

Notes on Discrete Subgroups of $PU(1, 2; \mathbf{C})$ with Heisenberg Translations

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0. Recently Parker, Basmajian and Miner have independently given some conditions for a subgroup of $PU(1, 2; \mathbf{C})$ to be non-discrete. In this paper we show that under some conditions Parker's theorem leads to some Basmajian and Miner's result.

1. To introduce Parker's theorem and Basmajian-Miner's theorem, we need some definitions and notation. Let $\mathbf{C}$ be the field of complex numbers. Let $V = V^{1,2}(\mathbf{C})$ denote the vector space $\mathbf{C}^3$, together with the unitary structure defined by the Hermitian form

$$\tilde{\Phi}(z^*, w^*) = -(\overline{z_0^*}w_1^* + \overline{z_1^*}w_0^*) + \overline{z_2^*}w_2^*$$

for $z^* = (z_0^*, z_1^*, z_2^*)$, $w^* = (w_0^*, w_1^*, w_2^*)$ in V .

An automorphism g of V , that is a linear bijection such that $\tilde{\Phi}(g(z^*), g(w^*)) = \tilde{\Phi}(z^*, w^*)$ for z^*, w^* in V , will be called a unitary transformation. We denote the group of all unitary transformations by $U(1, 2; \mathbf{C})$. Set $PU(1, 2; \mathbf{C}) = U(1, 2; \mathbf{C})/(\text{center})$. An element g in $PU(1, 2; \mathbf{C})$ acts on the Siegel domain

$$H^2 = \{w = (w_1, w_2) \in \mathbf{C}^2 \mid \operatorname{Re}(w_1) > \frac{1}{2}|w_2|^2\}$$

and its boundary ∂H^2 . Denote $H^2 \cup \partial H^2$ by $\overline{H^2}$. We define a new coordinate system in $\overline{H^2} - \{\infty\}$. To $q = (w_1, w_2) \in \overline{H^2} - \{\infty\}$ we can correspond the 3-tuple $(k, t, w_2) \in (\mathbf{R}^+ \cup \{0\}) \times \mathbf{R} \times \mathbf{C}$, where $k = \operatorname{Re}(w_1) - \frac{1}{2}|w_2|^2$ and $t = \operatorname{Im}(w_1)$. This 3-tuple $(k, t, w_2)_H$ is called the H -coordinates of q . For simplicity, we use $(t_1, w')_H$ for $(0, t_1, w')_H$.

The Cygan metric $\rho(p, q)$ for $p = (k_1, t_1, w')_H$ and $q = (k_2, t_2, W')_H$ is given by

$$\rho(p, q) = |\{\frac{1}{2}|W' - w'|^2 + |k_2 - k_1| + i\{t_1 - t_2 + \operatorname{Im}(\overline{w'}W')\}\}|^{\frac{1}{2}}.$$

We note that this Cygan metric ρ is a generalization of the Heisenberg metric δ in ∂H^2 (see [7]). Let $f = (a_{ij})_{1 \leq i, j \leq 3} \in PU(1, 2; \mathbf{C})$ with $f(\infty) \neq \infty$. We define the isometric sphere I_f of f by

$$I_f = \{w = (w_1, w_2) \in \overline{H^2} \mid |\tilde{\Phi}(W, Q)| = |\tilde{\Phi}(W, f^{-1}(Q))|\},$$

where $Q = (0, 1, 0)$, $W = (1, w_1, w_2)$ in V (see [3]). It follows that the isometric sphere I_f is the sphere in the Cygan metric with center $f^{-1}(\infty)$ and radius $R_f = \sqrt{1/|a_{12}|}$, that is,

$$I_f = \left\{ z = (k, t, w') \in (\mathbf{R}^+ \cup \{0\}) \times \mathbf{R} \times \mathbf{C} \mid \rho(z, f^{-1}(\infty)) = \sqrt{\frac{1}{|a_{12}|}} \right\}.$$

Remark 1.1. In $PU(1, 1; \mathbf{C})$, our radius of isometric sphere is the square root of the usual one.

We have the same formulae as in Möbius transformations (see [3]).

Proposition 1.2. *Let g and h be elements with $g(\infty) \neq \infty$ and $h(\infty) \neq \infty$. Then:*

$$\begin{aligned} (1) \quad R_{gh} &= \frac{R_g R_h}{\delta(g^{-1}(\infty), h(\infty))}. \\ (2) \quad R_h^2 &= \delta((gh)^{-1}(\infty), h^{-1}(\infty)) \delta(g^{-1}(\infty), h(\infty)). \end{aligned}$$

Now we are ready to state Parker's Theorem.

