

BLOCK MATRIX OPERATORS FOR p -HYPONORMALITY

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ABSTRACT. We introduce a new model of block matrix operator $M(\alpha, \beta)$ induced by two sequences α and β and characterize its p -hyponormality. The model induces a measurable transformation T on the set of nonnegative integers $\mathbb{N}_0$ with point mass and composition operator C_T on $l^2 := l^2(\mathbb{N}_0)$. The techniques via composition operators will be used to treat p -hyponormality of $M(\alpha, \beta)$ and provide some interesting theorems about p -hyponormality. Finally, we apply our results to obtain examples of p -hyponormal making distinct as usual.

1. Introduction and Preliminaries. This was talked at the 2008 RIMS conference: Inequalities on linear operators and its applications, which was held at Kyoto University on January 30-February 1 in 2008.

Let $\mathcal{H}$ be a separable, infinite dimensional complex Hilbert space and let $\mathcal{L}(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be p -hyponormal if $(T^*T)^p \geq (TT^*)^p$, $p \in (0, \infty)$. If $p = 1$, T is hyponormal and if $p = \frac{1}{2}$, T is semi-hyponormal ([Xi]). In particular, T is said to be ∞ -hyponormal if it is p -hyponormal for all $p > 0$ ([MS]). The Löwner-Heinz inequality implies that every p -hyponormal operators are q -hyponormal operators for $q \leq p$ and many operator theorists have studied properties in operators in those classes; for examples, spectral theory, operator inequalities, and invariant subspaces, etc. (cf. [BJ], [Fur], [IY], [JKP], [JLPa]). Also, the study of gaps between subnormality and hyponormality has been studied in several areas by many operator theorists, and whose study is growing up still. The p -hyponormality is contained in those studies, but new models for p -hyponormal operators need to be developed still. And also, Jung-Lee-Park constructed examples induced by some block matrix operators in [JLP] and [JLL], in which the classes of those operators are distinct with respect to any positive real number p . Recently Burnap-Jung-Lambert discussed some models via composition operator C_T on L^2 in [BJL] and [BJ], in which such classes of weak hyponormal operators are distinct for each p . Moreover, they used the notion of conditional expectations for studying of p -hyponormality of C_T , which will be also main tool of this note. Here are some terminologies for conditional expectation. Let $(X, \mathcal{F}, \mu)$ be a σ finite measure space and let $T : X \rightarrow X$ be a transformation such that $T^{-1}\mathcal{F} \subset \mathcal{F}$ and $\mu \circ T^{-1} \ll \mu$. It is assumed that the Radon-Nikodym derivative $h = d\mu \circ T^{-1} / d\mu$ is in L^∞ . The composition operator C_T acting on $L^2 := L^2(X, \mathcal{F}, \mu)$ is defined by $C_T f = f \circ T$.

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The condition $h \in L^\infty$ assures that C_T is bounded. And we denote $Ef = E(f|T^{-1}\mathcal{F})$ for the conditional expectation of f with respect to $T^{-1}\mathcal{F}$. Some useful results will come from [L], [BJL], and [HWh]. In particular, in the proofs and examples below, we will have need of the following special case: if $\mathcal{A}$ is the purely atomic σ -subalgebra of $\mathcal{F}$ generated by the measurable partition of X into sets of positive measure $\{A_k\}_{k \geq 0}$, then

$$E(f|\mathcal{A}) = \sum_{k=0}^{\infty} \frac{1}{\mu(A_k)} \left(\int_{A_k} f(x) d\mu(x) \right) \chi_{A_k}.$$

The interested readers can find a more extensive list of properties for conditional expectations in [BJL] and [Ra].

