parallel and perpendicular to the defect:

$$\mathrm{GL}(q, \mathbf{R}) \times \mathrm{GL}(d+1-q, \mathbf{R}) = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in \mathrm{GL}(d+1, \mathbf{R}) \mid A \in \mathrm{GL}(q, \mathbf{R}), B \in \mathrm{GL}(d+1-q, \mathbf{R}) \right\} . \quad (\text{A.26})$$

Away from the defect ($x \notin N_q$), the presence of N_q is not visible locally, and the spacetime symmetry reduces to the full affine symmetry (A.6) and (A.7). Thus the defect partially breaks translations and linear transformations, leaving only those that preserve the embedded submanifold N_q .

Example 3. Defect conformal field theories

As a third example, let us consider the case of defect conformal field theories (defect CFTs), which provide a particularly important class of systems with reduced spacetime symmetry. We start from a Euclidean CFT on a $(d+1)$ -dimensional manifold M_{d+1} equipped with a conformal structure $[g]$, and introduce a $(p+1)$ -dimensional planar or spherical defect

$$\mathcal{D}_{p+1} \subset M_{d+1} . \quad (\text{A.27})$$

The background data are therefore given by the triple

$$(M_{d+1}, [g], \mathcal{D}_{p+1}) , \quad (\text{A.28})$$

and the natural notion of spacetime symmetry is the group of defect-preserving conformal diffeomorphisms,

$$\text{Aut}(M_{d+1}; \mathcal{D}_{p+1}, [g]) := \{ f \in \text{Diff}(M_{d+1}) \mid f(\mathcal{D}_{p+1}) = \mathcal{D}_{p+1}, f^*[g] = [g] \} . \quad (\text{A.29})$$

In words, we restrict to diffeomorphisms that preserve the conformal class and map the defect submanifold into itself.

In flat space with a flat defect, the Lie algebra of infinitesimal transformations that leave $\mathcal{D}_{p+1}$ invariant is well known and takes the form

$$\mathfrak{so}(p+2, 1) \oplus \mathfrak{so}(d-p) . \quad (\text{A.30})$$

The first factor is the conformal algebra acting along the defect worldvolume $\mathcal{D}_{p+1}$, while the second factor corresponds to rotations in the $(d-p)$ directions transverse to the defect. Away from the defect, where its presence is not visible locally, the symmetry algebra reverts to the full bulk conformal algebra (A.13). This matches the usual representation-theoretic picture: bulk local primary operators transform in representations of $\mathfrak{so}(d+2, 1)$, while defect local primaries are organized by representations of $\mathfrak{so}(p+2, 1) \oplus \mathfrak{so}(d-p)$.

Exponentiating these Lie algebras yields the corresponding local symmetry groups: the bulk Euclidean conformal group $\text{SO}(d+2, 1)$ away from the defect, and the product

$$\text{SO}(p+2, 1) \times \text{SO}(d-p) \quad (\text{A.31})$$

along the defect. From the viewpoint of G -structures, a defect CFT can thus be regarded as a field theory on a conformal manifold endowed with a distinguished lower-dimensional submanifold $\mathcal{D}_{p+1}$, such that the spacetime symmetry is reduced to the subgroup that preserves both the conformal structure and the defect geometry.

B Comments on higher codimension foliations

In the main text, we have mainly focused on codimension-one foliations, which can be described locally by a single nowhere-vanishing normal one-form ω obeying the Frobenius

integrability condition $\omega \wedge d\omega = 0$. Choosing an auxiliary one-form η with

$$d\omega = \eta \wedge \omega, \quad (\text{B.1})$$

one can define the Godbillon-Vey three-form $\eta \wedge d\eta$, whose de Rham cohomology class is invariant under the redundancies of the pair (ω, η) and characterizes the foliation structure. These redundancies were encoded in the gauge symmetries of the Godbillon-Vey BF field theory constructed in section 3.3.