Theorem 1.3 ([9]). *Let g be a Heisenberg translation with the form*

$$g = \begin{pmatrix} 1 & 0 & 0 \\ s & 1 & \bar{a} \\ a & 0 & 1 \end{pmatrix},$$

where $\operatorname{Re}(s) = \frac{1}{2}|a|^2$. Let f be any element of $PU(1, 2; \mathbf{C})$ with isometric sphere of radius R_f . If

$$R_f^2 > \delta(gf^{-1}(\infty), f^{-1}(\infty)) \delta(gf(\infty), f(\infty)) + 2|a|^2,$$

then the group $\langle f, g \rangle$ generated by f and g is not discrete.

Remark 1.4. Suppose that g is a vertical Heisenberg translation. As $a = 0$, this theorem is equivalent to the result in [5] and [6].

Let

$$B_r = \{z \in \partial H^2 \mid \rho(z, 0) = \delta(z, 0) < r\},$$

and let $\overline{B}_s^c = \partial H^2 \cup \{\infty\} - B_s$. For $0 < r < 1$, the pair of open sets $(B_r, \overline{B}_{1/r}^c)$ is said to be *stable* with respect to a set S of elements in $PU(1, 2; \mathbf{C})$ if for any element $g \in S$,

$$g(0) \in B_r \quad g(\infty) \in \overline{B}_{1/r}^c.$$

A loxodromic element f has a unique complex dilation $\lambda(f)$ such that $|\lambda(f)| > 1$. Let $S(r, \varepsilon(r))$ denote the family of loxodromic elements f with fixed points in B_r and $\overline{B}_{1/r}^c$, and satisfying $|\lambda(f) - 1| < \varepsilon(r)$.

For positive real numbers r with $r < 1/\sqrt{3 + \sqrt{3} - \sqrt{2}} = 0.549\dots$, we define $\varepsilon(r)$ by

$$(*) \quad \varepsilon(r) = \sup\{|\lambda(f) - 1| < \varepsilon\},$$

where $\lambda(f)$ satisfies the inequalities below

$$|\lambda(f) - 1| < \sqrt{2 + \left(\frac{1 - (3 + |\lambda(f) - 1|)r^2}{1 - 2r^2} \right)^2 \left(\frac{1 - 3r^2}{1 - r^2} \right)^2} - \sqrt{2},$$

$$|\lambda(f)| < \frac{1 - 2r^2}{r^2}.$$

We show the graph of $\varepsilon(r)$ below.

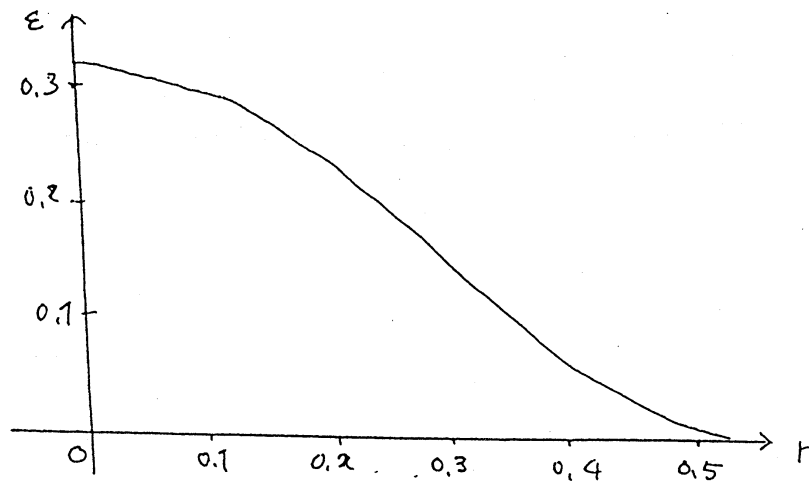


figure 1.

If $r < 1/\sqrt{3 + \sqrt{3} - \sqrt{2}}$ and $\varepsilon < \varepsilon(r)$, a pair of non-negative numbers (r, ε) is called a *stable basin point*.