This article consists of five sections. In Section 2, we construct a block matrix operator induced by two sequences α and β , which will make distinct classes of p -hyponormal operators with respect to $p > 0$ later section. A block matrix operator $M(\alpha, \beta)$ induced by two sequences α and β provides a measurable transformation T on $\mathbb{N}_0$ with point mass measure on $\mathbb{N}_0$ and its corresponding composition operator C_T on l^2 is equivalent to $M(\alpha, \beta)$. In Section 3, we characterize block matrix operators $M(\alpha, \beta)$ for p -hyponormality and construct a useful form for distinction examples. In Section 4, we discuss a flatness of p -hyponormality about block matrix operator $M(\alpha, \beta)$: the ∞ -hyponormality of $M(\alpha, \beta)$ is equivalent to any[some] p -hyponormality under some conditions. Finally, in Section 5, we give some examples being distinct the classes of p -hyponormal operators.

This article will be appeared in other journal as the full version. And so we skip the detail proofs here.

2. Relationships. Let $\alpha := \{a_i^{(n)}\}_{\substack{1 \leq i \leq r \\ 0 \leq n < \infty}}$ and $\beta := \{b_j^{(n)}\}_{\substack{1 \leq j \leq s \\ 0 \leq n < \infty}}$ be bounded sequences of positive real numbers. Let $M = [A_{ij}]_{0 \leq i, j < \infty}$ be a *block matrix operator* whose blocks are $(r+s) \times (s+1)$ matrices such that $A_{ij} = 0$, $i \neq j$, and

$$A_n := A_{nn} = \begin{pmatrix} a_1^{(n)} & & & & \\ & \ddots & & & \\ & & \bigcirc & & \\ a_r^{(n)} & & & & \\ & & b_1^{(n)} & & \\ & & & \ddots & \\ \bigcirc & & & & b_s^{(n)} \end{pmatrix}, \quad (2.1)$$

where other entries are 0 except $a_*^{(n)}$ and $b_*^{(n)}$ indicated in (2.1). Obviously such block matrix operator M is bounded.

Definition 2.1. For two bounded sequences $\alpha := \{a_i^{(n)}\}_{\substack{1 \leq i \leq r \\ 0 \leq n < \infty}}$ and $\beta := \{b_j^{(n)}\}_{\substack{1 \leq j \leq s \\ 0 \leq n < \infty}}$, the block matrix operator $M := M(\alpha, \beta)$ satisfying (2.1) is called a *block matrix operator with weight sequence* (α, β) .

Let M be a block matrix operator with weight sequence (α, β) and let $W_{\alpha, \beta}$ be its corresponding operator on l^2 relative to some orthonormal bases. Then $W_{\alpha, \beta}$ has a duplicate form; for example, if we take $r = 3$, $s = 2$ and $a_i^{(n)} = b_j^{(n)} = 1$ for all $i, j, n \in \mathbb{N}$, then the

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block matrix operator with (α, β) is unitarily equivalent to the following operator $W_{\alpha, \beta}$ on l^2 defined by

$$W_{\alpha, \beta}(x_1, x_2, x_3, x_4, x_5, \dots) = (\underbrace{x_1, x_1, x_1}_{(3)}, x_2, x_3, \underbrace{x_4, x_4, x_4}_{(3)}, x_5, x_6, \underbrace{x_7, x_7, x_7}_{(3)}, \dots).$$

For arbitrary block matrix operator M with weight sequence (α, β) , since M is p -hyponormal if and only if αM is p -hyponormal for any[some] positive real number α , we may assume $a_1^{(0)} = 1$, which will be assumed throughout this note.

We now return to our work, in particular, consider $X = \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and the power set $\mathcal{P}(X)$ of X for the σ -algebra $\mathcal{F}$. Define a non-singular measurable transformation T on $\mathbb{N}_0$ such that

$$T^{-1}(k(s+1)) = \{k(r+s) + i - 1 : 0 \leq i \leq r\}, \quad k = 0, 1, 2, \dots, \quad (2.2)$$

$$T^{-1}(k(s+1) + i) = k(r+s) + r - 1 + i, \quad 1 \leq i \leq s, \quad k = 0, 1, 2, \dots.$$

We write $m(\{i\}) := m_i$ for a point mass measure on X .