In this appendix we briefly comment on foliations of higher codimension, where the normal bundle has rank greater than one. Let $(M, \mathcal{F})$ be a codimension- q foliation with $q > 1$. Locally, one can choose a set of q linearly independent normal one-forms

$$\{\omega_a\}_{a=1,\dots,q}, \quad (\text{B.2})$$

whose common kernel defines the tangent distribution of the leaves. The Frobenius integrability condition can be written in terms of these forms as

$$d\omega_a \wedge \omega_1 \wedge \dots \wedge \omega_q = 0, \quad a = 1, \dots, q, \quad (\text{B.3})$$

which is equivalent to the existence of a matrix-valued one-form Θ such that

$$d\omega_a = -\Theta_{ab} \wedge \omega_b, \quad (\text{B.4})$$

where Θ_{ab} denotes the (a, b) -component of Θ . Geometrically, the collection $\{\omega_a\}$ gives a local coframe on the rank- q normal bundle $N\mathcal{F}$, while Θ plays the role of a connection one-form on this bundle. The defining equation (B.4) also exhibits an ω -shift redundancy analogous to the codimension-one case. Indeed, the equation is preserved under

$$\Theta_{ab} \longrightarrow \Theta_{ab} + u_a \omega_b, \quad (\text{B.5})$$

where u_a is a function.

A key difference from the codimension-one case is that the choice of normal coframe is now redundant up to non-Abelian frame rotations. At each point, one may perform a local change of basis

$$\omega_a \longrightarrow \omega'_a = R_{ab}(x) \omega_b, \quad (\text{B.6})$$

with $R(x) \in \text{GL}^+(q, \mathbf{R})$ a smooth matrix-valued function. To preserve (B.4), the matrix-valued one-form Θ must then transform as

$$\Theta \longrightarrow \Theta' = R \Theta R^{-1} + R dR^{-1}, \quad (\text{B.7})$$

where Θ denotes the matrix of one-forms whose (a, b) component is Θ_{ab} . This transformation is the familiar non-Abelian gauge transformation law for a $\text{GL}^+(q)$ connection (indeed, Θ is called the Bott connection). Thus, while in the codimension-one case, the redundancy

of (ω, η) is governed by an Abelian rescaling of the single normal one-form, in higher codimension the pair (ω_a, Θ) naturally carries a non-commutative gauge structure inherited from local rotations in the normal bundle. From this viewpoint, the appearance of non-Abelian structures is not an additional dynamical ingredient but an intrinsic feature of the redundancy in choosing a basis of normal one-forms.

It is natural to ask whether there exists an analogue of the Godbillon-Vey characteristic class for such higher-codimension foliations. For codimension one, the Godbillon-Vey class is represented by the three-form $\eta \wedge d\eta$ and is invariant under the redundancies $(\omega, \eta) \sim (\rho\omega, \eta + \xi)$ discussed in section 3.1. In the codimension- q case, a natural generalization of the Godbillon-Vey class is given by the de Rham cohomology class

$$[X \wedge (dX)^q], \quad X := \text{tr}(\Theta) \in \Omega^1(M), \quad (\text{B.8})$$

where Θ is the Bott connection on the normal bundle and X is its trace. We now sketch why this cohomology class is invariant under both (B.5) and (B.7).

First, we can characterize X in another way. Let $\omega_1, \dots, \omega_q$ be a local basis of defining one-forms for the codimension- q foliation, and set

$$\Omega := \omega_1 \wedge \omega_2 \wedge \dots \wedge \omega_q. \quad (\text{B.9})$$

The structure equation for the Bott connection implies

$$d\Omega = X \wedge \Omega. \quad (\text{B.10})$$

Applying the exterior derivative once more and using $d^2 = 0$, we obtain

$$0 = d^2\Omega = d(X \wedge \Omega) = (dX) \wedge \Omega - X \wedge d\Omega = (dX) \wedge \Omega, \quad (\text{B.11})$$

since $X \wedge X = 0$. Hence

$$dX \wedge \omega_1 \wedge \omega_2 \wedge \dots \wedge \omega_q = 0. \quad (\text{B.12})$$

This implies that dX can be written (non-uniquely) in the form

$$dX = \sum_{a=1}^q \alpha_a \wedge \omega_a, \quad (\text{B.13})$$

for some one-forms α_a .