For four points q_1, q_2, q_3, q_4 in ∂H^2 , define the *real cross ratio* $||[q_1, q_2, q_3, q_4]||$ by

$$||[q_1, q_2, q_3, q_4]|| = \frac{\delta^2(q_3, q_1)\delta^2(q_4, q_2)}{\delta^2(q_4, q_1)\delta^2(q_3, q_2)}.$$

Note that this real cross ratio is invariant under $PU(1, 2; \mathbb{C})$.

We shall state Basmajian-Miner's result.

Theorem 1.4 ([1]). *Fix a stable basin point (r, ε) . Let g be a parabolic element with fixed point ∞ . If f is a loxodromic element with fixed points 0 and q satisfying $|\lambda(f) - 1| < \varepsilon$. If $||[0, q, g(0), g(q)]|| < r^4$, then the group $\langle f, g \rangle$ generated by f and g is not discrete.*

2. In this section we show that Parker's Theorem leads to Basmajian-Miner's theorem under some conditions. First we treat a simple case.

Theorem 2.1. *Fix a stable basin point (r, ε) . Let g be a Heisenberg translation with the form*

$$g = \begin{pmatrix} 1 & 0 & 0 \\ s & 1 & \bar{a} \\ a & 0 & 1 \end{pmatrix},$$

where $\operatorname{Re}(s) = \frac{1}{2}|a|^2$. Let f be a loxodromic element with fixed points $a_f = (0, 0)$ and $b_f = (it, 0)$ ($t > 0$) such that $|\lambda(f) - 1| < \varepsilon$. If $||[a_f, b_f, g(a_f), g(b_f)]| < r^4$, then the group $\langle f, g \rangle$ generated by f and g is not discrete.

Lemma 2.2 immediately leads to

Corollary 2.3. *Fix a stable basin point (r, ε) . Let f and g be the same elements as in Theorem 2.1. If $\delta(a_f, b_f) > \frac{\delta(a_f, g(a_f))}{r^2}(1 + r^2 + \sqrt{1 + r^2})$, then the group $\langle f, g \rangle$ generated by f and g is not discrete.*

When the condition on fixed points of a loxodromic element is weakened, we obtain

Theorem 2.4. *Fix a stable basin point (r, ε) , where $r < 0.48$. Let g be the same element as in Theorem 2.1. Let f be a loxodromic element with fixed point 0 and $q (\neq \infty)$ satisfying $|\lambda(f) - 1| < \varepsilon$. If $\delta(0, q) > \frac{\delta(0, g(0))}{r^2}(1 + r^2 + \sqrt{1 + r^2})$, then the group $\langle f, g \rangle$ generated by f and g is not discrete.*

For our proof of Theorem 2.4, we need

Proposition 2.5. *Let f be a loxodromic element with the attracting fixed point a_f and the repelling fixed point b_f . Then:*

- (1) $||[f(z), z, b_f, a_f]| = |\lambda(f)|^2$ for any $z \in \partial H^2$.
- (2) $\delta(f(z), f(w)) = \frac{R_f^2}{\delta(z, f^{-1}(\infty))\delta(w, f^{-1}(\infty))} \delta(z, w)$ for $z, w \in \partial H^2$.
- (3) $R_f^2 = \delta(a_f, f^{-1}(\infty))\delta(b_f, f^{-1}(\infty)) = \delta(a_f, f(\infty))\delta(b_f, f(\infty))$.
- (4) $\frac{\delta(a_f, f^{-1}(\infty))}{\delta(b_f, f^{-1}(\infty))} = \frac{\delta(b_f, f(\infty))}{\delta(a_f, f(\infty))} = |\lambda(f)|$.
- (5) $\delta(a_f, f(\infty)) = \delta(b_f, f^{-1}(\infty)) = R_f |\lambda(f)|^{-\frac{1}{2}}$.
- (6) $\delta(a_f, f^{-1}(\infty)) = \delta(b_f, f(\infty)) = R_f |\lambda(f)|^{\frac{1}{2}}$.
- (7) $R_f(|\lambda(f)|^{\frac{1}{2}} - |\lambda(f)|^{-\frac{1}{2}}) \leq \delta(a_f, b_f) \leq R_f(|\lambda(f)|^{\frac{1}{2}} + |\lambda(f)|^{-\frac{1}{2}})$.

Remark 2.6. If f is an element of $PU(1, 1; \mathbb{C})$, then

$$R_f(|\lambda(f)| - |\lambda(f)|^{-1})^{\frac{1}{2}} = \delta(a_f, b_f).$$

But this is not true for an element of $PU(1, 2; \mathbb{C})$.

3. The details of this paper will be published elsewhere.

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