Proposition 2.2. *Under the above notation, the composition operator C_T on l^2 defined by $C_T f = f \circ T$ is unitarily equivalent to the block matrix operator $M(\alpha, \beta)$, where $\alpha : a_i^{(n)} = \sqrt{\frac{m_{n(r+s)+i-1}}{m_{n(s+1)}}}$ ($1 \leq i \leq r$) and $\beta : b_j^{(n)} = \sqrt{\frac{m_{n(r+s)+r+j-1}}{m_{n(s+1)+j}}}$ ($1 \leq j \leq s$), $n \in \mathbb{N}_0$.*

Proposition 2.3. *Let $M(\alpha, \beta)$ be a block matrix with weight sequence (α, β) , where $\alpha := \{a_i^{(n)}\}_{\substack{1 \leq i \leq r \\ 0 \leq n < \infty}}$, $\beta := \{b_j^{(n)}\}_{\substack{1 \leq j \leq s \\ 0 \leq n < \infty}}$, and $a_1^{(0)} = 1$. Then there exists a measurable transformation T on a σ finite measure space $(\mathbb{N}_0, \mathcal{P}(\mathbb{N}_0), m)$ such that $M(\alpha, \beta)$ is unitarily equivalent to a composition operator C_T on l^2 .*

3. Some Characterizations. Let T be a non-singular measurable transformation on l^2 as in (2.2) and let $m(\{i\}) = m_i$ be the point mass on $\mathbb{N}_0$.

Theorem 3.1. *Let $p \in (0, \infty)$. Then the following assertions are equivalent:*

- (i) C_T is p -hyponormal on l^2 ;
- (ii) the block matrix operator $M(\alpha, \beta)$ as in Proposition 2.2 is p -hyponormal;
- (iii) $E(1/h^p)(n) \leq 1/(h^p \circ T)(n)$
- (iv) it holds that

$$\frac{1}{m(T^{-1}(T(n)))} \sum_{j \in T^{-1}(T(n))} \frac{m_j^p m_j}{m(T^{-1}(j))^p} \leq \left(\frac{m_{T(n)}}{m(T^{-1}(T(n)))} \right)^p, \quad n \in \mathbb{N}_0.$$

Remark 3.2. By some formulas in the proof of Theorem 3.1, we have the following assertions:

- (i) $M(\alpha, \beta)$ is ∞ -hyponormal if and only if $m(T^{-1}(n))/m_n \geq m(T^{-1}(T(n)))/m(T(n))$ for all $n \in \mathbb{N}_0$.
- (ii) $M(\alpha, \beta)$ is quasinormal if and only if $m(T^{-1}(n))/m_n = m(T^{-1}(T(n)))/m(T(n))$ for all $n \in \mathbb{N}_0$.

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To obtain more useful and simpler form for p -hyponormality of $M(\alpha, \beta)$, we consider a block matrix operator M as following:

$$\begin{aligned} M(\alpha, \beta) : A &\equiv A_1 = A_2 = \dots \text{ (with notation in (2.1)) with} \\ \alpha : a_i^{(n)} &= a_i, \quad n \in \mathbb{N}_0, \quad 1 \leq i \leq r; \\ \beta : b_j^{(n)} &= b_j, \quad n \in \mathbb{N}_0, \quad 1 \leq j \leq s. \end{aligned} \quad (3.1)$$

This type will be used usefully to obtain examples being distinct classes of p -hyponormal operators in Section 5.