Under the transformation (B.5), the Bott connection changes by a tensor proportional to the normal one-forms, and the trace transforms as

$$X \longrightarrow X' := X + \sum_{a=1}^q u_a \omega_a, \quad (\text{B.14})$$

for some functions u_a . Let us consider the one-parameter family

$$X_t := X + t \sum_{a=1}^q u_a \omega_a , \quad t \in [0, 1] , \quad (\text{B.15})$$

with $X_0 = X$ and $X_1 = X'$. A straightforward computation, using the decomposition $dX_t = \sum_a \alpha_a(t) \wedge \omega_a$ and the fact that each dX_t contains exactly one factor of ω_a , shows that

$$\frac{d}{dt} \left(X_t \wedge (dX_t)^q \right) = d(\text{a } (2q)\text{-form depending on } u_a \text{ and } X_t) . \quad (\text{B.16})$$

In particular, the derivative is an exact form, so integrating from $t = 0$ to $t = 1$ implies

$$X' \wedge (dX')^q - X \wedge (dX)^q = d(\text{a } (2q)\text{-form}) . \quad (\text{B.17})$$

Thus the de Rham cohomology class $[X \wedge (dX)^q]$ is invariant under the Bott ω -shift (B.5).

Next, we study the effect of (B.7). From (B.10), the transverse volume form of the foliation transforms as

$$\omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_q \longrightarrow (\det R) \omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_q , \quad (\text{B.18})$$

under the linear change of normal frame, where $R \in \text{GL}^+(q, \mathbf{R})$ is the transition matrix. Using (B.10) for both the original and transformed frames, we find

$$\begin{aligned} d((\det R) \Omega) &= d(\det R) \wedge \Omega + (\det R) d\Omega \\ &= \left(d(\log \det R) + X \right) \wedge (\det R) \Omega . \end{aligned} \quad (\text{B.19})$$

On the other hand, by definition of X' we also have

$$d((\det R) \Omega) = X' \wedge (\det R) \Omega . \quad (\text{B.20})$$

Comparing the two expressions, we obtain

$$X' = X - d(\log(\det R)) . \quad (\text{B.21})$$

Therefore, under the gauge transformation (B.7) the trace X changes by an exact one-form. Consequently,

$$X' \wedge (dX')^q - X \wedge (dX)^q = d(\text{a } (2q)\text{-form}) , \quad (\text{B.22})$$

and the de Rham cohomology class $[X \wedge (dX)^q]$ is also invariant under (B.7).

In summary, the generalized Godbillon-Vey class

$$\text{GV}_q(\mathcal{F}) := [X \wedge (dX)^q] \in H_{\text{dR}}^{2q+1}(M) \quad (\text{B.23})$$

is well defined for a codimension- q foliation and is independent of the choices of Bott connection and normal frame.

The generalized Godbillon-Vey class is the simplest example of a *secondary characteristic class* of a foliation. More systematically, the characteristic classes of codimension- q foliations can be described in terms of Haefliger structures and a universal differential graded algebra WO_q [101]. For each codimension- q foliation $(M, \mathcal{F})$ one can associate a classifying map $f_{\mathcal{F}} : M \rightarrow B\Gamma_q$, where $B\Gamma_q$ is the Haefliger classifying space of codimension- q foliations. Universal characteristic classes of foliations are then elements of $H^\bullet(B\Gamma_q; \mathbf{R})$, and their pullbacks $f_{\mathcal{F}}^* : H^\bullet(B\Gamma_q; \mathbf{R}) \rightarrow H^\bullet(M; \mathbf{R})$ give characteristic classes of $(M, \mathcal{F})$.

Bott and Haefliger showed that $H^\bullet(B\Gamma_q; \mathbf{R})$ can be computed as the cohomology of a certain universal differential graded algebra WO_q . This algebra is generated by elements c_i of degree $2i$ and h_i of degree $2i - 1$ (corresponding, respectively, to Chern-Weil polynomial invariants and their Chern-Simons transgressions), subject to the differential

$$dh_i = c_i, \quad dc_i = 0. \quad (\text{B.24})$$

The Bott connection Θ on the normal bundle of $\mathcal{F}$ defines a natural differential graded algebra (DGA) morphism

$$\Delta_{\mathcal{F}} : WO_q \longrightarrow \Omega^\bullet(M), \quad (\text{B.25})$$

whose induced map in cohomology coincides with the pullback $f_{\mathcal{F}}^*$. In this language, the generalized Godbillon-Vey class corresponds to the universal element $h_1 c_1^q \in WO_q$ of total degree $2q + 1$, and $\Delta_{\mathcal{F}}(h_1 c_1^q)$ is represented precisely by $X \wedge (dX)^q$.

