

A note on global dynamics for nonlinear Schrödinger equations on star graphs

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Abstract

In this article, we give an announcement of our recent result on the dynamics of nonlinear Schrödinger equation on star graphs. We first discuss the related minimizing problem of the action functional. After that, we present the main statement, and then mention the profile decomposition for the Schrödinger propagator on star graph, which is the main ingredient of the proof.

§ 1. Introduction

The present article is an announcement of our result (Theorem 4.1) in the preprint [19], which is later published in *Partial Differ. Appl.* **5**, (2024), no. 5, Paper No. 4. This article will mainly discuss more technical details about the backgrounds omitted in [19].

A *star graph* $\mathcal{G}$ with N edges is a union of N half-lines whose origins are joined at one vertex. This is one of the most simplest examples of metric graphs, which are simplified models of quasi-one-dimensional region with nontrivial junctions (see [12, 13]). We identify $\mathcal{G}$ with a union of N copies of half-line $[0, \infty)$, and formulate a function on

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$\mathcal{G}$ by a vector-valued function $\mathbf{f} : [0, \infty) \rightarrow \mathbb{C}^N$. We express the interaction at the origin in the definition of the *Laplace operator*:

$$\mathcal{D}(\Delta_{\mathcal{G}}^{\gamma}) := \left\{ \mathbf{f} = (f_k)_{k=1}^N \in H^2(0, \infty)^N \left| \begin{array}{l} f_1(0) = \cdots = f_N(0), \\ \sum_{k=1}^N f'_k(0+) = N\gamma f_1(0) \end{array} \right. \right\},$$

$$\Delta_{\mathcal{G}}^{\gamma} \mathbf{f} := (\partial_{xx} f_k)_{1 \leq k \leq N}.$$

When $\gamma = 0$, the boundary condition at the origin is called the *Kirchhoff condition*, which corresponds to the case of free interaction. When $\gamma > 0$ (resp. $\gamma < 0$), it is called *repulsive* (resp. *attractive*) *delta condition*, which corresponds to the case when the Dirac delta potential is present at the origin. It is known that $\Delta_{\mathcal{G}}^{\gamma}$ is a self-adjoint operator on $L^2(\mathcal{G})$, where $L^2(\mathcal{G}) := L^2(0, \infty)^N$ (see [27]). In general, a pair of metric graph and the Hamiltonian operator defined on it is referred to as *quantum graph*, and there are many studies on its spectral properties ([12, 13]). In this article, we particularly consider the case when $\gamma \geq 0$, where no point spectrum of $\Delta_{\mathcal{G}}^{\gamma}$ exists.

We introduce some notations here. We define the Lebesgue spaces and Sobolev spaces on the star graph $\mathcal{G}$ as follows:

$$L^q(\mathcal{G}) := L^q(0, \infty)^N, \quad W^{s,q}(\mathcal{G}) := W^{s,q}(0, \infty)^N,$$

whose norms are defined by

$$\|\mathbf{f}\|_{L^q(\mathcal{G})} := \left[\sum_{k=1}^N \|f_k\|_{L^q(0, \infty)}^q \right]^{1/q}, \quad \|\mathbf{f}\|_{W^{s,q}(\mathcal{G})} := \left[\sum_{k=1}^N \|f_k\|_{W^{s,q}(0, \infty)}^q \right]^{1/q}$$

for $1 \leq q \leq \infty$ and $s = 1, 2$. We use the standard notation $H^s(\mathcal{G}) := W^{s,2}(\mathcal{G})$, equipped with the inner product

$$(\mathbf{f}, \mathbf{g})_{H^s(\mathcal{G})} := \sum_{k=1}^N (f_k, g_k)_{H^s(0, \infty)}.$$

We use $H_c^s(\mathcal{G})$ ($s = 1, 2$) to denote the set of the functions in $H^s(\mathcal{G})$ which are continuous at the origin.

The study of linear and nonlinear Schrödinger equations on star graphs is a topic attracting growing interest. The dispersive estimate for the Schrödinger propagator has been established in [9, 10, 18]. The existence of standing waves and their stability are investigated, for example, in [1, 2, 3, 24, 23, 6, 5, 22, 17]. There are several recent studies on the dynamics of nonlinear Schrödinger equation. For example, [7] constructs a solution with modified scattering asymptotics when the nonlinearity is long-range.

It is also shown by [8] that no solution scatters to a standing wave for the long-range nonlinearities.

§ 2. Nonlinear Schrödinger equation

In this article, we consider the focusing nonlinear Schrödinger equation (NLS) on $\mathcal{G}$ with Kirchhoff boundary or repulsive delta boundary condition:

$$(2.1) \quad i\partial_t \mathbf{u} + \Delta_{\mathcal{G}}^{\gamma} \mathbf{u} = \mathcal{N}(\mathbf{u}), \quad (t, x) \in \mathbb{R} \times (0, \infty),$$

where $\mathbf{u}(t, x) = (u_k(t, x))_{k=1}^N : \mathbb{R} \times [0, \infty) \rightarrow \mathbb{C}^N$ is the unknown function, and where we suppose $p > 5$, $\gamma \geq 0$, $N \geq 3$. We also set $\mathcal{N}(\mathbf{u}) := (-|u_k|^{p-1}u_k)_{k=1}^N$. Note that the condition $p > 5$ implies that (2.1) is L^2 -supercritical problem. Also the sign of the nonlinearity corresponds to the focusing case. If the solution $\mathbf{u}$ is in $C(\mathbb{R}; H_c^1(\mathcal{G}))$, $\mathbf{u}$ enjoys the conservation laws:

$$(\text{Mass}) \quad M(\mathbf{u}(t)) := \|\mathbf{u}(t)\|_{L^2(\mathcal{G})}^2,$$

$$(\text{Energy}) \quad E_{\gamma}(\mathbf{u}(t)) := \frac{1}{2} \|\partial_x \mathbf{u}(t)\|_{L^2(\mathcal{G})}^2 + \frac{N\gamma}{2} |u_1(0)|^2 - \frac{1}{p+1} \|\mathbf{u}(t)\|_{L^{p+1}(\mathcal{G})}^{p+1}.$$

It is known that (2.1) is locally well-posed in the energy class:

Theorem 2.1 (Local well-posedness of (2.1) [1, 5, 14]). *Let $p > 1$ and $\gamma \geq 0$. For any $\mathbf{u}_0 \in H_c^1(\mathcal{G})$, there exist $T_{\max} \in (0, \infty]$ and $T_{\min} \in [-\infty, 0)$ such that (2.1) has a unique solution $\mathbf{u} \in C((T_{\min}, T_{\max}); H_c^1(\mathcal{G}))$, where $(T_{\min}, T_{\max})$ is called maximal lifespan of $\mathbf{u}$. For each compact interval $I \subset (T_{\min}, T_{\max})$, the mapping $H_c^1(\mathcal{G}) \ni \mathbf{u}_0 \mapsto \mathbf{u} \in C(I; H_c^1(\mathcal{G}))$ is continuous. Moreover, the solution $\mathbf{u}$ has the following blow-up alternative: If $T_{\max} < \infty$ (resp. $T_{\min} > -\infty$), then*

$$\lim_{t \nearrow T_{\max}} \|\mathbf{u}(t)\|_{H^1(\mathcal{G})} = \infty \quad \left(\text{resp.} \quad \lim_{t \searrow T_{\min}} \|\mathbf{u}(t)\|_{H^1(\mathcal{G})} = \infty \right).$$

Furthermore, the solution $\mathbf{u}$ to (2.1) preserves the mass M and energy E_{γ} with respect to time t .

In this article, we are particularly interested in the global-in-time behavior of the solutions to (2.1). For the sake of comparison, let us recall the corresponding result in the $\mathbb{R}$ -case:

$$(2.2) \quad i\partial_t u + \Delta_{\mathbb{R}}^{\gamma} u = -|u|^{p-1}u, \quad (t, x) \in \mathbb{R} \times \mathbb{R},$$

where $p > 5$, $\gamma \geq 0$ are constants, $u = u(t, x) : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{C}$ is unknown, and $\Delta_{\mathbb{R}}^{\gamma}$ is defined as

$$\begin{aligned} \mathcal{D}(\Delta_{\mathbb{R}}^{\gamma}) &:= \{f \in H^1(\mathbb{R}) \cap H^2(\mathbb{R} \setminus \{0\}) : f'(0+) - f'(0-) = 2\gamma f(0)\}, \\ \Delta_{\mathbb{R}}^{\gamma} f &:= \partial_{xx} f. \end{aligned}$$

(2.2) also enjoys the conservation of the mass and the energy:

$$\begin{aligned} M^{\text{line}}(f) &:= \|f\|_{L^2(\mathbb{R})}^2, \\ E_\gamma^{\text{line}}(f) &:= \frac{1}{2} \|\partial_x f\|_{L^2(\mathbb{R})}^2 + \gamma |f(0)|^2 - \frac{1}{p+1} \|f\|_{L^{p+1}(\mathbb{R})}^{p+1}. \end{aligned}$$

In terms of the dynamics of (2.2), the action $S_{\omega,\gamma}^{\text{line}}$ and the virial functional K_γ^{line} defined below play an important role:

$$\begin{aligned} S_{\omega,\gamma}^{\text{line}}(f) &:= E_\gamma^{\text{line}}(f) + \frac{\omega}{2} M^{\text{line}}(f), \\ K_\gamma^{\text{line}}(f) &:= 2 \|\partial_x f\|_{L^2(\mathbb{R})}^2 + 2\gamma |f(0)|^2 - \frac{p-1}{p+1} \|f\|_{L^{p+1}(\mathbb{R})}^{p+1} \end{aligned}$$

for $\omega > 0$ and $\gamma \geq 0$. Let us first consider the following minimizing problem:

$$(2.3) \quad \mathbf{n}_{\omega,\gamma}^{\text{line}} := \inf \{ S_{\omega,\gamma}^{\text{line}}(\phi) : \phi \in H^1(\mathbb{R}) \setminus \{0\}, K_\gamma^{\text{line}}(\phi) = 0 \}.$$

Concerning this, the following is known:

Proposition 2.2 ([16]). *Let $\gamma \geq 0$, and $\omega > 0$. Then the following holds:*

(1) *If $\gamma = 0$, $\mathbf{n}_{\omega,0}^{\text{line}}$ is attained by*

$$Q_{\omega,0}(x) := \left[\frac{(p+1)\omega}{2} \operatorname{sech}^2 \left(\frac{(p-1)\sqrt{\omega}}{2} |x| \right) \right]^{\frac{1}{p-1}}.$$

(2) *If $\gamma > 0$, $\mathbf{n}_{\omega,\gamma}^{\text{line}} = \mathbf{n}_{\omega,0}^{\text{line}}$ holds and $\mathbf{n}_{\omega,\gamma}^{\text{line}}$ is not attained.*

Remark 1. (i) $w = Q_{\omega,0}$ is the unique positive solution, up to the translation, of the elliptic equation:

$$(2.4) \quad -\partial_{xx} w + \omega w = |w|^{p-1} w.$$

If w is a solution to (2.4), then $u(x) = e^{i\omega t} w(x)$ is a solution to (2.2) with $\gamma = 0$. The solutions of this form are called the *standing waves*, and $e^{i\omega t} Q_{\omega,0}$ corresponds to a nontrivial standing wave with least action.

(ii) When $\gamma > 0$, the infimum (2.3) is *virtually* attained when the function $Q_{\omega,0}$ lies in one side of the half line. (See [16] for the detail.)

Then the following dichotomy has been proven:

Theorem 2.3 ([15, 4] when $\gamma = 0$, [20] when $\gamma > 0$). *Let $p > 5$ and $\gamma \geq 0$. Let u be a solution to (2.2) with initial data $u_0 \in H^1(\mathbb{R})$. We assume that $S_{\omega,\gamma}^{\text{line}}(u_0) < \mathbf{n}_{\omega,\gamma}^{\text{line}} (= \mathbf{n}_{\omega,0}^{\text{line}})$ for some $\omega > 0$. Then the following holds:*

- (Scattering) If $K_\gamma^{\text{line}}(u_0) \geq 0$, then the solution u to (2.2) exists globally-in-time and scatters in both time directions; Namely, there exist $\psi_\pm \in H^1(\mathbb{R})$ such that

$$\lim_{t \rightarrow \pm\infty} \left\| u(t) - e^{it\Delta_\mathbb{R}} \psi_\pm \right\|_{H^1(\mathbb{R})} = 0.$$

- (Blow-up or grow-up) If $K_\gamma^{\text{line}}(u_0) < 0$, then the solution u to (2.2) blows up or grows up in both time directions:

$$\limsup_{t \nearrow T_{\max}} \|u(t)\|_{H^1(\mathbb{R})} = \limsup_{t \searrow T_{\min}} \|u(t)\|_{H^1(\mathbb{R})} = \infty.$$

The theorem says if the action is less than the (virtual) ground state level, the dynamics of the solutions to (2.1) can be characterized by the sign of the virial functional. (We can show that the sign does not change under the above restriction of action.) If $\gamma = 0$, this threshold is critical since the ground state solutions neither scatter, blow up nor grow up.

