

On ruled 3-folds

阪大 数理学部 満洲 俊樹

近年 Itaka, Ueno, Viehweg, Kawamata, Fujita らによつて高次元代数多様体の分類に関する著しい結果が得られ始めた。そこで我々は $\kappa(X) \leq 0$ の 3 次元の 4 次元 compact complex manifolds を調べてみた。

(1.1) 以下 $X, Y, Y_1, Y_2, \dots$ 等はすべて irreducible reduced compact complex space (単に complex variety と呼ぶ) を示すものとする。Surjective meromorphic maps $f_i: X \rightarrow Y_i, i=1,2$ に対し次の様子を定義を行う。

$Y_1 \overset{\times}{\ast} Y_2 :=$ meromorphic image of X under the meromorphic map $x \in X \mapsto (f_1(x), f_2(x)) \in Y_1 \times Y_2$

$f_1 \ast f_2 :=$ surjective meromorphic map from X to $Y_1 \ast Y_2$ defined by $x \mapsto (f_1(x), f_2(x))$

Proposition (1.1.1)

Assume that $\dim X \leq 4$. Then

- (i) $\kappa(Y_1) \geq 0$ and $\kappa(Y_2) \geq 0$ imply $\kappa(Y_1 \overset{\times}{\ast} Y_2) \geq 0$.
- (ii) $\kappa(Y_1) = \dim Y_1$ and $\kappa(Y_2) = \dim Y_2$ imply $\kappa(Y_1 \overset{\times}{\ast} Y_2) = \dim(Y_1 \overset{\times}{\ast} Y_2)$

証明のポイント: (i) $\dim Y_1, \dim Y_2, \dim Y_1 * Y_2 \leq 4$ なの
 で. case 毎に調べていくことによって, 結局
 Viehweg により示された curve を fibre とする
 fibration に於ける小平次元の劣加法性に帰着さ
 れる. (ii) 一たん (i) がわかれば, Kawamata によ
 って示された base の general type であるような
 fibration に於ける小平次元の加法性から従う。

Definition (1.1.2) For a complex variety X , we define

$$\mathcal{B}_X := \left\{ (Y, f) \mid \begin{array}{l} f: X \rightarrow Y \text{ is a surjective meromorphic} \\ \text{map to a complex variety } Y \text{ with } \chi(Y) \geq 0 \end{array} \right\} / \sim$$

$$\mathcal{B}'_X := \left\{ (Y, f) \mid \begin{array}{l} f: X \rightarrow Y \text{ is a surjective meromorphic} \\ \text{map to a complex variety } Y \text{ of general type} \end{array} \right\} / \sim$$

where $(Y_1, f_1) \sim (Y_2, f_2)$ if there exists a bimeromorphic map
 $\tau: Y_1 \rightarrow Y_2$ such that $f_2 = \tau \circ f_1$.

Proposition (1.1.3)

Assume that $\dim X \leq 4$. Then

- (i) There exists a unique element in $\mathcal{B}_X$ (which we denote
 by $(B(X), \pi_X)$) such that for every $(Y, f) \in \mathcal{B}_X$, there
 exists a surjective meromorphic map from $B(X)$ to Y which

makes the following diagram commutative :

$$\begin{array}{ccc} & X & \\ \pi_X \swarrow & \circlearrowright & \searrow f \\ B(X) & \longrightarrow & Y \end{array}$$

(ii) There exists a unique element in $\mathcal{B}_X'$ (which we denote by $(B'(X), \pi_X')$) such that for every $(Y, f) \in \mathcal{B}_X'$, there exists a surjective meromorphic map from $B'(X)$ to Y which makes the following diagram commutative :

$$\begin{array}{ccc} & X & \\ \pi_X' \swarrow & \circlearrowright & \searrow f \\ B'(X) & \longrightarrow & Y \end{array}$$

証明のポイント: proposition (1.1.1) の容易な帰結である。

Corollary (1.1.4)