Theorem 3.3. *Let $M(\alpha, \beta)$ be as in (3.1). Then the block matrix operator $M(\alpha, \beta)$ is p -hyponormal if and only if the following two cases hold:*

(i) for $n = k(r + s) + i - 1$ ($1 \leq i \leq r$),

$$\begin{aligned} \sum_{\substack{j \in T^{-1}(T(n)) \\ j \equiv 0 \pmod{s+1}}} \left(\frac{1}{\sum_{1 \leq i \leq r} a_i^2} \right)^p \frac{a_{i_j}^2}{\sum_{1 \leq i \leq r} a_i^2} + \sum_{\substack{j \in T^{-1}(T(n)) \\ j \not\equiv 0 \pmod{s+1}}} \frac{1}{b_{l_j}^{2p}} \cdot \frac{a_{i_j}^2}{\sum_{1 \leq i \leq r} a_i^2} \\ \leq \left(\frac{1}{\sum_{1 \leq i \leq r} a_i^2} \right)^p, \quad 1 \leq i_j \leq r, \quad 1 \leq l_j \leq s, \end{aligned} \quad (3.2)$$

(ii) for $n = k(r + s) + r + j - 1$ ($1 \leq j \leq s$),

$$(ii-a) \quad b_j^2 \leq \sum_{1 \leq i \leq r} a_i^2 \quad \text{if } n \equiv 0 \pmod{s+1}$$

$$(ii-b) \quad b_j^2 \leq b_{t_n}^2 \quad \text{if } n \not\equiv 0 \pmod{s+1} \text{ and for some } t_n \text{ (} 1 \leq t_n \leq s \text{)}.$$

The following is a special case of Theorem 3.3, which provides a simple form.

Corollary 3.4. *Let $M := M(\alpha, \beta)$ be as in (3.1) with $a_i^{(n)} = a$ ($1 \leq i \leq r$) and $b_j^{(n)} = b$ ($1 \leq j \leq s$). Then M is p -hyponormal if and only if the following two cases hold:*

(i) for $n = k(r + s) + i - 1$ ($1 \leq i \leq r$),

$$\frac{1}{r} \left[\sum_{\substack{j \in T^{-1}(T(n)) \\ j \equiv 0 \pmod{s+1}}} \left(\frac{1}{ra^2} \right)^p + \sum_{\substack{j \in T^{-1}(T(n)) \\ j \not\equiv 0 \pmod{s+1}}} \frac{1}{b^{2p}} \right] \leq \left(\frac{1}{ra^2} \right)^p,$$

(ii) for $n = k(r + s) + r + j - 1$ ($1 \leq j \leq s$), $b^2 \leq ra^2$ holds.

Note that if we are under type of Theorem 3.3 (which will be called "type I") it will be important to know *which* j in $T^{-1}(T(n))$ have various $j \equiv t_j \pmod{s+1}$ which if we are under type of Corollary 3.4 (which will be called "type II") it is only important to know *how many* j are of various $j \equiv t_j \pmod{s+1}$. Then we have the following remark.

Remark 3.5 (Special case of Corollary 3.4 with $r = N(s + 1)$). In this case for $n = n(r + s) + i - 1$, $1 \leq i \leq r$, the set of l in $T^{-1}(T(n))$ contains exactly N elements of each modulus, $\pmod{s+1}$. So under type II the test (3.2) for such n becomes

$$N \left(\frac{1}{ra^2} \right)^p \frac{1}{r} + (r - N) \left(\frac{1}{b^2} \right)^p \frac{1}{r} \leq \left(\frac{1}{ra^2} \right)^p.$$

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For $n = k(r + s) + r - 1 + j$, and under type II we either get a condition trivially satisfied for all p , or $1/(ra^2) \leq 1/b^2$, the latter only if there is at least one n so that $n = K(r + s) + r - 1 + j$ and $n = Q(s + 1)$. But since $r = N(s + 1)$, this is $(K + 1)N(s + 1) + Ks + j - 1 = Q(s + 1)$ for some K, Q, j , and take $K = s + 1$ and $j = 1$ to obtain a solution, so $1/(ra^2) \leq 1/b^2$.

Remark 3.6. We can apply the idea of Theorem 3.3 to the model of general block matrix operator in the Definition 2.1 by the same method; the result formula will be slight complete than that of Theorem 3.3. We leave the exact formula to interested readers.

4. ∞ -hyponormality and Flatness. We begin this section with the following fundamental lemma.