§ 3. The minimizing problem on the star graph

Let us go back to the star graph case. The action and virial functionals in this case are

$$\begin{aligned} S_{\omega,\gamma}(\mathbf{f}) &:= E_\gamma(\mathbf{f}) + \frac{\omega}{2}M(\mathbf{f}), \\ K_\gamma(\mathbf{f}) &:= 2\|\partial_x \mathbf{f}\|_{L^2(\mathcal{G})}^2 + N\gamma|f_1(0)|^2 - \frac{p-1}{p+1}\|\mathbf{f}\|_{L^{p+1}(\mathcal{G})}^{p+1} \end{aligned}$$

for $\omega > 0$ and $\gamma \geq 0$. From the analogy of $\mathbb{R}$ -case, we consider the minimization problem:

$$(3.1) \quad \mathfrak{n}_{\omega,\gamma} := \inf\{S_{\omega,\gamma}(\mathbf{f}) : \mathbf{f} \in H_c^1(\mathcal{G}) \setminus \{\mathbf{0}\}, K_\gamma(\mathbf{f}) = 0\}.$$

Then we have the following:

Proposition 3.1. *Let $p > 5$, $\gamma \geq 0$, and $\omega > 0$. Then $\mathfrak{n}_{\omega,\gamma} = \mathfrak{n}_{\omega,0}^{\text{line}}$ holds and $\mathfrak{n}_{\omega,\gamma}$ is not attained.*

It is worth noting that the non-attainability happens also in the Kirchhoff boundary condition. The proof is essentially given by Adami et al. [1], where the constraint is replaced by the Nehari functional, instead of the virial functional. We present a direct proof here.

Proof. For simplicity, we only prove in the case $N = 3$. We divide the proof into two steps.

Step 1. We first show $\mathbf{n}_{\omega,\gamma} = \mathbf{n}_{\omega,0}^{\text{line}}$. By the constraint, we may substitute $S_{\omega,\gamma}$ by

$$\begin{aligned} U_{\omega,\gamma}(\mathbf{f}) &:= S_{\omega,\gamma}(\mathbf{f}) - \frac{1}{4}K_{\gamma}(\mathbf{f}) \\ &= \frac{\omega}{2}\|\mathbf{f}\|_{L^2(\mathcal{G})}^2 + \frac{3\gamma}{4}|f_1(0)|^2 + \frac{p-5}{4(p+1)}\|\mathbf{f}\|_{L^{p+1}(\mathcal{G})}^{p+1}. \end{aligned}$$

For later use, we also define the corresponding functional on $\mathbb{R}$ as

$$U_{\omega,\gamma}^{\text{line}}(f) := \frac{\omega}{2}\|f\|_{L^2(\mathbb{R})}^2 + \frac{\gamma}{2}|f(0)|^2 + \frac{p-5}{4(p+1)}\|f\|_{L^{p+1}(\mathbb{R})}^{p+1}.$$

Now we introduce the following lemma, which is shown later.

Lemma 3.2. *The following holds:*

$$\mathbf{n}_{\omega,\gamma} = \inf\{U_{\omega,\gamma}(\mathbf{f}) : \mathbf{f} \in H_c^1(\mathcal{G}) \setminus \{\mathbf{0}\}, K_{\gamma}(\mathbf{f}) \leq 0\}.$$

Since $U_{\omega,\gamma}(\mathbf{f}) \geq U_{\omega,0}(\mathbf{f})$ and $K_{\gamma}(\mathbf{f}) \geq K_0(\mathbf{f})$ hold for all $\mathbf{f} \in H_c^1(\mathcal{G})$, we have $\mathbf{n}_{\omega,\gamma} \geq \mathbf{n}_{\omega,0}$ by Lemma 3.2. Therefore, it suffices to show

$$(A) \quad \mathbf{n}_{\omega,\gamma} \leq \mathbf{n}_{\omega,0}^{\text{line}}, \quad \text{and} \quad (B) \quad \mathbf{n}_{\omega,0} \geq \mathbf{n}_{\omega,0}^{\text{line}}.$$

Let $\chi \in C^\infty(\mathbb{R})$ satisfy $\chi(x) = 0$ for $x < 1$ and $\chi(x) = 1$ for $x > 2$. We define

$$\mathbf{v}_n(x) = \begin{pmatrix} \{Q_{\omega,0}(x-n)\chi(x)\}\mathbb{1}_{>0} \\ 0 \\ 0 \end{pmatrix}, \quad (n \in \mathbb{Z}).$$

Then $\mathbf{v}_n \in H_c^1(\mathcal{G})$ for any $n \in \mathbb{N}$, and we have $S_{\omega,\gamma}(\mathbf{v}_n) \rightarrow S_{\omega,\gamma}^{\text{line}}(Q_{\omega,0}) = \mathbf{n}_{\omega,0}^{\text{line}}$. Moreover,

$$(3.2) \quad K_{\gamma}(\mathbf{v}_n) \rightarrow K_0^{\text{line}}(Q_{\omega,0}) = 0,$$

where we use spatial decay of $Q_{\omega,0}$. By (3.2), there exists λ_n such that $K_{\gamma}(\lambda_n \mathbf{v}_n) = 0$ and $\lambda_n \rightarrow 1$ as $n \rightarrow \infty$. Then we also have $S_{\omega,\gamma}(\lambda_n \mathbf{v}_n) \rightarrow \mathbf{n}_{\omega,0}^{\text{line}}$. This means that $\mathbf{n}_{\omega,\gamma} \leq \mathbf{n}_{\omega,0}^{\text{line}}$, and hence (A) follows. Next we shall prove (B). By Lemma 3.2, we can take a sequence $\{\varphi_n\}_n$ such that $U_{\omega,0}(\varphi_n) \rightarrow \mathbf{n}_{\omega,0}$ and $K_0(\varphi_n) \leq 0$. By symmetric rearrangement [1, Proposition A.1 and Theorem 6], we get $\varphi_n^* \in H_{c,\text{rad}}^1(\mathcal{G})$ such that

