

Global structure of plane closed elastic curves

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1 Introduction

This is a joint work with Waichiro Matsumoto and Shoji Yotsutani (Ryukoku University).

Let Γ be a plane closed elastic curve with length 2π . We denote arc-length and curvature by s and $\kappa(s)$, respectively. Let M be the signed area defined by

$$M := \frac{1}{2} \int_{\Gamma} xdy - ydx,$$

where $(x, y) = (x(s), y(s)) \in \Gamma$ with $(x(0), y(0)) := (0, 0)$. Let us consider the following variational problem (VP) :

Find a curve Γ (the curvature $\kappa(s)$) which minimize $\frac{1}{2} \int_0^{2\pi} \kappa(s)^2 ds$ subject to $\pi > M$ and $\omega\pi \neq M$, where ω is the winding number.

K. Watanabe ([1, 2]) considered this variational problem (VP) with $\omega = 1$. He derived the Euler-Lagrange equation to (VP) and showed the existence of the minimizer and investigate the profile near the disk.

The Euler-Lagrange equation to (VP) is

$$(P^\omega) \begin{cases} \kappa_{ss} + \frac{1}{2}\kappa^3 + \mu\kappa - \nu = 0, & s \in [0, 2\pi], \\ \kappa(0) = \kappa(2\pi), \quad \kappa_s(0) = \kappa_s(2\pi), \\ \frac{1}{2\pi} \int_0^{2\pi} \kappa(s) ds = \omega, \\ \frac{4\mu\pi^2 + \pi \int_0^{2\pi} \kappa(s)^2 ds}{4\pi\omega\mu + \int_0^{2\pi} \kappa(s)^3 ds} = M, \end{cases}$$

where μ and ν are some constants. We can obtain the following proposition by using the argument of K.Watanabe [1, Lemma 3 and Lemma 4]

Proposition 1.1 *Suppose that $\kappa(s)$ is a solution of (P^ω) , then the following properties hold:*

- (i) $\kappa(s) \in C^\infty([0, 2\pi])$.
- (ii) *There exists a positive integer m such that $\kappa(s)$ is periodic function*

with period $s = 2\pi/m$ and axially symmetric with respect to $s = \pi/m$ and m denotes the number of minimum points of $\kappa(s)$ by normalizing $\kappa(0) := \max_{0 \leq s \leq 2\pi} \kappa(s)$. (We call this solution “ m – mode solution”.)

Let us normalize $\kappa(s)$ as $\kappa(0) := \max_{0 \leq s \leq 2\pi} \kappa(s)$. For n -mode solution $\kappa(s)$, we may consider the following differential equation:

$$(P_n^\omega) \begin{cases} \kappa_{ss} + \frac{1}{2}\kappa^3 + \mu\kappa - \nu = 0, & s \in \left[0, \frac{\pi}{n}\right], & (1.1) \\ \kappa_s(0) = \kappa_s\left(\frac{\pi}{n}\right) = 0, \quad \kappa_s(s) < 0 & s \in \left(0, \frac{\pi}{n}\right), & (1.2) \\ \int_0^{\pi/n} \kappa(s) ds = \frac{\omega\pi}{n}, & & (1.3) \\ \frac{2\mu\pi^2 + n\pi \int_0^{\pi/n} \kappa(s)^2 ds}{2\pi\omega\mu + n \int_0^{\pi/n} \kappa(s)^3 ds} = M. & & (1.4) \end{cases}$$

We introduce the following auxiliary problem. Let $\kappa(s)$ be unknown function, and μ, ν be unknown constants. Find $(\kappa(s), \mu, \nu)$ such that

$$(E_n) \begin{cases} \kappa_{ss} + \frac{1}{2}\kappa^3 + \mu\kappa - \nu = 0, & s \in \left[0, \frac{\pi}{n}\right], & (1.5) \\ \kappa_s(0) = \kappa_s\left(\frac{\pi}{n}\right) = 0, \quad \kappa_s(s) < 0 & \text{for } s \in \left(0, \frac{\pi}{n}\right). & (1.6) \end{cases}$$

First we represent all solution $(\kappa(s), \mu, \nu)$ of (E_n) . Next we give the representation of the constraint (1.3) and (1.4).