A fundamental structural result is that the cohomology of WO_q vanishes in low degrees:

$$H^k(WO_q) = 0 \quad \text{for all } k \leq 2q. \quad (\text{B.26})$$

Equivalently, any universal characteristic class of codimension- q foliations represented by differential forms of degree $\leq 2q$ is cohomologous to zero. Nontrivial secondary characteristic classes can only appear in degrees strictly larger than $2q$, and the generalized Godbillon-Vey class in degree $2q + 1$ is the first such example. Higher-degree classes (sometimes called “exotic” or residual classes) are obtained from other monomials in the generators h_i and c_j , but all of them necessarily have degree $\geq 2q + 1$.

From the viewpoint of the present work, this general theorem explains why the natural foliation-dependent topological densities built from the Bott connection in codimension q begin at degree $2q + 1$. In particular, the generalized Godbillon-Vey form $X \wedge (dX)^q$ is the lowest-degree universal secondary characteristic form associated with a codimension- q foliation, and higher-degree invariants necessarily involve more complicated combinations of the curvature and Chern-Simons forms of the Bott connection.

There are some natural candidates for quantum field theories associated with higher-codimension foliations. Let us briefly discuss a simple toy model in the case of a codimension-two foliation, defined by a pair of normal one-forms ω_1, ω_2 and the associated trace one-form

$X = \text{tr}\Theta$ of the Bott connection, satisfying $d(\omega_1 \wedge \omega_2) = X \wedge \omega_1 \wedge \omega_2$. Inspired by the generalized Godbillon-Vey form $X \wedge (dX)^2$, one can consider a four-dimensional topological action of the form

$$S = \frac{k}{2\pi} \int_{M_4} [b^{(1)} \wedge dc^{(2)} - \lambda^{(1)} \wedge \omega_1 \wedge \omega_2 \wedge (d\phi + b^{(1)})] . \quad (\text{B.27})$$

Here $b^{(1)}$ and $\lambda^{(1)}$ are one-form gauge fields, $c^{(2)}$ is a two-form gauge field, and ϕ is a compact scalar field; ω_1, ω_2 and X are regarded as fixed background data encoding the codimension-two foliation.

This action is invariant (up to total derivatives) under the following set of local gauge transformations:

$$c^{(2)} \longrightarrow c^{(2)} + u \omega_1 \wedge \omega_2, \quad \lambda^{(1)} \longrightarrow \lambda^{(1)} - u X - du, \quad (\text{B.28})$$

$$c^{(2)} \longrightarrow c^{(2)} + d\xi^{(1)}, \quad (\text{B.29})$$

$$b^{(1)} \longrightarrow b^{(1)} + d\theta, \quad \phi \longrightarrow \phi - \theta, \quad (\text{B.30})$$

$$\lambda^{(1)} \longrightarrow \lambda^{(1)} + v \omega_1, \quad (\text{B.31})$$

$$\lambda^{(1)} \longrightarrow \lambda^{(1)} + v' \omega_2, \quad (\text{B.32})$$

$$\omega_a \longrightarrow \omega'_a = R_{ab} \omega_b, \quad \lambda^{(1)} \longrightarrow \lambda^{(1)'} = (\det R)^{-1} \lambda^{(1)}, \quad R \in \text{GL}^+(2, \mathbf{R}), \quad (\text{B.33})$$

with u, v, v' arbitrary functions and $\xi^{(1)}, \theta$ arbitrary one- and zero-forms. The transformation (B.28) implements the redundancy of shifting $c^{(2)}$ along the transverse volume form $\omega_1 \wedge \omega_2$, compensated by a shift of $\lambda^{(1)}$ involving the trace X of the Bott connection, while (B.29)-(B.32) encode the ordinary higher-form gauge symmetries and the freedom to add components of ω_1 and ω_2 to $\lambda^{(1)}$ without changing the constraint enforced by $\lambda^{(1)}$. Finally, (B.33) reflects the local change of normal frame, restricted here to the determinant part of $\text{GL}^+(2, \mathbf{R})$.