Lemma 4.1. Suppose $p > 1$ and $q > 1$ are relatively prime. Given any l_p , $0 \leq l_p \leq p - 1$, and any l_q , $0 \leq l_q \leq q - 1$, there exists $n \in \mathbb{N}$ so that $n \equiv l_p \pmod{p}$ and $n \equiv l_q \pmod{q}$.

Lemma 4.2. Suppose that

$$A := \begin{pmatrix} \sqrt{y_1} & & & & \\ & \ddots & & & \\ & & \sqrt{y_r} & & \\ & & & \sqrt{x_1} & \\ & & & & \ddots \\ O & & & & & \sqrt{x_s} \end{pmatrix} \quad \text{and} \quad M := \begin{pmatrix} A & & \\ & A & \\ & & \ddots \end{pmatrix}. \quad (4.1)$$

Assume that $\text{GCD}(r + s, s + 1) = 1$. If M is p -hyponormal for some $p \in (0, \infty)$, then

$$x_1 = x_2 = \cdots = x_s \leq \sum_{1 \leq i \leq r} y_i. \quad (4.2)$$

Proposition 4.3. Let A and M be as in (4.1). Suppose there exists $N \in \mathbb{N}$ such that $r = N(s + 1)$ and $\text{GCD}(r + s, s + 1) = 1$. Then the following assertions are equivalent:

- (i) M is p -hyponormal for some $p \in (0, \infty)$;
- (ii) M is ∞ -hyponormal;
- (iii) $x_1 = x_2 = \cdots = x_s = \sum_{1 \leq i \leq r} y_i$.

5. Examples. Let A and M be as in (4.1) with $r + s = N(s + 1)$ for some $N \in \mathbb{N}$ and we will see this is the "opposite" of $r = N(s + 1)$ and $\text{GCD}(r + s, s + 1) = 1$.

Proposition 5.1. Let M be the block matrix operator as in (4.1). Then M is p -hyponormal if and only if the following inequality holds:

$$\sum_{\substack{j \not\equiv 0 \pmod{s+1} \\ j \in T^{-1}(T(n))}} \left(\frac{1}{x_{t_j \pmod{s+1}}} \right)^p y_{j+1} \leq \frac{1}{(\sum_{1 \leq i \leq r} y_i)^p} \sum_{\substack{j \not\equiv 0 \pmod{s+1} \\ j \in T^{-1}(T(n))}} y_{j+1}. \quad (5.1)$$

The following corollaries come immediately from Proposition 5.1.

Corollary 5.2. Let M be the block matrix operator as in (4.1) with $x_1 = x_2 = \cdots = x_s = x$. Then (5.1) is trivially satisfied as long as $x \geq \sum_{1 \leq i \leq r} y_i$ with no conditions on the y_j .

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Corollary 5.3. *Let M be the block matrix operator as in (4.1) such that the y_{j+1} for $j \equiv 0 \pmod{s+1}$ occur only in $\sum_{1 \leq i \leq r} y_i$. Thus if we consider some y'_{j+1} for $j \equiv 0 \pmod{s+1}$, as long as $\sum_{j \equiv 0} y'_{j+1} = \sum_{j \equiv 0} y_{j+1}$, then M' is p -hyponormal if and only if M is p -hyponormal.*

Now we close this paper with the following example.

Example 5.4. Let

$$A := \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & \sqrt{x_1} \\ & & & & & \sqrt{x_2} \end{pmatrix} \quad \text{and} \quad M := \begin{pmatrix} A & & \\ & A & \\ & & \ddots \end{pmatrix}.$$