$$\|\varphi_n^*\|_{L^q(\mathcal{G})} = \|\varphi_n\|_{L^q(\mathcal{G})}, \quad \|\partial_x \varphi_n\|_{L^2(\mathcal{G})} \geq \frac{2}{3}\|\partial_x \varphi_n^*\|_{L^2(\mathcal{G})},$$

for $q \in [1, \infty)$, where $H_{c,\text{rad}}^1(\mathcal{G})$ is the space of all radially symmetric functions in $H_c^1(\mathcal{G})$. Therefore, we have $U_{\omega,0}(\varphi_n^*) = U_{\omega,0}(\varphi_n)$ and

$$K_0(\varphi_n) \geq 2 \left(\frac{2}{3} \right)^2 \|\partial_x \varphi_n^*\|_{L^2(\mathcal{G})}^2 - \frac{p-1}{p+1} \|\varphi_n^*\|_{L^{p+1}(\mathcal{G})}^{p+1}.$$

Hence we obtain

$$\mathbf{n}_{\omega,0} \geq \inf\{U_{\omega,0}(\mathbf{f}) : \mathbf{f} \in H_{c,\text{rad}}^1(\mathcal{G}) \setminus \{\mathbf{0}\}, 2\left(\frac{2}{3}\right)^2 \|\partial_x \mathbf{f}\|_{L^2(\mathcal{G})}^2 - \frac{p-1}{p+1} \|\mathbf{f}\|_{L^{p+1}(\mathcal{G})}^{p+1} \leq 0\}.$$

Setting $\mathbf{f}_\lambda(x) := \lambda^{\frac{1}{p+1}} \mathbf{f}(\lambda x)$ with $\lambda^{\frac{p+3}{p+1}} = \left(\frac{3}{2}\right)^2$, we obtain

$$\begin{aligned} & 2\left(\frac{2}{3}\right)^2 \|\partial_x \mathbf{f}_\lambda\|_{L^2(\mathcal{G})}^2 - \frac{p-1}{p+1} \|\mathbf{f}_\lambda\|_{L^{p+1}(\mathcal{G})}^{p+1} \\ &= 2\left(\frac{2}{3}\right)^2 \lambda^{\frac{p+3}{p+1}} \|\partial_x \mathbf{f}\|_{L^2(\mathcal{G})}^2 - \frac{p-1}{p+1} \|\mathbf{f}\|_{L^{p+1}(\mathcal{G})}^{p+1} \\ &= K_0(\mathbf{f}) \end{aligned}$$

and

$$U_{\omega,0}(\mathbf{f}_\lambda) = \frac{\omega}{2} \lambda^{-\frac{p-1}{p+1}} \|\mathbf{f}\|_{L^2(\mathcal{G})}^2 + \frac{p-5}{4(p+1)} \|\mathbf{f}\|_{L^{p+1}(\mathcal{G})}^{p+1} = U_{\omega \lambda^{-\frac{p-1}{p+1}},0}(\mathbf{f}).$$

Thus it holds that

$$\begin{aligned} & \inf\{U_{\omega,0}(\mathbf{f}) : \mathbf{f} \in H_{c,\text{rad}}^1(\mathcal{G}) \setminus \{\mathbf{0}\}, 2\left(\frac{2}{3}\right)^2 \|\partial_x \mathbf{f}\|_{L^2(\mathcal{G})}^2 - \frac{p-1}{p+1} \|\mathbf{f}\|_{L^{p+1}(\mathcal{G})}^{p+1} \leq 0\} \\ &= \inf\{U_{\omega \lambda^{-\frac{p-1}{p+1}},0}(\mathbf{f}) : \mathbf{f} \in H_{c,\text{rad}}^1(\mathcal{G}) \setminus \{\mathbf{0}\}, K_0(\mathbf{f}) \leq 0\}, \end{aligned}$$

where λ satisfies $\lambda^{\frac{p+3}{p+1}} = \left(\frac{3}{2}\right)^2$. By the symmetry, we obtain

$$\begin{aligned} & \inf\{U_{\omega \lambda^{-\frac{p-1}{p+1}},0}(\mathbf{f}) : \mathbf{f} \in H_{c,\text{rad}}^1(\mathcal{G}) \setminus \{\mathbf{0}\}, K_0(\mathbf{f}) \leq 0\} \\ &= \frac{3}{2} \inf\{U_{\omega \lambda^{-\frac{p-1}{p+1}},0}^{\text{line}}(f) : f \in H_{\text{rad}}^1(\mathbb{R}) \setminus \{0\}, K_0^{\text{line}}(f) \leq 0\} \\ &= \frac{3}{2} \mathbf{n}_{\omega \lambda^{-\frac{p-1}{p+1}},0}^{\text{line}}, \end{aligned}$$

where we have used the well-known fact that $\inf\{U_{\omega,0}^{\text{line}}(f) : f \in H_{\text{rad}}^1(\mathbb{R}) \setminus \{0\}, K_0^{\text{line}}(f) \leq 0\} = \mathbf{n}_{\omega,0}^{\text{line}}$, which is shown in the same way as Lemma 3.2. It is also well-known that

$$\mathbf{n}_{\omega,0}^{\text{line}} = \omega^{\frac{p+3}{2(p-1)}} \mathbf{n}_{1,0}^{\text{line}}.$$

Therefore, we get

$$\frac{3}{2} \mathbf{n}_{\omega \lambda^{-\frac{p-1}{p+1}},0}^{\text{line}} = \frac{3}{2} \left(\lambda^{-\frac{p-1}{p+1}}\right)^{\frac{p+3}{2(p-1)}} \mathbf{n}_{\omega,0}^{\text{line}} = \frac{3}{2} \lambda^{-\frac{p+3}{2(p+1)}} \mathbf{n}_{\omega,0}^{\text{line}} = \mathbf{n}_{\omega,0}^{\text{line}}.$$

Therefore, we obtain (B).