The equations of motion derived from S are

$$dc^{(2)} + \lambda^{(1)} \wedge \omega_1 \wedge \omega_2 = 0, \quad (\text{B.34})$$

$$db^{(1)} = 0, \quad (\text{B.35})$$

$$\omega_1 \wedge \omega_2 \wedge (d\phi + b^{(1)}) = 0, \quad (\text{B.36})$$

$$d(\lambda^{(1)} \wedge \omega_1 \wedge \omega_2) = 0. \quad (\text{B.37})$$

Equation (B.36) expresses a transversality condition for the combination $d\phi + b^{(1)}$ with respect to the two-dimensional normal distribution spanned by ω_1 and ω_2 , while (B.34) and (B.37) state that $dc^{(2)}$ is constrained to lie along $\omega_1 \wedge \omega_2$ and that the corresponding Lagrange multiplier $\lambda^{(1)}$ is conserved along the leaves.

As in the codimension-one case, one can construct nontrivial gauge-invariant operators. A natural electric-type surface operator is

$$\mathcal{W}_\Sigma(\beta) := \exp\left(i\beta \int_\Sigma c^{(2)}\right), \quad (\text{B.38})$$

for a closed surface $\Sigma \subset M_4$, and a Wilson-line-like operator built from the compact scalar and the one-form gauge field is

$$\mathcal{O}_\gamma(\gamma) := \exp(-i\gamma \phi(x)) \exp\left(i\gamma \int_{x'}^x b^{(1)}\right) \exp(i\gamma \phi(x')) , \quad (\text{B.39})$$

where the path from x' to x is taken within a leaf of the foliation. However, the interpretation of these operators in terms of subdimensional particle trajectories is more subtle than in the codimension-one model. In particular, the gauge symmetry associated with (B.28) constrains the two-form $c^{(2)}$ so that its support cannot carry an area component along the transverse 2-plane dual to $\omega_1 \wedge \omega_2$. This condition is weaker than the usual notion of “subdimensionality” in foliated fracton phases, where one demands that extended operators be entirely tangent to the leaves and thus have no extension in *either* of the normal directions picked out by ω_1 and ω_2 separately.

The toy model above is sensitive only to the determinant part of $\text{GL}(2, \mathbf{R})$ through the trace one-form $X = \text{tr}\Theta$, and hence realizes an effectively Abelian transverse gauge structure. From the perspective of the general theory of foliation characteristic classes, this corresponds to retaining only the generalized Godbillon-Vey class associated with the element $h_1 c_1^2 \in WO_2$. It is natural to expect that more intricate and genuinely non-Abelian examples can be constructed by coupling quantum field theories to the full set of secondary characteristic classes arising from the universal algebra WO_q , thereby probing more refined aspects of the transverse $\text{GL}(q, \mathbf{R})$ structure of the foliation.

C Canonical quantization in (2+1) dimensions

In section 3.3 we constructed the $U(1)$ BF-like field theory (3.42), motivated by the Godbillon-Vey invariant. In this appendix we give a detailed account of the canonical quantization of the (2+1)-dimensional theory (3.59), whose main steps were only sketched in the body of the thesis for reasons of space. Our treatment closely follows the canonical quantization of (2+1)-dimensional Chern-Simons theory presented in [102].

For the canonical analysis it is convenient to rewrite the Lagrangian density of (3.59) in a symmetric form, choosing local coordinates (x^0, x^1, x^2) and a normal one-form of the foliation $\omega = \omega_1 dx^1$:

$$\mathcal{L} = \frac{k}{4\pi} (\epsilon^{\mu\nu\rho} b_\mu \partial_\nu c_\rho + \epsilon^{\mu\nu\rho} c_\mu \partial_\nu b_\rho - \epsilon^{\mu\nu 1} \omega_1 \lambda_\mu \partial_\nu \phi + \epsilon^{\mu\nu 1} \partial_\nu (\omega_1 \lambda_\mu) \phi - 2 \epsilon^{\mu\nu 1} \omega_1 \lambda_\mu b_\nu) . \quad (\text{C.1})$$

