

Properties of Operator L^λ in the Classes $J_\alpha^*(k)$ and $E_\alpha(k)$

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Abstract . We investigate the relationship between $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $L^\lambda f(z) = z + \sum_{n=2}^{\infty} n^{-\lambda} a_n z^n$, λ real, when $f(z)$ is analytic and univalent in the unit disk, and when $f(z)$ is in the classes $J_\alpha^*(k)$ and $E_\alpha(k)$ of analytic univalent functions defined in terms of certain operators of fractional calculus .

1 . Introduction

Let S , S^* and K denote the classes consisting of analytic functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in U = \{z: |z| < 1\} \quad (1)$$

that are, respectively, univalent, starlike, and convex in U . For an analytic function $f(z)$ given by (1), Komatu [2] defined the linear integral transformation $L^\lambda f$ by

$$L^\lambda f(z) = z + \sum_{n=2}^{\infty} n^{-\lambda} a_n z^n \quad (\lambda \text{ real}, z \in U). \quad (2)$$

The function $L^\lambda f(z)$ is clearly analytic in U . Questions arise as to when $L^\lambda f$ will be in the same class as f . For example, what is the smallest λ for which $L^\lambda f \in S$ whenever f is? In [2], Komatu proved that if $f \in S$, then $L^\lambda f \in S^*$ at least for $\lambda \geq \lambda_0$, where $\lambda_0 \in (3, 4)$ is the unique root of equation $\zeta(\lambda-2)=2$ (ζ denotes the Riemann zeta function), and conjectured that

(I) If $f \in S$, then $L^\lambda f \in S$ at least for $\lambda \geq 1$;

(II) If $f \in K$ (or, more generally, $f \in S^*$), then $L^\lambda f \in K$ at least for $\lambda \geq 1$.

Lewis [3] essentially showed that the conjecture (II) is true (cf.[5]). In the case $\lambda=1$, the conjecture (I) reduces to the Biernarcki conjecture which is false [1,P.257]. We note that the conjecture (I) is also false in the case $\lambda=2$. In fact, for the function $f(z)=z(1-z)^{(1+i)} \in S$, we have $z(L^2 f(z))' = L^1 f(z) = -i((1-z)^{-1} - 1)$. Hence $z(L^2 f(z))' = 0$ at $z = 1 - e^{-2\pi i} \in U$, this shows that $L^2 f(z) \notin S$.

Owa [4] and Silverman [5] investigated the relationship between $f(z)$ and $L^\lambda f(z)$, λ real, when $f(z)$ is in the subsets of S, S^* and K .

Let $J_\alpha^*(k)$ denote the class of functions

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \quad (a_n \geq 0, z \in U) \quad (3)$$

which are analytic and univalent in U and satisfy the condition

$$\operatorname{Re} \left\{ \frac{\Gamma(2-\alpha) z^\alpha D_z^\alpha f(z)}{f(z)} \right\} > k \quad (z \in U) \quad (4)$$

for $0 \leq \alpha < 1$ and $0 \leq k < 1$, where $D_z^\alpha f(z)$ denotes the fractional derivative of $f(z)$ of order α (cf.[6]). Furthermore, let $E_\alpha(k)$ denote the class of analytic univalent functions $f(z)$ defined by (3) such that $\Gamma(2-\alpha) z^\alpha D_z^\alpha f(z) \in J_\alpha^*(k)$.

Srivastava and Owa [6] investigated Komatu's conjectures for two general classes $J_\alpha^*(k)$ and $E_\alpha(k)$. In [6], the following results, supporting conjectures (I)(II), were established.

Theorem A. If $f(z) \in J_\alpha^*(k)$, then (i) $L^\lambda f(z) \in J_\alpha^*(k)$ for $\lambda \geq 2$; (ii) $L^\lambda f(z) \in E_\alpha(k)$ for $\lambda \geq 3$; (iii) $L^\lambda f(z) \in J_\alpha^*(0)$ for $\lambda \geq 2$.

Theorem B. If $f(z) \in E_\alpha(k)$, then (i) $L^\lambda f(z) \in E_\alpha(k)$ for $\lambda \geq 2$; (ii) $L^\lambda f(z) \in J_\alpha^*(k)$ for $\lambda \geq \ln(4-\alpha)/\ln 2$; (iii) $L^\lambda f(z) \in J_\alpha^*(0)$ for $\lambda \geq 1 + \{\ln(4-\alpha-4k+2\alpha k) - \ln(2-2k+\alpha k)\}/\ln 2$; (iv) $L^\lambda f(z) \in E_\alpha(0)$.

for $\lambda \geq 2$.

In the present paper, we shall improve the results of Theorem A and Theorem B further.

2. Main Results

In our investigation of Komatu's conjectures for the classes $J_\alpha^*(k)$ and $E_\alpha(k)$, we need the following lemmas (cf. [6]).