Assume that $\dim X \leq 4$. Let $f: X \rightarrow Y$ be a surjective meromorphic map. Then there exists a surjective meromorphic map $f_b: B(X) \rightarrow B(Y)$ (resp. $f_b': B'(X) \rightarrow B'(Y)$), unique up to bimeromorphic equivalence, such that the following diagram commutes :

$$\begin{array}{ccc}
 X & \xrightarrow{f} & Y \\
 \pi_X \downarrow & \circlearrowright & \downarrow \pi_Y \\
 B(X) & \xrightarrow{f_b} & B(Y)
 \end{array}
 \quad \left(\text{resp.} \quad \begin{array}{ccc}
 X & \xrightarrow{f} & Y \\
 \pi'_X \downarrow & \circlearrowright & \downarrow \pi'_Y \\
 B'(X) & \xrightarrow{f'_b} & B'(Y)
 \end{array} \right)$$

Corollary (1.1.5)

For a complex variety X of dimension ≤ 4 , the surjective meromorphic map $\pi_X : X \rightarrow B(X)$ canonically induces a bimeromorphic map $(\pi_X)_b' : B'(X) \xrightarrow{\sim} B'(B(X))$.

Corollary (1.1.6)

Assume that X is a Moishezon variety (or more generally $X \in \mathcal{C}$) with $\dim X \leq 4$. Let $f : X \rightarrow Y$ be a surjective morphism whose general fibre is irreducible with $\chi = 0$.

Then f_b' is a bimeromorphic map : $B'(X) \xrightarrow{\sim} B'(Y)$. In particular, if $\varphi : X \rightarrow X_0$ is the Iitaka fibration, then φ_b' is a bimeromorphic map : $B'(X) \xrightarrow{\sim} B'(X_0)$.

Corollary (1.1.7)

Assume that X is a Moishezon variety (or more generally $X \in \mathcal{C}$) with $\dim X \leq 4$. Then there exists a sequence

of surjective meromorphic maps

$$X = X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \rightarrow \cdots \xrightarrow{f_{m-1}} X_m \xrightarrow{f_m} X_{m+1}$$

with the following properties:

(1) $\dim X_i > \dim X_{i+1}$, $i=0, 1, 2, \dots, m$.

(2) X_{m+1} is of general type.

(3) For each i ,

(i) if $\kappa(X_i) = -\infty$, then $X_{i+1} = B(X_i)$ and $f_i = \pi_{X_i}$, i.e.,

$f_i: X_i \rightarrow X_{i+1}$ is the fibration defined in (i) of (1.1.3).

(ii) if $\kappa(X_i) \geq 0$, then $f_i: X_i \rightarrow X_{i+1}$ is the Iitaka fibration

of X_i . (In particular, if $\kappa(X_i) = 0$ for some $0 \leq i < m$,

then $i=m-1$ and X_m is a singleton.)

Such a sequence of surjective meromorphic maps is unique up to bimeromorphic equivalence. Furthermore, for each i ($0 \leq i \leq m$),

$f_m \circ f_{m-1} \circ \cdots \circ f_{i+1} \circ f_i: X_i \rightarrow X_{m+1}$ is the fibration defined in

(ii) of (1.1.3), i.e., $X_{m+1} = B(X_i)$ and $f_m \circ f_{m-1} \circ \cdots \circ f_i = \pi_{X_i}'$.

(1.2) さて以下、 $\dim X \leq 3$ の complex variety を主に考える。Recall that a complex variety X is of class $\mathcal{C}$ (記号 $X \in \mathcal{C}$) if there exists a morphism from a compact Kähler manifold onto X .

Definition (1.2.1): a complex variety X is called of Castelnuovo's type (or shortly "of CNT") if there exists no surjective meromorphic map $\star$ from X to a complex variety Y of $\dim Y > 0$ and $\kappa(Y) \geq 0$.

Remark (1.2.2): If $\dim X \leq 4$, then X is of CNT if and only if $\dim B(X) = 0$.

Proposition (1.2.3): Let $g: X \rightarrow Z$ be a surjective morphism of complex varieties such that (1) Z is of CNT and that (2) general fibres of g are irreducible and of CNT. Then X is of CNT.

証明の方針: $f: X \rightarrow Y$ を surjective meromorphic map with $\kappa(Y) \geq 0$ とする。条件 (2) から f は $X \xrightarrow{f} Y$ と surjective meromorphic maps の composition の形に factor する。とすると (1) により Y は 1 点でなければならぬ。

Proposition (1.2.4)

Let X be a complex variety of CNT.