Step 2. Finally we show the non-attainability. Suppose that there exists a minimizer $\varphi \in H_c^1(\mathcal{G})$, i.e., $S_{\omega,\gamma}(\varphi) = \mathbf{n}_{\omega,\gamma}$ and $K_\gamma(\varphi) = 0$. By Lemma 3.2, $U_{\omega,\gamma}(\varphi) = \mathbf{n}_{\omega,\gamma} = \mathbf{n}_{\omega,0}^{\text{line}}$. Recalling $U_{\omega,\gamma}(\varphi) = \frac{\omega}{2} \|\varphi\|_{L^2(\mathcal{G})}^2 + \frac{3\gamma}{4} |\varphi_1(0)|^2 + \frac{p-5}{4(p+1)} \|\varphi\|_{L^{p+1}(\mathcal{G})}^{p+1}$, we have

$$(3.3) \quad \mathbf{n}_{\omega,0}^{\text{line}} \geq \sum_{k \in \{k_1, k_2\}} \frac{\omega}{2} \|\varphi_k\|_{L^2(0,\infty)}^2 + \frac{3\gamma}{4} |\varphi_{k_1}(0)|^2 + \frac{p-5}{4(p+1)} \|\varphi_k\|_{L^{p+1}(0,\infty)}^{p+1}$$

for arbitrary two elements $k_1, k_2 \in \{1, 2, 3\}$. Suppose that $\varphi_k = 0$ on $(0, \infty)$ for some $k \in \{1, 2, 3\}$. Without loss of generality, we may assume that $\varphi_3 = 0$ on $(0, \infty)$. Then $\varphi_{1,2}$, which denotes the function on the line which is generated by combination of φ_1 and φ_2 , satisfies

$$(3.4) \quad U_{\omega,\gamma}^{\text{line}}(\varphi_{1,2}) \leq \mathbf{n}_{\omega,0}^{\text{line}} \quad \text{and} \quad K_\gamma^{\text{line}}(\varphi_{1,2}) = 0.$$

By the result of the minimization problem on the line, we get $\varphi_{1,2}(x) = e^{i\theta} Q_{\omega,0}(x-y)$ for some $\theta \in \mathbb{R}$ and $y \in \mathbb{R}$. Since $Q_{\omega,0}(x) \neq 0$ for any x , φ is not continuous at the origin. This is a contradiction. Hence the function φ_k is not identically zero for all $k \in \{1, 2, 3\}$. Therefore, the equality in (3.3) does not occur for arbitrary $k_1, k_2 \in \{1, 2, 3\}$. By the result of the minimization problem on the line, we obtain

$$(3.5) \quad K_\gamma^{\text{line}}(\varphi_{k_1, k_2}) > 0$$

for arbitrary two elements $k_1, k_2 \in \{1, 2, 3\}$, where φ_{k_1, k_2} denotes the function on the line which is generated by combination of φ_{k_1} and φ_{k_2} . Then we have

$$(3.6) \quad 0 < \sum_{1 \leq k_1 < k_2 \leq 3} K_\gamma^{\text{line}}(\varphi_{k_1, k_2}) = 2K_\gamma(\varphi).$$

This is a contradiction. □

Proof of Lemma 3.2. It suffices to show

$$\mathbf{n}_{\omega,\gamma} \leq \inf\{U_{\omega,\gamma}(\mathbf{f}) : \mathbf{f} \in H_c^1(\mathcal{G}) \setminus \{\mathbf{0}\}, K_\gamma(\mathbf{f}) \leq 0\} =: \mathbf{l}_{\omega,\gamma}.$$

By definition, there exists $\{\varphi_n\}_{n \in \mathbb{N}} \subset H_c^1(\mathcal{G}) \setminus \{0\}$ such that $K_\gamma(\varphi_n) \leq 0$ and $\mathbf{l}_{\omega,\gamma} \leq U_{\omega,\gamma}(\varphi_n) \leq \mathbf{l}_{\omega,\gamma} + \frac{1}{n}$. Now we set

$$\lambda_n := \left(\frac{2\|\partial_x \varphi_n\|_{L^2(\mathcal{G})}^2 + 3\gamma|\varphi_{n,1}(0)|^2}{\frac{p-1}{p+1} \|\varphi_n\|_{L^{p+1}(\mathcal{G})}^{p+1}} \right)^{\frac{1}{p-1}}.$$

Then we have $K_\gamma(\lambda_n \varphi_n) = 0$ and $\lambda_n \in [0, 1]$. Hence it follows that

$$\mathbf{n}_{\omega,\gamma} \leq S_{\omega,\gamma}(\lambda_n \varphi_n) = U_{\omega,\gamma}(\lambda_n \varphi_n) \leq U_{\omega,\gamma}(\varphi_n) \leq \mathbf{l}_{\omega,\gamma} + \frac{1}{n}.$$

Since n is arbitrary, we get $\mathfrak{n}_{\omega,\gamma} \leq \mathfrak{l}_{\omega,\gamma}$. $\square$

§ 4. Global dynamics for NLS on star graphs

We announce that we proved an analogous result to Theorem 2.3 in the star graph case:

Theorem 4.1 ([19]). *Let $p > 5$ and $\gamma \geq 0$. Let $\mathbf{u}$ be a solution to (2.1) with initial data $\mathbf{u}_0 \in H_c^1(\mathcal{G})$. We assume that $S_{\omega,\gamma}(\mathbf{u}_0) < \mathfrak{n}_{\omega,\gamma} = \mathfrak{n}_{\omega,0}^{\text{line}}$ for some $\omega > 0$. Then the following holds:*

- (Scattering) *If $\mathbf{u}_0$ satisfies $K_\gamma(\mathbf{u}_0) \geq 0$, then the solution $\mathbf{u}$ to (2.1) exists globally-in-time, and scatters in both time directions; Namely, $\mathbf{u}$ satisfies $T_{\max} = \infty$ and $T_{\min} = -\infty$, and there exist $\psi_\pm \in H_c^1(\mathcal{G})$ such that*

$$\lim_{t \rightarrow \pm\infty} \left\| \mathbf{u}(t) - e^{it\Delta_{\mathcal{G}}^\gamma} \psi_\pm \right\|_{H^1(\mathcal{G})} = 0.$$