We prepare notations to state our theorems.

Definition 1.1 We define the complete elliptic integral of first, second and third kind by

$$\begin{aligned} K(k) &:= \int_0^1 \frac{d\xi}{\sqrt{(1-\xi^2)(1-k^2\xi^2)}}, \\ E(k) &:= \int_0^1 \sqrt{\frac{1-k^2\xi^2}{1-\xi^2}} d\xi, \\ \Pi(\ell, k) &:= \int_0^1 \frac{d\xi}{(1+\ell\xi^2)\sqrt{(1-\xi^2)(1-k^2\xi^2)}}. \end{aligned}$$

Definition 1.2 *Jacobi's sn function is defined by*

$$z = \int_0^{\text{sn}(z,k)} \frac{d\xi}{\sqrt{(1-\xi^2)(1-k^2\xi^2)}}$$

and Jacobi's cn function is defined by

$$\text{cn}(z, k) := \sqrt{1 - \text{sn}^2(z, k)}$$

for $z \in [-K(k), K(k)]$. These elliptic functions are extended to $(-\infty, \infty)$ by using the relation $\text{sn}(z + 2K(k), k) = -\text{sn}(z, k)$ and $\text{cn}(z + 2K(k), k) = -\text{cn}(z, k)$.

1.1 Main Result

Theorem 1.1 *All solutions $(\kappa(s), \mu, \nu)$ of (E_n) are represented by the following (i), (ii) and (iii):*

(i) $\kappa(s) = \bar{\kappa}_n(s; k, h)$, $\mu = \bar{\mu}_n(k, h)$ and $\nu = \bar{\nu}_n(k, h)$ for $(k, h) \in \bar{\Sigma}$, where

$$\bar{\Sigma} := \Sigma_{S^*} \cup \Sigma_S, \quad (1.7)$$

$$\Sigma_{S^*} := \{(k, h); -1 < k \leq 0, 2 < h < 3\}, \quad (1.8)$$

$$\Sigma_S := \{(k, h); 0 \leq k < 1, 0 < h \leq 3 - 2k^2\}, \quad (1.9)$$

$$\bar{\kappa}_n(s; k, h) := \begin{cases} \kappa_n^{S^*}(s; k, v(k, h)) & \text{for } (k, h) \in \Sigma_{S^*}, \\ \kappa_n^S(s; k, u(k, h)) & \text{for } (k, h) \in \Sigma_S, \end{cases} \quad (1.10)$$

$$\bar{\mu}_n(k, h) := \begin{cases} \mu_n^{S^*}(k, v(k, h)) & \text{for } (k, h) \in \Sigma_{S^*}, \\ \mu_n^S(k, u(k, h)) & \text{for } (k, h) \in \Sigma_S, \end{cases} \quad (1.11)$$

and

$$\bar{\nu}_n(k, h) := \begin{cases} \nu_n^{S^*}(k, v(k, h)) & \text{for } (k, h) \in \Sigma_{S^*}, \\ \nu_n^S(k, u(k, h)) & \text{for } (k, h) \in \Sigma_S. \end{cases} \quad (1.12)$$