Using this expression, we can read off the canonical momenta conjugate to each dynamical field (with $\dot{X} \equiv \partial_0 X$ and $\epsilon^{012} = +1$)¹²:

$$\begin{aligned} \pi_b^\mu &:= \frac{\partial \mathcal{L}}{\partial \dot{b}_\mu} = \frac{k}{4\pi} \epsilon^{0\mu\nu} c_\nu , & \pi_c^\mu &:= \frac{\partial \mathcal{L}}{\partial \dot{c}_\mu} = \frac{k}{4\pi} \epsilon^{0\mu\nu} b_\nu , \\ \pi_\lambda^0 &:= \frac{\partial \mathcal{L}}{\partial \dot{\lambda}_0} = 0 , & \pi_\lambda^2 &:= \frac{\partial \mathcal{L}}{\partial \dot{\lambda}_2} = \frac{k}{4\pi} \omega_1 \phi , & \pi_\phi &:= \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = -\frac{k}{4\pi} \omega_1 \lambda_2 . \end{aligned} \quad (\text{C.2})$$

¹²We adopt the convention $\epsilon^{012} = +1$ throughout this appendix.

The equal-time Poisson brackets $\{\cdot, \cdot\}_{\text{PB}}$ among these canonical variables take the standard form

$$\begin{aligned} \{X_\mu(\mathbf{x}), \pi_X^\nu(\mathbf{y})\}_{\text{PB}} &= \delta_\mu^\nu \delta^2(\mathbf{x} - \mathbf{y}), \quad X = b, c, \\ \{\lambda_0(\mathbf{x}), \pi_\lambda^0(\mathbf{y})\}_{\text{PB}} &= \{\lambda_2(\mathbf{x}), \pi_\lambda^2(\mathbf{y})\}_{\text{PB}} = \delta^2(\mathbf{x} - \mathbf{y}), \\ \{\phi(\mathbf{x}), \pi_\phi(\mathbf{y})\}_{\text{PB}} &= \delta^2(\mathbf{x} - \mathbf{y}), \end{aligned} \quad (\text{C.3})$$

where $\mathbf{x}$ and $\mathbf{y}$ denote the spatial coordinates (x^1, x^2) . In the following we will use these relations to identify the constraint structure of the theory and to construct the reduced phase space appropriate for the (2+1)-dimensional Godbillon-Vey BF system.

C.1 Primary and secondary constraints

We first need to find out all constraints in the U(1) BF-like theory, which can be mainly classified into primary and secondary constraints. From the expressions of canonical momenta (C.2), we can obtain the following primary constraints ($i = 1, 2$):

$$\pi_b^0 \approx 0, \quad \pi_c^0 \approx 0, \quad \pi_\lambda^0 \approx 0, \quad (\text{C.4})$$

$$\begin{aligned} \chi_b^i &:= \pi_b^i - \frac{k}{4\pi} \epsilon^{0ij} c_j \approx 0, & \chi_c^i &:= \pi_c^i - \frac{k}{4\pi} \epsilon^{0ij} b_j \approx 0, \\ \chi_\lambda^2 &:= \pi_\lambda^2 - \frac{k}{4\pi} \omega_1 \phi \approx 0, & \chi_\phi &:= \pi_\phi + \frac{k}{4\pi} \omega_1 \lambda_2 \approx 0. \end{aligned} \quad (\text{C.5})$$

Here, the symbol “ $\approx$ ” denotes the *weak* equality. In other words, $f \approx g$ for two functions f and g defined on the phase space means that f and g are equal to each other only if the constraints are satisfied. On the one hand, we refer to the ordinary equality “=” as the *strong* equality, meaning that it holds at every point in the phase space. In the rest of this section, we distinguish these two different equalities. We can then define the Hamiltonian H as follows:

$$\begin{aligned} H = \int d^2x & \left[-\frac{k}{4\pi} (\epsilon^{0ij} (b_0 \partial_i c_j + b_i \partial_j c_0 + c_0 \partial_i b_j + c_i \partial_j b_0)) \right. \\ & \quad \left. + \omega_1 \lambda_0 \partial_2 \phi - \partial_2 (\omega_1 \lambda_0) \phi + 2\omega_1 \lambda_0 b_2 - 2\omega_1 \lambda_2 b_0 \right. \\ & \quad \left. + \sum_{X=b,c} (\theta_0^X \pi_X^0 + \theta_i^X \chi_X^i) + \theta_0^\lambda \pi_\lambda^0 + \theta_2^\lambda \chi_\lambda^2 + \theta^\phi \chi_\phi \right], \end{aligned} \quad (\text{C.6})$$

where the θ 's in the final line denote the Lagrange multipliers associated to the primary constraints (C.4)-(C.5).