Write Y for $\sum_{1 \leq i \leq 4} y_i$. Then the condition of

$$\frac{1}{Y^p} \frac{y_1}{Y} + \frac{1}{x_1^p} \frac{y_2}{Y} + \frac{1}{x_2^p} \frac{y_3}{Y} + \frac{1}{Y^p} \frac{y_4}{Y} \leq \frac{1}{Y^p}$$

is equivalent to

$$\frac{y_2}{x_1^p} + \frac{y_3}{x_2^p} \leq \frac{y_2 + y_3}{4^p}.$$

Inserting the $y_i \equiv 1$, $1 \leq i \leq 4$, we get

$$\left(\frac{4}{x_1}\right)^p + \left(\frac{4}{x_2}\right)^p \leq 2, \quad (5.2)$$

which is equivalent to M is p -hyponormal. Note that (5.2) keeps distinct the classes of p -hyponormal operators with respect to $0 < p < \infty$. To obtain region for ∞ -hyponormality of M we use Remark 3.2 and formulas in proof of Theorem 3.3, and there are three cases, Cases 1a, 1b, and 2b, which imply that $m_{3k_1} \geq m_{3k}$, $x_1 \geq 4$ & $x_2 \geq 4$, and $x_1 \geq x_1$ & $x_2 \geq x_2$, respectively. Thus we obtain that

$$M \text{ is } \infty\text{-hyponormal} \iff x_1 \geq 4 \text{ and } x_2 \geq 4.$$

Of course, since (5.2) is equivalent to $x_2 \geq 4 \cdot (2 - (4/x_1)^p)^{-1/p}$ for $x_1 > 4 \cdot 2^{-1/p}$, taking $p \rightarrow \infty$, we may check easily the obtaining conditions ∞ -hyponormality of M are $x_1 \geq 4$ and $x_2 \geq 4$. On the other hand, applying Remark 3.2 and formulas in proof of Theorem 3.3 for quasinormality of M , we also obtain that M is quasinormal if and only if $(x_1, x_2) = (4, 4)$.

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REFERENCES

- [BJ] C. Burnap and I. Jung, *Composition operators with weak hyponormality*, J. Math. Anal. Appl., to appear.
- [BJL] C. Burnap, I. Jung and A. Lambert, *Separating partial normality classes with composition operators*, J. Operator Theory, **53**(2005), 381-397.
- [CHol] J. Campbell and W. Hornor, *Seminormal composition operators*, J. Operator Theory, **29**(1993), 323-343.
- [EJL] G. Exner, I. Jung and M. Lee, *Block matrix operators and p -hyponormality*, preprint.
- [Fur] T. Furuta, *Invitation to linear operators*, Taylor & Francis Inc., 2001.
- [L] A. Lambert, *Hyponormal composition operators*, Bull. London Math. Soc., **18**(1986), 395-400.
- [HWh] D. Harrington and R. Whitley, *Seminormal Composition Operators*, J. Operator Theory, **11**(1984), 125-135.
- [IY] M. Ito and T. Yamazaki, *Relations between two inequalities $(B^{r/2} A^p B^{r/2})^{r/(p+r)} \geq B^r$ and $A^p \geq (A^{p/2} B^r A^{p/2})^{p/(p+r)}$ and their applications*, Integral Equations Operator Theory, **44**(2002) 442-450.
- [JKP] I. Jung, E. Ko, and C. Pearcy, *Aluthge transforms of operators*, Integral Equations Operator Theory, **37**(2000), 437-448.
- [JLL] I. Jung, M. Lee and P. Lim, *Gaps of operators, II*, Glasgow Math. J., **47**(2005), 461-469.
- [JLP] I. Jung, P. Lim and S. Park, *Gaps of operators*, J. Math. Anal. Appl., **304**(2005), 87-95.
- [JLPa] I. Jung, M. Lee and S. Park, *Separating classes of composition operators via subnormal condition*, Proc. A.M.S., to appear
- [MS] S. Miyajima and I. Saito, *∞ -hyponormal operators and their spectral properties*, Acta Sci. Math. (Szeged) **67**(2001), 357-371.
- [Ra] M. Rao, *Conditional measures and Applications*, Marcel Decker, New York 1993.
- [Xi] D. Xia, *Spectral Theory of hyponormal Operators*, Birkhäuser, Boston, 1983.