Here the functions $\kappa_n^*(s; k, v)$, $\mu_n^*(k, v)$, $\nu_n^*(k, v)$ and $v(k, h)$ are defined by

$$\begin{aligned} \kappa_n^{S^*}(s; k, v) := & - \frac{\sqrt{1-v}\sqrt{(1-k^2)v+1+k^2}}{\sqrt{v+1}\left(2-(1+v)\text{cn}^2\left(\frac{n}{\pi}K(k)\left(\frac{\pi}{n}-s\right), k\right)\right)} \left(\frac{4\sqrt{2}n}{\pi}K(k)\right) \\ & + \frac{4-(1-k^2)(1-v)^2+4k^2(1-v)}{\sqrt{1-v^2}\sqrt{(1-k^2)v+1+k^2}} \left(\frac{n}{\sqrt{2}\pi}K(k)\right), \end{aligned} \quad (1.13)$$

$$\mu_n^{S^*}(k, v) := \left(\frac{-3(4 - (1 - k^2)(1 - v)^2)^2}{(1 - v^2)((1 - k^2)v + 1 + k^2)} + 8(2 - k^2) \right) \left(\frac{n}{2\pi} K(k) \right)^2, \quad (1.14)$$

$$\begin{aligned} \nu_n^{S^*}(k, v) := & \frac{-2\sqrt{2}(4 - (1 - k^2)(1 - v)^2)}{(1 - v^2)^{3/2}((1 - k^2)v + 1 + k^2)^{3/2}} \\ & \left((1 + v)^2((1 - k^2)v + 1 + k^2)^2 - k^4(1 - v)^2 \right) \left(\frac{n}{2\pi} K(k) \right)^3 \end{aligned} \quad (1.15)$$

and

$$v(k, h) := \frac{-2 + (2 - k^2)(2 - h) + \sqrt{(2 - k^2)^2(2 - h)^2 + 4k^4(3 - h)}}{2(1 - k^2)} \quad (1.16)$$

and the functions $\kappa_n(s; k, u)$, $\mu_n(k, u)$, $\nu_n(k, u)$ and $u(k, h)$ are also defined by

$$\begin{aligned} \kappa_n^S(s; k, u) := & -\frac{(1 - k^2)(1 - ku) + k((1 - k^2)u + k) \operatorname{cn}\left(\frac{2n}{\pi} K(k) \left(\frac{\pi}{n} - s\right), k\right)}{(1 - k^2)u + k - k(1 - ku) \operatorname{cn}\left(\frac{2n}{\pi} K(k) \left(\frac{\pi}{n} - s\right), k\right)} \\ & \frac{\sqrt{u}\sqrt{(1 - 2k^2)u + 2k}}{\sqrt{(1 - k^2)u^2 + 1}} \cdot \left(\frac{4nK(k)}{\pi} \right) \\ & + \frac{(1 - ku)((1 - k^2)u + k)}{\sqrt{u}\sqrt{(1 - 2k^2)u + 2k}\sqrt{(1 - k^2)u^2 + 1}} \left(\frac{2nK(k)}{\pi} \right), \end{aligned} \quad (1.17)$$

$$\begin{aligned} \mu_n^S(k, u) := & \left(\frac{-3(1 - ku)^2((1 - k^2)u + k)^2}{2u((1 - 2k^2)u + 2k)((1 - k^2)u^2 + 1)} + 1 - 2k^2 \right) \\ & \left(\frac{2nK(k)}{\pi} \right)^2, \end{aligned} \quad (1.18)$$

$$\begin{aligned} \nu_n^S(k, u) := & \frac{-(1 - ku)((1 - k^2)u + k)}{4u^{3/2}((1 - 2k^2)u + 2k)^{3/2}((1 - k^2)u^2 + 1)^{3/2}} \\ & \left(4k^2((1 - k^2)u^2 + 1)^2 + (1 - k^2)u^2((1 - 2k^2)u + 2k)^2 \right) \\ & \left(\frac{2nK(k)}{\pi} \right)^3 \end{aligned} \quad (1.19)$$

and

$$\begin{aligned} u(k, h) := & \frac{1}{4k(1 - k^2)} \cdot \left(2 - h + \right. \\ & \left. \frac{(1 - 2k^2)((2 - h)^2 + 16k^2(1 - k^2))}{8k^2(1 - k^2) + \sqrt{(1 - 2k^2)^2(2 - h)^2 + 16k^2(1 - k^2)}} \right). \end{aligned} \quad (1.20)$$