- (Blow-up or grow-up) *If $\mathbf{u}_0$ satisfies $K_\gamma(\mathbf{u}_0) < 0$, then the solution $\mathbf{u}$ to (2.1) blows up or grows up in both time directions; Namely,*

$$\limsup_{t \nearrow T_{\max}} \|\mathbf{u}(t)\|_{H^1(\mathcal{G})} = \limsup_{t \searrow T_{\min}} \|\mathbf{u}(t)\|_{H^1(\mathcal{G})} = \infty.$$

We give a remark on the above theorem. The threshold level of action is that of the ground state in the $\mathbb{R}$ -case. It is natural to ask whether this is the actual threshold of the dichotomy. Indeed, when $\gamma = 0$ and $N \geq 3$, it is known that $\mathfrak{n}_{\omega,0}^{\text{line}}$ is strictly less than that of any of the known standing waves:

Theorem 4.2 ([1]). *Suppose $\gamma = 0$ and $\omega > 0$. Then the following holds:*

- (i) *If N is odd, then (2.1) has a standing wave solution $e^{i\omega t} \varphi_\omega^0(x)$ with*

$$\varphi_\omega^0 = (\varphi_\omega^0)_{k=1}^N, \quad \varphi_\omega^0(x) = Q_{\omega,0}(x) \quad (x > 0).$$

Moreover, we have $S_{\omega,0}(\varphi_\omega^0) = \frac{N}{2} \mathfrak{n}_{\omega,0}^{\text{line}} (> \mathfrak{n}_{\omega,0}^{\text{line}})$.

- (ii) *If N is even, then there is a one parameter family of standing wave solutions $e^{i\omega t} \varphi_\omega^{0,a}(x)$ ($a > 0$), where*

$$\varphi_\omega^{0,a} = (\varphi_{\omega,k}^{0,a})_{k=1}^N, \quad \varphi_{\omega,k}^{0,a}(x) = \begin{cases} Q_{\omega,0}(x-a) & k = 1, \dots, \frac{N}{2}, \\ Q_{\omega,0}(x+a) & k = \frac{N}{2} + 1, \dots, N. \end{cases}$$

Moreover, we have $S_{\omega,0}(\varphi_\omega^{0,a}) = \frac{N}{2} \mathfrak{n}_{\omega,0}^{\text{line}} (> \mathfrak{n}_{\omega,0}^{\text{line}})$.

The global-in-time behavior of the solutions of (2.1) with $\mathbf{n}_{\omega,0}^{\text{line}} \leq S_{\omega,0}(\mathbf{u}) < \frac{N}{2} \mathbf{n}_{\omega,0}^{\text{line}}$ is currently an open problem.

§ 5. Profile decomposition for the Schrödinger propagator

For the proof of the scattering part, we use the concentration-compactness and rigidity argument by Kenig and Merle [25]. The key ingredient of the argument is the profile decomposition for the Schrödinger propagator. Let us first recall the $\mathbb{R}$ -case. The underlying fact is that the bounded linear map

$$(5.1) \quad H^1(\mathbb{R}) \ni \varphi \mapsto e^{it\Delta_{\mathbb{R}}^\gamma} \varphi \in L_{t,x}^\infty(\mathbb{R} \times \mathbb{R})$$

is non-compact due to the spatial and temporal translation invariance. In detail, let us consider

$$\{\tau_{y_n} \varphi\}_{n=1}^\infty \quad \text{with} \quad y_n \rightarrow \pm\infty, \quad \text{or} \quad \{e^{it_n \Delta_{\mathbb{R}}^\gamma} \varphi\}_{n=1}^\infty \quad \text{with} \quad t_n \rightarrow \pm\infty$$

for $\varphi \in H^1(\mathbb{R})$, where we denote $\tau_y f = f(\cdot - y)$ for $f : \mathbb{R} \rightarrow \mathbb{C}$ and $y \in \mathbb{R}$. Then the corresponding images of these sequences have no convergent subsequence in $L_t^\infty L_x^\infty$. The idea of the profile decomposition is to describe *how non-compact* a bounded sequence is, and for (5.1), this is expressed by the combination of spatially and temporally shifting sequences.

Theorem 5.1 ([26] when $\gamma = 0$, [11] when $\gamma > 0$). *Let $\{\varphi_n\}_{n \in \mathbb{N}}$ be a bounded sequence in $H^1(\mathbb{R})$. Then there exist*

$$t_n^j \in \mathbb{R}, \quad y_n^j \in \mathbb{R}, \quad \psi^j \in H^1(\mathbb{R}), \quad (j \geq 1, n \in \mathbb{N}),$$

such that, by taking subsequence if necessary, we can write

$$\varphi_n = \sum_{j=1}^J e^{-it_n^j \Delta_{\mathbb{R}}^\gamma} \tau_{y_n^j} \psi^j + w_n^J$$

with the following properties:

(i) *For each $j \in \mathbb{N}$,*

$$\begin{aligned} & \text{either } t_n^j = 0 \quad (n \in \mathbb{N}), & \text{or} & \quad t_n^j \rightarrow \pm\infty \quad (n \rightarrow \infty), \\ & \text{either } y_n^j = 0 \quad (n \in \mathbb{N}), & \text{or} & \quad y_n^j \rightarrow \pm\infty \quad (n \rightarrow \infty); \end{aligned}$$

(ii) *Pairwise asymptotic orthogonality of parameters:*

$$|t_n^j - t_n^k| + |y_n^j - y_n^k| \rightarrow \infty, \quad (\forall j \neq k);$$

(iii) *Smallness of remainder:*

$$\lim_{J \rightarrow \infty} \limsup_{n \rightarrow \infty} \left\| e^{it\Delta_{\mathbb{R}}^{\gamma}} w_n^J \right\|_{L_{t,x}^{\infty}} = 0;$$

(iv) *Asymptotic orthogonality of norms:* For $q \in [2, \infty)$,

$$\|\varphi_n\|_{L^q(\mathbb{R})}^q = \sum_{j=1}^J \|\psi^j\|_{L^q(\mathbb{R})}^q + \|w_n^J\|_{L^q(\mathbb{R})}^q + o_n(1), \quad (\forall J \in \mathbb{N}),$$

$$\|\varphi_n\|_{\dot{H}_{\gamma}^1(\mathbb{R})}^2 = \sum_{j=1}^J \|\psi^j\|_{\dot{H}_{\gamma}^1(\mathbb{R})}^2 + \|w_n^J\|_{\dot{H}_{\gamma}^1(\mathbb{R})}^2 + o_n(1), \quad (\forall J \in \mathbb{N}),$$

$$\text{where } \|f\|_{\dot{H}_{\gamma}^1(\mathbb{R})} = \|(-\Delta_{\mathbb{R}}^{\gamma})^{1/2} f\|_{L^2(\mathbb{R})}.$$