Lemma 1. The function $f(z)$ defined by (3) is in the class $J_\alpha^*(k)$ if and only if

$$\sum_{n=2}^{\infty} \left(\frac{\Gamma(n+1) \Gamma(2-\alpha)}{\Gamma(n+1-\alpha)} - k \right) a_n \leq 1 - k. \quad (5)$$

Lemma 2. The function $f(z)$ defined by (3) is in the class $E_\alpha(k)$ if and only if

$$\sum_{n=2}^{\infty} \frac{\Gamma(n+1) \Gamma(2-\alpha)}{\Gamma(n+1-\alpha)} \left(\frac{\Gamma(n+1) \Gamma(2-\alpha)}{\Gamma(n+1-\alpha)} - k \right) a_n \leq 1 - k. \quad (6)$$

Theorem 1. (i) If $f(z) \in J_\alpha^*(k)$, then $L^\lambda f(z) \in J_\alpha^*(k)$ for $\lambda \geq 0$.

(ii) If $f(z) \in E_\alpha(k)$, then $L^\lambda f(z) \in E_\alpha(k)$ for $\lambda \geq 0$.

These results are all sharp.

Proof. (i) Let $f(z) = z - \sum_{n=2}^{\infty} a_n z^n \in J_\alpha^*(k)$. By using Lemma 1, we show that (5) implies

$$\sum_{n=2}^{\infty} \left(\frac{\Gamma(n+1) \Gamma(2-\alpha)}{\Gamma(n+1-\alpha)} - k \right) \frac{a_n}{n^\lambda} \leq 1 - k \quad (7)$$

for $\lambda \geq 0$, where $0 \leq \alpha < 1$, $0 \leq k < 1$.

For any real $\lambda \geq 0$, it follows from (5) that

$$\begin{aligned} & \sum_{n=2}^{\infty} \left(\frac{\Gamma(n+1) \Gamma(2-\alpha)}{\Gamma(n+1-\alpha)} - k \right) \frac{a_n}{n^\lambda} \\ & \leq \sum_{n=2}^{\infty} \left(\frac{\Gamma(n+1) \Gamma(2-\alpha)}{\Gamma(n+1-\alpha)} - k \right) a_n \end{aligned}$$

$$\leq 1 - k. \quad (8)$$

Hence $L^\lambda f(z) \in J_\alpha^*(k)$ for $\lambda \geq 0$.

To show sharpness, set

$$f_n(z) = z - \frac{(1-k) \Gamma(n+1-\alpha)}{\Gamma(n+1) \Gamma(2-\alpha) - k \Gamma(n+1-\alpha)} z^n,$$

and observe that $f_n(z) \in J_\alpha^*(k)$, but $L^\lambda f_n(z) \notin J_\alpha^*(k)$ for $\lambda < 0$.

(ii) The proof of (ii) is much akin to that of (i) detailed already : indeed, instead of Lemma 1 and $f_n(z)$, it uses Lemma 2 and

$$g_n(z) = z - \frac{(1-k) \{\Gamma(n+1-\alpha)\}^2}{\Gamma(n+1) \Gamma(2-\alpha) \{\Gamma(n+1) \Gamma(2-\alpha) - k \Gamma(n+1-\alpha)\}} z^n \in E_\alpha(k).$$

The proof of Theorem 1 is completed.

Corollary 1. (i) If $f(z) \in J_\alpha^*(k)$, then $L^\lambda f(z) \in J_\alpha^*(0)$ for $\lambda \geq 0$, and $L^\lambda f(z) \notin J_\alpha^*(0)$ for $\lambda < \lambda_0 = \{\ln(2-2k) - \ln(2-2k+k\alpha)\} / \ln 2$. (ii) If $f(z) \in E_\alpha(k)$, then $L^\lambda f(z) \in E_\alpha(0)$ for $\lambda \geq 0$, and $L^\lambda f(z) \notin E_\alpha(0)$ for $\lambda < \lambda_0 = \{\ln(2-2k) - \ln(2-2k+k\alpha)\} / \ln 2$.

Proof. Since $J_\alpha^*(k) \subset J_\alpha^*(0)$, $E_\alpha(k) \subset E_\alpha(0)$, we have from Theorem 1 that if $f(z) \in J_\alpha^*(k)$ then $L^\lambda f(z) \in J_\alpha^*(0)$ for $\lambda \geq 0$; if $f(z) \in E_\alpha(k)$ then $L^\lambda f(z) \in E_\alpha(0)$ for $\lambda \geq 0$. For the functions $f_2(z)$ and $g_2(z)$, we have $L^\lambda f_2(z) \notin J_\alpha^*(0)$ and $L^\lambda g_2(z) \notin E_\alpha(0)$ for $\lambda < \lambda_0$. The proof of Corollary 1 is completed.

Theorem 2. If $f(z) \in J_\alpha^*(k)$, then $L^\lambda f(z) \in E_\alpha(k)$ for $\lambda \geq \alpha$, and $L^\lambda f(z) \notin E_\alpha(k)$ for $\lambda < 1 - \ln(2-\alpha)/\ln 2$.