(ii) $\kappa(s) = \underline{\kappa}_n(s; k, h)$, $\mu = \underline{\mu}_n(k, h)$ and $\nu = \underline{\nu}_n(k, h)$ for $(k, h) \in \underline{\Sigma}$, where

$$\underline{\Sigma} := \Sigma_{R^*} \cup \Sigma_R, \quad (1.21)$$

$$\Sigma_{R^*} := \{(k, h); -1 < k \leq 0, -3 < h < -2\}, \quad (1.22)$$

$$\Sigma_R := \{(k, h); 0 \leq k < 1, 2k^2 - 3 \leq h < 0\}. \quad (1.23)$$

Here

$$\underline{\kappa}_n(s; k, h) := -\bar{\kappa}_n\left(\frac{\pi}{n} - s, k, -h\right), \quad (1.24)$$

$$\underline{\mu}_n(k, h) := \bar{\mu}_n(k, -h), \quad \underline{\nu}_n(k, h) := -\bar{\nu}_n(k, -h).$$

(iii) $\kappa(s) = \bar{\kappa}_n(s; k, h)$, $\mu = \bar{\mu}_n(k, h)$ and $\nu = \bar{\nu}_n(k, h)$ for $(k, h) \in \Sigma_0$, where

$$\Sigma_0 := \{(k, h); 0 < k < 1, h = 0\}. \quad (1.25)$$

Here

$$\bar{\kappa}_n(s; k, h) = \frac{4nkK(k)}{\pi} \operatorname{cn}\left(\frac{2n}{\pi}K(k)s, k\right),$$

$$\bar{\mu}_n(k, h) = (1 - 2k^2) \left(\frac{2nK(k)}{\pi}\right)^2, \quad \bar{\nu}_n(k, h) = 0.$$

We show the domains $\bar{\Sigma} \cup \underline{\Sigma} \cup \Sigma_0$ in Figure 1.

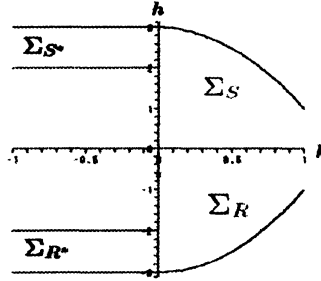


Figure 1: The domain of $\bar{\Sigma} \cup \underline{\Sigma} \cup \Sigma_0$

Remark 1.1 *It is more useful by using the parameter (k, u) and (k, v) than (k, h) . Let us set*

$$\Sigma_v := \{(k, v); -1 < k \leq 0, -1 < v < 1/k\}, \quad (1.26)$$

$$\Sigma_u := \{(k, u); 0 < k < 1, 0 < u < 1/k\}. \quad (1.27)$$

Then the following (i), (ii), (iii) and (iv) hold:

(i) Changing the parameters from (k, h) to (k, v) by $k = k$ and $v = v(k, h)$, Σ_{S^*} becomes Σ_v .

(ii) Changing the parameter from (k, h) to (k, u) by $k = k$ and $u = u(k, h)$, Σ_S becomes Σ_u .

(iii) Changing the parameter from (k, h) to (k, u) by $k = k$ and $u = u(k, -h)$, Σ_R becomes Σ_u .

(iv) Changing the parameter from (k, h) to (k, v) by $k = k$ and $v = v(k, -h)$, Σ_{R^*} becomes Σ_v .

We note that all changing the parameters are bijective.

We show the domains Σ_v and Σ_u in Figure 2.