We also must require that all constraints satisfy, what is called, the *consistency conditions*, i.e., constraints must be invariant under the time evolution. These consistency conditions may give rise to new types of constraints, which are often referred to as the secondary

constraints. Indeed, by requiring that the primary constraints π_b^0 , π_c^0 and π_λ^0 are invariant under the time evolution (e.g., $\dot{\pi}_b^0 := \{\pi_b^0, H\}_{\text{PB}} \approx 0$), we can obtain the following secondary constraints:

$$\begin{aligned}\xi_b^0 &:= \dot{\pi}_b^0 = \frac{k}{2\pi} \epsilon^{0ij} \partial_i c_j - \frac{k}{2\pi} \omega_1 \lambda_2 \approx 0 , \\ \xi_c^0 &:= \dot{\pi}_c^0 = \frac{k}{2\pi} \epsilon^{0ij} \partial_i b_j \approx 0 , \quad \xi_\lambda^0 := \omega_1^{-1} \dot{\pi}_\lambda^2 = \frac{k}{2\pi} \partial_2 \phi + \frac{k}{2\pi} b_2 \approx 0 .\end{aligned}\tag{C.7}$$

We should notice that the consistency conditions on the primary constraint functions χ 's in (C.5) do not produce new constraints if we choose nice Lagrange multipliers from the beginning. For instance, the consistency condition on b_1 asserts

$$\dot{\chi}_b^1 = \{\chi_b, H\}_{\text{PB}} = \frac{k}{2\pi} (\partial_2 c_0 - \theta_2^c) \approx 0 .\tag{C.8}$$

This implies that if we take a Lagrange multiplier θ_2^c such that $\theta_2^c \approx \partial_2 c_0$, the consistency condition is automatically satisfied and no secondary constraints appear. As the same reasoning, we can conclude that if we choose some of the Lagrange multipliers such that

$$\theta_i^c \approx \partial_i c_0 - \delta_i^1 \omega_1 \lambda_0 , \quad \theta_i^b \approx \partial_i b_0 , \quad \theta_2^\lambda \approx \partial_2 \lambda_0 + \lambda_0 \partial_2 \log \omega_1 , \quad \theta^\phi \approx -b_0 ,\tag{C.9}$$

there are no secondary constraints stemming from the consistency conditions of χ 's.

Although we seem to have non-trivial constraints from the consistency conditions on secondary constraints (C.7), we have no further meaningful constraints. This is because the consistency conditions on secondary constraints are automatically satisfied thanks to (C.9). Indeed, the time derivatives of second constraint functions can be computed as

$$\dot{\xi}_b^0 \approx \frac{k}{2\pi} (\epsilon^{0ij} \partial_i \theta_j^c - \omega_1 \theta_2^\lambda) , \quad \dot{\xi}_c^0 \approx \frac{k}{2\pi} \epsilon^{0i\ell} \partial_i \theta_\ell^b , \quad \dot{\xi}_\lambda^0 \approx \frac{k}{2\pi} \omega_1 (\partial_2 \theta^\phi + \theta_2^b) ,\tag{C.10}$$

and we can readily check that all these vanish on the constrained surface by using (C.9). From the discussions so far, we can conclude that all the constraints appearing in the U(1) BF-like theory (3.59) are the primary constraints (C.4)-(C.5), and secondary constraints (C.7).

C.2 First, second class constraints and Dirac bracket

In the previous subsection, we succeeded in finding out the all constraints of the U(1) BF-like theory. We next classify the obtained constraints into the first class and second one to introduce the Dirac bracket [103]. Firstly, the primary constraints (C.4) clearly belong to the first class constraints since they weakly commute with the all other constraints. Also, at first glance, the secondary constraints (C.7) seem to be second class. However, they can be

linearly mapped to the first class constraints after the following linear transformations:

$$\begin{aligned}\tilde{\xi}_b^0 &:= \xi_b^0 + \partial_1 \chi_b^1 + \partial_2 \chi_b^2 + \chi_\phi = \frac{k}{4\pi} \epsilon^{0ij} \partial_i c_j + \partial_i \pi_b^i - \frac{k}{4\pi} \omega_1 \lambda_2 + \pi_\phi , \\ \tilde{\xi}_c^0 &:= \xi_c^0 + \partial_1 \chi_c^1 + \partial_2 \chi_c^2 = \frac{k}{4\pi} \epsilon^{0ij} \partial_i b_j + \partial_i \pi_c^i , \\ \tilde{\xi}_\lambda^0 &:= \xi_\lambda^0 + \partial_2 (\omega_1^{-1} \chi_c^1) + \chi_c^1 = \frac{k}{4\pi} \partial_2 \phi + \partial_2 (\omega_1^{-1} \pi_\lambda^2) + \pi_c^1 + \frac{k}{4\pi} b_2 .\end{aligned}\tag{C.11}$$

Indeed, we can check that the Poisson brackets between $\tilde{\xi}_b^0$, $\tilde{\xi}_c^0$ and $\tilde{\xi}_\lambda^0$, and all constraint functions vanishes. This phenomenon can be also seen in the contest of the canonical quantization of the Chern-Simons theory [102]. In summary, the all second class constraints of the U(1) BF-like theory are χ 's listed as (C.5).

Under the preparation in the above, we can compute the Dirac bracket $\{\cdot, \cdot\}_{\text{DB}}$ defined by

$$\{A(\mathbf{x}), B(\mathbf{y})\}_{\text{DB}} := \{A(\mathbf{x}), B(\mathbf{y})\}_{\text{PB}} - \int d^2 \mathbf{z} d^2 \mathbf{w} \{A(\mathbf{x}), \chi_I(\mathbf{z})\}_{\text{PB}} C_{IJ}^{-1}(\mathbf{z}, \mathbf{w}) \{\chi_J(\mathbf{w}), B(\mathbf{y})\}_{\text{PB}} ,\tag{C.12}$$

where I and J run over the all second class constraint functions, and $C_{IJ}(\mathbf{x}, \mathbf{y})$ is given by

$$C_{IJ}(\mathbf{x}, \mathbf{y}) := \{\chi_I(\mathbf{x}), \chi_J(\mathbf{y})\}_{\text{PB}} .\tag{C.13}$$

Exploiting the canonical relations (C.3), some of the important Dirac brackets can be computed as

$$\{b_i(\mathbf{x}), c_j(\mathbf{y})\}_{\text{DB}} = \frac{2\pi}{k} \epsilon_{0ij} \delta^2(\mathbf{x} - \mathbf{y}) , \quad \{\phi(\mathbf{x}), \lambda_2(\mathbf{y})\}_{\text{DB}} = \frac{2\pi}{k} \omega_1^{-1} \delta^2(\mathbf{x} - \mathbf{y}) ,\tag{C.14}$$

$$\{b_0(\mathbf{x}), \pi_b^0(\mathbf{y})\}_{\text{DB}} = \{c_0(\mathbf{x}), \pi_c^0(\mathbf{y})\}_{\text{DB}} = \{\lambda_0(\mathbf{x}), \pi_\lambda^0(\mathbf{y})\}_{\text{DB}} = \delta^2(\mathbf{x} - \mathbf{y}) .\tag{C.15}$$

We next deal with the first class constraints (C.4). These constraints are closely related to the gauge redundancies, hence we impose the following gauge fixing conditions:

$$b_0 \approx 0 , \quad c_0 \approx 0 , \quad \lambda_0 \approx 0 .\tag{C.16}$$

Remark that once we fix the gauge, the first class constraints turn into the second ones in general. As a result, we need to introduce a new Dirac bracket where the constraint functions in (C.4) also contribute to the matrix C_{IJ} . After some calculations in this new brackets, we find that all Dirac brackets in (C.15) vanish. To pass from the classical theory to the quantum one, we must replace the Dirac bracket to the commutator, and arrive at the following canonical commutation relations:

$$[b_i(\mathbf{x}), c_j(\mathbf{y})] = \frac{2\pi i}{k} \epsilon_{0ij} \delta^2(\mathbf{x} - \mathbf{y}) , \quad [\phi(\mathbf{x}), \lambda_2(\mathbf{y})] = \frac{2\pi i}{k} \omega_1^{-1} \delta^2(\mathbf{x} - \mathbf{y}) .\tag{C.17}$$

This completes the rigorous proof of the formula (3.67) used in the main text.

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