- (i) If $\dim X = 1$, then X is a rational curve.
 (ii) Assume that X is smooth and $\dim X = 2$. Then X is either a rational surface or a surface of class VII. (In particular, if $X \in \mathcal{C}$, then X is a rational surface.)

証明の方針: Kodaira の classification による。

Proposition (1.2.5)

Let X be a compact complex manifold of class $\mathcal{C}$ with $\dim X \leq 3$. Then there exists a Zariski open dense subset U of $B(X)$ such that for every $y \in U$, the fibre $\pi_X^{-1}(y)$ is of CNT, i.e., $B(\pi_X^{-1}(y))$ is a singleton.

証明の方針: $\dim(\text{general fibre of } \pi_X)$ が 1 か 2 のときを考えればよいから。1 のときは Curve を fibre とするような fibration に於ける小平次元の可加法性 (by Viehweg) により明らか; 2 のときは $\begin{cases} \dim X = 3 \\ \dim B(X) = 1 \end{cases}$ と仮定してよいから。Viehweg によって示された小平次元の可加法性 (C_3) によって general fibre of

π_X は rational surface か ruled surface of genus $g \geq 1$ のいずれかになる。ところが後者の場合は relative albanese map

$$\begin{array}{ccc} X & \longrightarrow & B(X) \\ & \searrow \swarrow & \uparrow \\ & \text{Alb}(X/B(X)) & \end{array}$$

が構成され (due to Fujiki).

しかも $\text{Alb}(X/B(X))$ は general fibre を genus $g \geq 1$ の curve としてもつような $B(X)$ 上の fibre space となっている。よって $\chi(\text{Alb}(X/B(X))) \geq 0$ である。これは $B(X)$ の定義に反する。故に general fibre of π_X は rational surface であり得ない。

(1.3) をこいさい compact complex manifold of dimension 3 with $\kappa = -\infty$ の大雑把なクラスわけを行う。まず Kawai, Ueno, Fujiki の結果から次のことが出てくるのはほぼ明らかである。

Proposition (1.2.6): Let X be a compact complex manifold with $\kappa(X) = -\infty$, $\dim X = 3$, and $X \in \mathcal{C}$. Then we have one of the following:

- i) $\dim B(X) = \alpha(X) = 0$. In this case, X is of CNT and is also simple in Fujiki's sense.

ii) $\dim B(X) = 0$ and $X_{\text{alg}} = \mathbb{P}^1$. Then for a suitable birational model of X , the algebraic reduction $X \rightarrow X_{\text{alg}}$ has general fibres of one of the following form: (due to Fujiki)

- a) relatively minimal Kähler K3-surface
- b) complex torus
- c) almost homogeneous ruled surface of genus 1.

iii) $\dim B(X) = 0$, and $X_{\text{alg}} \cong \mathbb{P}^2$. Then the algebraic reduction $X \rightarrow X_{\text{alg}}$ is an elliptic fibration.

iv) $\dim B(X) = 0$ and $a(X) = 3$, i.e., X is a Moishezon manifold ~~and~~ and is of CNT.

v) $B(X)$ is a curve of genus $g \geq 1$, and over a Zariski open dense subset of $B(X)$, every fibre $X \rightarrow B(X)$ is a rational surface.

vi) $\dim B(X) = 2$ and over a Zariski open dense subset of $B(X)$, every fibre of $X \rightarrow B(X)$ is a rational curve.

次に X を $\mathbb{C}$ として $\chi(X) = -\infty$ の compact complex manifold を調べてみる。そういった X の構造は ~~ほとんど~~ $\mathbb{C}$ に属する場合と異ってもう少し複雑になる。

Proposition (1.2.7)

Let X be a compact complex manifold of dimension 3 with $\kappa(X) = -\infty$ and $X \notin \mathcal{C}$. Then, replacing X by its appropriate bimeromorphic model, we have one of the following:

- (i) X is simple in Fujiki's sense.
- (ii) (due to Fujiki) There exists a surface S of class $\mathcal{M}_0$ with $\kappa(S) = -\infty$ and $a(S) = 0$, and also exists a fibration $\pi: X \rightarrow S$ with the following property:

For every surjective meromorphic map $f: X \rightarrow Y$ with $\dim Y > 0$, there exists a generically finite meromorphic map $\tilde{f}: S \rightarrow Y$ such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \pi \searrow & \circlearrowleft & \nearrow \tilde{f} \\ & S & \end{array}$$

commutes. (In particular, Y is a surface.)