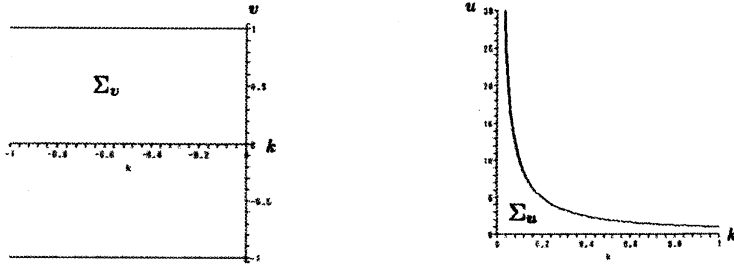


Figure 2: The domains Σ_v and Σ_u

Theorem 1.2 Let $\kappa(s)$ be given by Theorem 1.1 and

$$Z(k, h) := \int_0^{\pi/n} \kappa(s) ds.$$

Then $Z(k, h)$ is given by the following (i), (ii) and (iii):

(i) $Z(k, h) = \bar{Z}(k, h)$ for $(k, h) \in \bar{\Sigma}$, where

$$\bar{Z}(k, h) := \begin{cases} Z^{S^*}(k, v(k, h)) & \text{for } (k, h) \in \Sigma_{S^*}, \\ Z^S(k, u(k, h)) & \text{for } (k, h) \in \Sigma_S. \end{cases}$$

Here the function $Z^{S^*}(k, v)$ is defined by

$$Z^{S^*}(k, v) := \frac{Z_\infty^*(k, v)}{Z_0^*(k, v)}, \quad (1.28)$$

$$Z_\infty^*(k, v) := ((1 - k^2)(1 + v)(3 - v) + 4k^2v)K(k) - 4(1 - k^2)(1 - v^2)\Pi\left(\frac{1}{2}(1 - k^2)(1 - v) - 1, k\right), \quad (1.29)$$

$$Z_0^*(k, v) := \sqrt{2}\sqrt{1 - v^2}\sqrt{(1 - k^2)v + 1 + k^2} \quad (1.30)$$

and the function $Z^S(k, u)$ is also defined by

$$Z^S(k, u) := \frac{2((1 - k^2)u + k)Z_\infty(k, u)}{Z_0(k, u)}, \quad (1.31)$$

$$Z_\infty(k, u) := (2(1 - k^2)u^2 + 2 - (1 - ku)^2)K(k) - 2((1 - k^2)u^2 + 1)\Pi\left(\frac{k^2(1 - ku)^2}{u((1 - 2k^2)u + 2k)}, k\right), \quad (1.32)$$

$$Z_0(k, u) := (1 - ku)\sqrt{u}\sqrt{(1 - 2k^2)u + 2k}\sqrt{(1 - k^2)u^2 + 1}. \quad (1.33)$$

$$(1.34)$$

(ii) $Z(k, h) = \underline{Z}(k, h)$ for $(k, h) \in \underline{\Sigma}$, where

$$\underline{Z}(k, h) := -\overline{Z}(k, -h).$$

(iii) $Z(k, h) = 0$ for $(k, h) \in \Sigma_0$.

The relation $Z(k, h)$ is represented by two forms $\overline{Z}(k, h)$ and $\underline{Z}(k, h)$. The relation $Z(k, h) = \frac{\omega\pi}{n}$ is equivalent to (1.3). For example, we show the level curves of $Z(k, h) = 0$ in the case $\omega = 0$ in Figure 3.

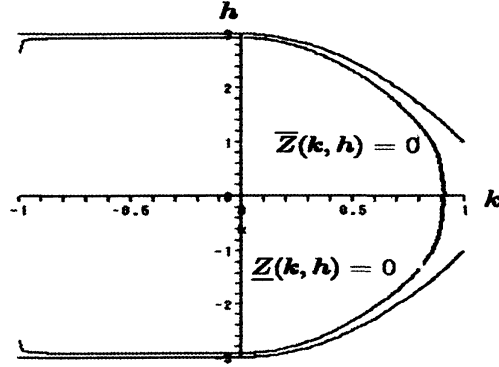


Figure 3: The level curves of $Z(k, h) = 0$

Theorem 1.3 Let $\kappa(s)$ be given by Theorem 1.1 and

$$\mathcal{E}_n(k, h) := n \int_0^{\pi/n} \kappa(s)^2 ds.$$

Then $\mathcal{E}_n(k, h)$ is given by the following (i), (ii) and (iii):