Let us turn to the star graph case. We first note that by [18], the same Strichartz estimate is known to be true for $e^{it\Delta_{\mathcal{G}}^{\gamma}}$. For simplicity, let us consider the case where $N = 3$. The technical difficulty is that it is not clear how to represent spatial translation. Indeed, we cannot define the spatial shift as an operator on $H_c^1(\mathcal{G})$. For this, we define such sequences in the following way:

Definition 5.2. For $f \in H^1(\mathbb{R})$, we define the operator $\Lambda_{k,y}^* : H^1(\mathbb{R}) \rightarrow H_c^1(\mathcal{G})$ by

$$\begin{aligned} \Lambda_{1,y}^* f &:= \begin{pmatrix} \tau_y f \mathbb{1}_{>0} \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -\frac{1}{3} \tau_{-y} \mathcal{R} f \mathbb{1}_{>0} \\ \frac{2}{3} \tau_{-y} \mathcal{R} f \mathbb{1}_{>0} \\ \frac{2}{3} \tau_{-y} \mathcal{R} f \mathbb{1}_{>0} \end{pmatrix}, \\ \Lambda_{2,y}^* f &:= \begin{pmatrix} 0 \\ \tau_y f \mathbb{1}_{>0} \\ 0 \end{pmatrix} + \begin{pmatrix} \frac{2}{3} \tau_{-y} \mathcal{R} f \mathbb{1}_{>0} \\ -\frac{1}{3} \tau_{-y} \mathcal{R} f \mathbb{1}_{>0} \\ \frac{2}{3} \tau_{-y} \mathcal{R} f \mathbb{1}_{>0} \end{pmatrix}, \\ \Lambda_{3,y}^* f &:= \begin{pmatrix} 0 \\ 0 \\ \tau_y f \mathbb{1}_{>0} \end{pmatrix} + \begin{pmatrix} \frac{2}{3} \tau_{-y} \mathcal{R} f \mathbb{1}_{>0} \\ \frac{2}{3} \tau_{-y} \mathcal{R} f \mathbb{1}_{>0} \\ -\frac{1}{3} \tau_{-y} \mathcal{R} f \mathbb{1}_{>0} \end{pmatrix}, \end{aligned}$$

where $\mathcal{R}f(\cdot) := f(-\cdot)$ for $f : \mathbb{R} \rightarrow \mathbb{C}$. Moreover, for $\psi_1, \psi_2, \psi_3 \in H^1(\mathbb{R})$, $t \in \mathbb{R}$ and $y > 0$, we define

$$\mathcal{T}_{t,y}(\psi_1, \psi_2, \psi_3) = e^{-it\Delta_{\mathcal{G}}^{\gamma}} (\Lambda_{1,y}^* \psi_1 + \Lambda_{2,y}^* \psi_2 + \Lambda_{3,y}^* \psi_3).$$

We announce that we established the profile decomposition for $e^{it\Delta_{\mathcal{G}}^{\gamma}}$.

Proposition 5.3 (Linear profile decomposition on star graphs [19]). *Let $\{\varphi_n\}_{n=1}^\infty$ be a bounded sequence in $H_c^1(\mathcal{G})$. Then there exist*

$$t_n^j \in \mathbb{R}, \quad y_n^j \geq 0, \quad \psi_k^j \in H^1(\mathbb{R}), \quad (j \geq 1, k = 1, 2, 3, n \in \mathbb{N}),$$

such that, by taking subsequence if necessary, we have

$$(5.2) \quad \varphi_n = \sum_{j=0}^J \mathcal{T}_{t_n^j, y_n^j}(\psi_1^j, \psi_2^j, \psi_3^j) + \mathbf{w}_n^J, \quad \forall J \in \mathbb{N},$$

where all of the following hold:

(i) For each $j \in \mathbb{N}$,

$$\begin{aligned} & \text{either } t_n^j = 0 \quad (n \in \mathbb{N}), & \text{or } t_n^j \rightarrow \pm\infty \quad (n \rightarrow \infty), \\ & \text{either } y_n^j = 0 \quad (n \in \mathbb{N}), & \text{or } y_n^j \rightarrow \infty \quad (n \rightarrow \infty); \end{aligned}$$

(ii) Pairwise asymptotic orthogonality of parameters:

$$|t_n^j - t_n^k| + |y_n^j - y_n^k| \rightarrow \infty, \quad \forall j \neq k;$$

(iii) Smallness of remainder:

$$\lim_{J \rightarrow \infty} \limsup_{n \rightarrow \infty} \left\| e^{it\Delta_{\mathcal{G}}^\gamma} \mathbf{w}_n^J \right\|_{L_{t,x}^\infty} = 0;$$

(iv) Asymptotic orthogonality of norms: For $q \in [2, \infty)$,

$$\|\varphi_n\|_{L^q(\mathcal{G})}^q = \sum_{j=1}^J \left\| \mathcal{T}_{t_n^j, y_n^j}(\psi_1^j, \psi_2^j, \psi_3^j) \right\|_{L^q(\mathcal{G})}^q + \|\mathbf{w}_n^J\|_{L^q(\mathcal{G})}^q + o_n(1), \quad \forall J \in \mathbb{N},$$

$$\|\varphi_n\|_{\dot{H}_\gamma^1(\mathcal{G})}^2 = \sum_{j=1}^J \left\| \mathcal{T}_{t_n^j, y_n^j}(\psi_1^j, \psi_2^j, \psi_3^j) \right\|_{\dot{H}_\gamma^1(\mathcal{G})}^2 + \|\mathbf{w}_n^J\|_{\dot{H}_\gamma^1(\mathcal{G})}^2 + o_n(1), \quad \forall J \in \mathbb{N},$$

$$\text{where } \|\mathbf{f}\|_{\dot{H}_\gamma^1(\mathcal{G})} := \|(-\Delta_{\mathcal{G}}^\gamma)^{1/2} \mathbf{f}\|_{L^2(\mathbb{R})}.$$