~~Moreover~~ Furthermore, a general fibres of π are either elliptic curves or rational curves.

- (iii) $X_c = X_{\text{alg}} =$ a curve and $\dim B(X) \leq 1$ and a general fibre of the algebraic reduction $X \rightarrow X_{\text{alg}}$ is ~~one~~

~~is birational~~ is birational to one of the following:

- a) K3-surface, b) complex torus, c) hyperelliptic surface,
 d) Enriques surface, e) elliptic surface with trivial
 canonical bundle, f) surface of class III_0 , g) rational
 surface ($\neq \mathbb{P}^2$), h) ruled surface of genus 1.

(iv) $B(X_c) = B(X) =$ either a curve of genus ≥ 1 or a singleton,
 and X_c is a ruled surface, and the $\mathbb{C}$ -reduction $X \rightarrow X_c$ is
 an elliptic fibration.

(v) $B(X_c) = X_c =$ a curve, and $B(X)$ is an elliptic surface with
 odd first Betti number fibred over the curve $B(X)_c$, and a
 general fibre of $X \rightarrow B(X)$ is $\mathbb{P}^1$.

(vi) $B(X)_c =$ a curve, and there is a generically finite
 surjective morphism $f: X \rightarrow B(X) \times_{B(X)_c} X_c$ making the
 following diagram commute:

$$\begin{array}{ccccc}
 & & X & & \\
 \pi_X \swarrow & & \downarrow f & \searrow \text{\textit{C-reduction}} & \\
 B(X) & \xleftarrow{\text{pr}_1} & B(X) \times_{B(X)_c} X_c & \xrightarrow{\text{pr}_2} & X_c
 \end{array}$$

Furthermore X_c is a ruled surface over the curve $B(X)_c$ and
 $B(X)$ is an elliptic surface fibred over the curve $B(X)_c$.

2. Fibrations associated with holomorphic 2-forms

Proposition (2.1.1) Let X be a projective algebraic manifold with

$\dim X = 3$ and $\dim B(X) > 0$. Then $\pi_X: X \rightarrow B(X)$ induces

$$H^0(X, S^m(\Omega_X^2)) \cong H^0(B(X), S^m(\Omega_{B(X)}^2)) \quad m=1, 2, \dots$$

$$H^0(X, \Omega_X^1) \cong H^0(B(X), \Omega_{B(X)}^1)$$

Furthermore

i) If $\dim B(X) \geq 2$, then $H^0(X, S^m(\Omega_X^1)) \cong H^0(B(X), S^m(\Omega_{B(X)}^1))$
 $m=1, 2, \dots$

ii) $\dim B(X) = 1$. Then every $\omega \in H^0(X, S^m(\Omega_X^1))$ is written
 in the form $\pi_X^*(\omega')$ for some $\omega' \in H^0(B(X), \mathcal{O}(B(X)) \cdot S^m(\Omega_{B(X)}^1))$
 s.t. $\int_{B(X)} (\omega' \wedge \overline{\omega'})^{\frac{1}{m}} < +\infty$.

証明の要旨: $\dim B(X) > 0$ に注意すると Proposition (1.2.5)
 により π_X の fibre は rational variety になる。このこと
 から容易に π_X^* が成る。

Remark (2.1.2)

上のことより $m \geq 2$ に対しては isomorphism が成り立つ
 部分があるのは Ueno の指摘による。更に ii) の

$\dim B(X)=1$ の時に $H^0(X, S^m(\Omega_X^1)) \cong H^0(B(X), S^m(\Omega_{B(X)}^1))$ が成り立つことも十分可能性がある。たとえは rational surface を smooth fibre とし B を base space を nonsingular curve とし π を fibration とすると、singular fibre の所に multiplicity が 1 となる component が含まれていれば同型が成立する。
(= Ueno の注意)

Conjecture (2.1.3)

$$\dim B(X)=0 \rightarrow H^0(X, S^m(\Omega_X^1)) = H^0(X, S^m(\Omega_X^2)) = 0? \quad \forall m=1, 2, \dots$$

最後に holomorphic 2-forms から定義される fibration を考えてみる。

X : 3 dim projective algebraic manifold with $\kappa(X) \leq 0$ とする。

r_{def} rank of the subsheaf of Ω_X^2 generated by the global sections of Ω_X^2

このとき $r=1$ or 2 or 3 .