(i) $\mathcal{E}_n(k, h) = \bar{\mathcal{E}}_n(k, h)$ for $(k, h) \in \bar{\Sigma}$, where

$$\bar{\mathcal{E}}_n(k, h) := \begin{cases} \mathcal{E}_n^{S^*}(k, v(k, h)) & \text{for } (k, h) \in \Sigma_{S^*}, \\ \mathcal{E}_n^S(k, u(k, h)) & \text{for } (k, h) \in \Sigma_S. \end{cases} \quad (1.35)$$

Here the function $\mathcal{E}_n^{S^*}(k, v)$ is defined by

$$\begin{aligned} \mathcal{E}_n^{S^*}(k, v) &:= \frac{2n^2}{\pi} K(k) \cdot \\ &\left(\left(\frac{(4 - (1 - k^2)(1 - v)^2)^2}{(1 - v^2)((1 - k^2)v + k^2 + 1)} + 8k^2 - 16 \right) K(k) + 16E(k) \right) \end{aligned} \quad (1.36)$$

and the function $\mathcal{E}_n^S(k, u)$ is also defined by

$$\begin{aligned} \mathcal{E}_n^S(k, u) &:= \frac{4n^2}{\pi} K(k) \cdot \\ &\left(\frac{(1 - ku)^2((1 - k^2)u + k)^2 K(k)}{u((1 - 2k^2)u + 2k)((1 - k^2)u^2 + 1)} - 4((1 - k^2)K(k) - E(k)) \right). \end{aligned} \quad (1.37)$$

(ii) $\mathcal{E}_n(k, h) = \underline{\mathcal{E}}_n(k, h)$ for $(k, h) \in \underline{\Sigma}$, where

$$\underline{\mathcal{E}}_n(k, h) := \bar{\mathcal{E}}_n(k, -h).$$

(iii) $\mathcal{E}_n(k, h) = \bar{\mathcal{E}}_n(k, h)$ for $(k, h) \in \Sigma_0$, where

$$\bar{\mathcal{E}}_n(k, h) = \frac{-16n^2}{\pi} K(k) \cdot ((1 - k^2)K(k) - E(k)).$$

Theorem 1.4 Let $(\kappa(s), \mu, \nu)$ be given by Theorem 1.1 with (1.3) and $h \neq 0$ and

$$M_n(k, h) := \frac{2\mu\pi^2 + n\pi \int_0^{\pi/n} \kappa(s)^2 ds}{2\pi\omega\mu + n \int_0^{\pi/n} \kappa(s)^3 ds}.$$

Then $M_n(k, h)$ is given by the following (i) and (ii):

(i) $M_n(k, h) = \bar{M}_n(k, h)$ for $(k, h) \in \bar{\Sigma}$, where

$$\bar{M}_n(k, h) := \begin{cases} M_n^{S^*}(k, v(k, h)) & \text{for } (k, h) \in \Sigma_{S^*}, \\ M_n^S(k, u(k, h)) & \text{for } (k, h) \in \Sigma_S. \end{cases} \quad (1.38)$$

Here the function $M_n^{S^*}(k, v)$ is defined by

$$\underline{M}_n^{S^*}(k, v) := \frac{\sqrt{2}\pi^2}{n} \cdot \frac{\sqrt{1-v^2}\sqrt{(1-k^2)v+k^2+1}}{((1-k)v+1+k)} \cdot \frac{\varphi_1(k, v)}{\varphi_2(k, v)} \cdot \frac{1}{K(k)^2}, \quad (1.39)$$

$$\begin{aligned} \varphi_1(k, v) &:= \left(-(1-k^2)(1-v)^2 + 4 \right)^2 K(k) \\ &\quad - 8(1-v^2)((1-k^2)v+k^2+1)E(k), \end{aligned} \quad (1.40)$$