The key idea of the proof is to introduce an operator $\mathcal{P}$ (denoted by $\mathcal{Q}$ in [19]) which maps a function in $H_c^1(\mathcal{G})$ to a pair of two odd functions and one even function on $\mathbb{R}$, satisfying

$$(5.3) \quad \mathcal{P} \Delta_{\mathcal{G}}^\gamma \mathbf{f} = \Delta_{\mathbb{R}}^\gamma \mathcal{P} \mathbf{f}, \quad (\mathbf{f} \in \mathcal{D}(\Delta_{\mathcal{G}}^\gamma)),$$

and accordingly,

$$(5.4) \quad \mathcal{P} e^{it\Delta_{\mathcal{G}}^\gamma} \mathbf{f} = e^{it\Delta_{\mathbb{R}}^\gamma} \mathcal{P} \mathbf{f}, \quad (\mathbf{f} \in H_c^1(\mathcal{G}), t \in \mathbb{R}).$$

We construct $\mathcal{P}$ such that $\mathcal{P}$ is invertible and preserves the Lebesgue and Sobolev norms up to multiples of constants. Therefore, Proposition 5.3 follows from the corresponding results on $\mathbb{R}$ under the restriction of odd/even functions.

The difficulty of the proof of (5.3) is to transform $\mathcal{D}(\Delta_{\mathcal{G}}^\gamma)$ to $\mathcal{D}(\Delta_{\mathbb{R}}^\gamma)$ with taking care of the boundary condition at the origin, without losing the regularity and quantitative information. Keeping this in mind, we define $\mathcal{P}$ in the following way. For $\mathbf{f} \in L^2(\mathcal{G})$, we first define $\alpha_k(\mathbf{f}) : [0, \infty) \rightarrow \mathbb{C}$ ($k = 1, 2, 3$) by

$$\alpha_1(\mathbf{f}) := \frac{-2f_1 + f_2 + f_3}{3}, \quad \alpha_2(\mathbf{f}) := \frac{-f_1 - f_2 + 2f_3}{3},$$

$$\alpha_3(\mathbf{f}) := \frac{f_1 + f_2 + f_3}{3}.$$

Note that if $\mathbf{f} \in \mathcal{D}(\Delta_{\mathcal{G}}^\gamma)$, we have

$$(5.5) \quad \alpha_1(\mathbf{f})(0) = \alpha_2(\mathbf{f})(0) = 0, \quad \alpha_3(\mathbf{f})'(0+) = \gamma\alpha_3(\mathbf{f})(0).$$

Then, we define $\mathcal{P}_k : L^2(\mathcal{G}) \rightarrow L^2(\mathbb{R})$ by

$$\mathcal{P}_1\mathbf{f}(x) := \begin{cases} -\alpha_1(\mathbf{f})(-x) & (x < 0) \\ \alpha_1(\mathbf{f})(x) & (x \geq 0) \end{cases}, \quad \mathcal{P}_2\mathbf{f}(x) := \begin{cases} -\alpha_2(\mathbf{f})(-x) & (x < 0) \\ \alpha_2(\mathbf{f})(x) & (x \geq 0) \end{cases},$$

$$\mathcal{P}_3\mathbf{f}(x) := \begin{cases} \alpha_3(\mathbf{f})(-x) & (x < 0) \\ \alpha_3(\mathbf{f})(x) & (x \geq 0) \end{cases}.$$

Note that $\mathcal{P}_1(\mathbf{f}), \mathcal{P}_2(\mathbf{f})$ are odd functions on $\mathbb{R}$, while $\mathcal{P}_3(\mathbf{f})$ is even. Moreover, (5.5) implies that if $\mathbf{f} \in \mathcal{D}(\Delta_{\mathcal{G}}^\gamma)$, then $\mathcal{P}_j(\mathbf{f}) \in \mathcal{D}(\Delta_{\mathbb{R}}^\gamma)$ for $j = 1, 2, 3$. (In fact, $\mathcal{P}_1(\mathbf{f}), \mathcal{P}_2(\mathbf{f}) \in H^2(\mathbb{R}) = \mathcal{D}(\Delta_{\mathbb{R}}^0)$, since the interaction at the origin vanishes due to (5.5).) Therefore, defining $\mathcal{P} := {}^t(\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3)$, we are able to prove that $\mathcal{P}$ satisfies the desired properties (see [19] for the detailed proof.) Note that

$$\mathcal{T}_{t,y}(\psi_1, \psi_2, \psi_3) = e^{-it\Delta_{\mathcal{G}}^\gamma} \mathcal{P}^{-1} \begin{pmatrix} \tau_y\psi_1 - \tau_{-y}\mathcal{R}\psi_1 \\ \tau_y\psi_2 - \tau_{-y}\mathcal{R}\psi_2 \\ \tau_y\psi_3 + \tau_{-y}\mathcal{R}\psi_3 \end{pmatrix}$$

for $t \in \mathbb{R}$, $y \geq 0$, $\psi_k \in L^2(\mathbb{R})$, $k = 1, 2, 3$. Thus $\mathcal{T}_{t,y}(\psi_1, \psi_2, \psi_3)$ is defined along with the corresponding shifting profiles for odd or even functions on $\mathbb{R}$ through $\mathcal{P}$ (see [20, 21]).

We conclude this section by discussing the idea of the definition of $\mathcal{P}$. For $\mathbf{f} \in L^2(\mathcal{G})$, we can write

$$\mathbf{f} = \alpha_1(\mathbf{f}) \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + \alpha_2(\mathbf{f}) \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} + \alpha_3(\mathbf{f}) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Thus we may interpret that $\mathcal{P}_1 \mathbf{f}$ and $\mathcal{P}_2 \mathbf{f}$ correspond to odd functions lying on the first and second edges of $\mathcal{G}$, and on the second and third edges of $\mathcal{G}$, respectively, while $\mathcal{P}_3 \mathbf{f}$ is a radially symmetric function on $\mathcal{G}$. The radial part is responsible for the boundary condition for $\mathbf{f} \in \mathcal{D}(\Delta_{\mathcal{G}}^{\gamma})$, while the gap from radial symmetry is compensated by the odd parts. This idea is easily generalized to star graphs with any number of edges, for which we omit the discussion.

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