Proposition (2.2.1) $r=3 \Rightarrow h^{2,0}(X)=3$

(その simplified proof は Ueno による。)

証明: $k^{2,0} > 3$ とする。 $\omega_1 \wedge \omega_2 \wedge \omega_3 \neq 0 \in H^0(X, \det(\Omega_X^2))$
 とする。よって $H^0(X, \Omega_X^2)$ の $\mathbb{C}$ -basis $\{\omega_1, \omega_2, \omega_3, \omega_4, \dots\}$
 をとる。Then $\omega_4 = f_1 \omega_1 + f_2 \omega_2 + f_3 \omega_3$ ($\exists f_1, f_2, f_3 \in \mathbb{C}(X)$)
 である。 $f_1 = \text{non-constant rational function}$ と仮定しても
 一般性を失わない。 Then

$$\text{Both } \omega_4 \wedge \omega_2 \wedge \omega_3 = f_1 \omega_1 \wedge \omega_2 \wedge \omega_3 \left\{ \begin{array}{l} \in H^0(X, \det(\Omega_X^2)) = H^0(X, K_X^{\otimes 2}) \\ \omega_1 \wedge \omega_2 \wedge \omega_3 \end{array} \right.$$

$\therefore \kappa(X) > 0 \quad \therefore \text{Contradiction.}$

Proposition (2.2.2)

Assume $r=1$ and $k^{2,0}(X) \geq 2$.

$\Rightarrow$ Then $\kappa(X) = -\infty$ and $\dim B(X) = 2$

証明の半分:

$r=1$ のとき $\{\omega_1, \omega_2, \dots, \omega_\nu\}$ を $H^0(X, \Omega_X^2)$ の $\mathbb{C}$ -basis
 とすると

$$\Phi: X \rightarrow \mathbb{P}^{\nu-1}$$

$$x \longmapsto (\omega_1(x), \omega_2(x), \dots, \omega_\nu(x))$$

$\mathbb{P}^{\nu-1}$ fibration を作ることにできる。 (但し X を適当
 な birational model にとりかえておいて Φ が最初か
 り morphism である と仮定しておいてよい。)

case 1) $\dim \bar{X}(X) \geq 2$. この時は Bogomolov の Lemma
により $\dim \bar{X}(X) = 2$ が成り立つ。Stein factorization

$$\begin{array}{ccc} X & \xrightarrow{\pi} & \bar{X}(X) \\ & \searrow \quad \nearrow & \\ & B & \end{array}$$

をとる。このとき $\kappa(X) = -\infty$, $B = B(X)$ となる事が示される。

case 2) $\dim \bar{X}(X) = 1$. 矢張り Stein factorization

$$\begin{array}{ccc} X & \xrightarrow{\pi} & \bar{X}(X) \\ & \searrow \quad \nearrow & \\ & C & \end{array}$$

をとる。このとき $\kappa(X) \leq 0$ に注意すると variation of Hodge
structure の一般論から relative albanese $\text{Alb}(X/C)$ を
構成すると surface になることがわかる。そして
結局 $\kappa(X) = -\infty$, $B(X) = \text{Alb}(X/C)$ が成り立つ。

さて以上の如く $r=1, 3$ の場合はうまくい
った。 $r=2$ の場合も同様の fibration (Grassmann
variety への map) が構成できる。例えばは
Lemo の予想「 X : projective algebraic 3-fold with

$\kappa(X)=0 \Rightarrow h^{2,0}(X) \leq 3$ 」を証明するには、ひとつだけ大きな gap が残っている。ここでは、これ以上細部にはたどり着かないで次の予想 ($\kappa=0$ の場合は Ueno による) をあげておわりとする。

Conjecture: Let X be a 3-dimensional projective alg manifold.

Then 1) $\kappa(X)=0 \Rightarrow h^{2,0}(X) \leq r$?

2) $\dim B(X)=0 \Rightarrow h^{2,0}(X) \leq r$?

(この2)は前で述べた conjecture (2.1.3) の weak version の特別なものである。)