Proof. By virtue of Lemma 1 and Lemma 2, we show that (5) implies

$$\sum_{n=2}^{\infty} \frac{\Gamma(n+1) \Gamma(2-\alpha)}{\Gamma(n+1-\alpha)} \left(\frac{\Gamma(n+1) \Gamma(2-\alpha)}{\Gamma(n+1-\alpha)} - k \right) \frac{a_n}{n^\lambda} \leq 1 - k \quad (9)$$

for $\lambda \geq \alpha$. It suffices to prove that $H(n) = \Gamma(n+1) \Gamma(2-\alpha) / (n^\lambda \Gamma(n+1-\alpha)) \leq 1$ for $\lambda \geq \alpha$ and $n \geq 2$. Let $h(n) = H(n+1)/H(n)$. Then $\lim_{n \rightarrow \infty} h(n) = 1$, $h'(n) \geq 0$ for $\lambda \geq \alpha$ and $n \geq 2$. Thus $H(n)$ is a decreasing function of n ($n \geq 2$) for $\lambda \geq \alpha$. Since $H(2) = 1 / ((2-\alpha) 2^{\lambda-1}) \leq 1$ when $\lambda \geq 1 - \ln(2-\alpha)/\ln 2$, and $\max\{\alpha, 1 - \ln(2-\alpha)/\ln 2\} = \alpha$, we have $H(n) \leq 1$ for $\lambda \geq \alpha$ and $n \geq 2$.

For the function $f_2(z) = z - (1-k)(2-\alpha)/(2-k(2-\alpha)) z^2 \in J_\alpha^*(k)$, we have $L^\lambda f_2(z) \notin E_\alpha(k)$ for $\lambda < 1 - \ln(2-\alpha)/\ln 2$. This completes the proof.

Theorem 3. If $f(z) \in E_\alpha(k)$, then $L^\lambda f(z) \in J_\alpha^*(k)$ for $\lambda \geq -2\alpha/(3-\alpha)$, and $L^\lambda f(z) \notin J_\alpha^*(k)$ for $\lambda < \ln(2-\alpha)/\ln 2 - 1$.

Proof. Since $f(z) = z - \sum_{n=2}^{\infty} a_n z^n \in E_\alpha(k)$, by using Lemma 2, we have

$$\sum_{n=2}^{\infty} \frac{\Gamma(n+1) \Gamma(2-\alpha)}{\Gamma(n+1-\alpha)} \left(\frac{\Gamma(n+1) \Gamma(2-\alpha)}{\Gamma(n+1-\alpha)} - k \right) a_n \leq 1 - k \quad (10)$$

for $0 \leq \alpha < 1$, $0 \leq k < 1$. Let $H(n) = \Gamma(n+1) \Gamma(2-\alpha) n^\lambda / \Gamma(n+1-\alpha)$, $h(n) = H(n+1)/H(n)$. Then $\lim_{n \rightarrow \infty} h(n) = 1$, $h'(n) \leq 0$ for $n \geq 2$ and $\lambda \geq -2\alpha/(3-\alpha)$. Thus $H(n)$ is an increasing function of n ($n \geq 2$) for $\lambda \geq -2\alpha/(3-\alpha)$. But $H(2) = 2^{\lambda+1}(2-\alpha) \geq 1$ for $\lambda \geq \{\ln(2-\alpha) - \ln 2\} / \ln 2$, and

$$\max \left\{ \frac{-2\alpha}{3-\alpha}, \frac{\ln(2-\alpha) - \ln 2}{\ln 2} \right\} = \frac{-2\alpha}{3-\alpha} \quad (0 \leq \alpha < 1), \quad (11)$$

we get $H(n) \geq 1$ for $\lambda \geq -2\alpha/(3-\alpha)$ and $n \geq 2$, i.e.,

$$\frac{\Gamma(n+1) \Gamma(2-\alpha)}{\Gamma(n+1-\alpha)} \geq \frac{1}{n^\lambda} \quad \left(\lambda \geq \frac{-2\alpha}{3-\alpha}, n \geq 2 \right). \quad (12)$$

It follows from (10) and (12) that

$$\sum_{n=2}^{\infty} \left(\frac{\Gamma(n+1) \Gamma(2-\alpha)}{\Gamma(n+1-\alpha)} - k \right) \frac{a_n}{n^\lambda} \leq 1 - k$$

for $\lambda \geq -2\alpha/(3-\alpha)$. Hence $L^\lambda f(z) \in J_\alpha^*(k)$.

For the function

$$g_2(z) = z - \frac{(1-k)(2-\alpha)^2}{2(2-k(2-\alpha))} z^2 \in E_\alpha(k),$$

we have $L^\lambda g_2(z) \notin J_\alpha^*(k)$ for $\lambda < (\ln(2-\alpha) - \ln 2)/\ln 2$. This completes the proof .

Corollary 2 . If $f(z) \in E_\alpha(k)$, then $L^\lambda f(z) \in J_\alpha^*(0)$ for $\lambda \geq -2\alpha/(3-\alpha)$ and $L^\lambda f(z) \notin J_\alpha^*(0)$ for $\lambda < \{\ln(2(1-k)-\alpha(1-k)) - \ln(2(1-k)+k\alpha)\}/\ln 2$.

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