$$\begin{aligned} \varphi_2(k, v) &:= ((1+k)v+1-k)((1-k^2)(v+1)^2+4k^2) \cdot \\ &\quad \left(-(1-k^2)(1-v)^2 + 4 \right) \end{aligned} \quad (1.41)$$

and the function $M_n^S(k, u)$ is also defined by

$$M_n^S(k, u) := \frac{-\pi^2}{2n} \cdot \frac{\sqrt{u}\sqrt{(1-k^2)u^2+1}\sqrt{(1-2k^2)u+2k}}{(1-ku)((1-k^2)u+k)} \cdot \frac{\varphi_3(k, u)}{\varphi_4(k, u)} \cdot \frac{1}{K(k)^2}, \quad (1.42)$$

$$\varphi_3(k, u) := - \left((1-ku)^2((1-k^2)u+k)^2 \right. \quad (1.43)$$

$$\begin{aligned} &\quad + u((1-2k^2)u+2k)((1-k^2)u^2+1)) K(k) \\ &\quad + 2u((1-2k^2)u+2k)((1-k^2)u^2+1)E(k), \\ \varphi_4(k, u) &:= k^2(1-ku)^2((1-k^2)u^2+1) \\ &\quad + u((1-2k^2)u+2k)((1-k^2)u+k)^2. \end{aligned} \quad (1.44)$$

(ii) $M_n(k, h) = \underline{M}_n(k, h)$ for $(k, h) \in \underline{\Sigma}$, where

$$\underline{M}_n(k, h) = -\overline{M}_n(k, -h).$$

For given M , the solutions of transcendental equations

$$Z(k, h) = \frac{\omega\pi}{n}, \quad M_n(k, h) = M \quad (1.45)$$

give the solution of (P_n^ω) by Theorem 1.1.

For example, let us determine the solution $\kappa(s)$ of (P_1^0) . Figure 4 shows 1-mode solution which is obtained by solving (1.45) with $\omega = 0$ and $n = 1$. Figure 5 shows the curve which is corresponding to Figure 4. The thick-line is corresponding to $0 \leq s \leq \pi$.

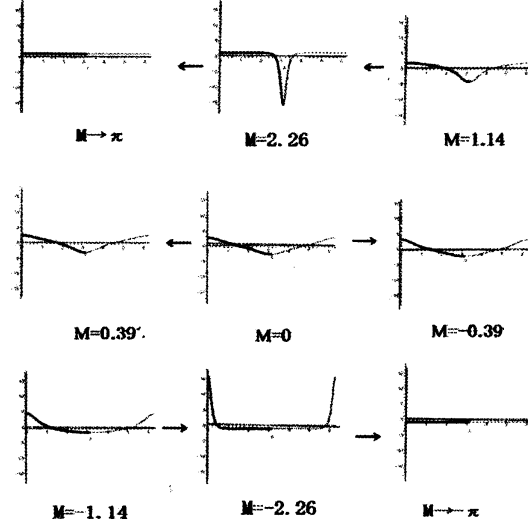


Figure 4: Curvatures of 1-mode solution for $\omega = 0$

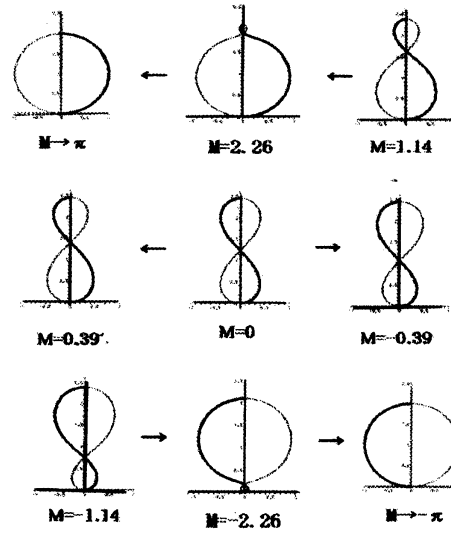


Figure 5: Curves of 1-mode solution for $\omega = 0$

We note that the other curves are not closed except for $k = k_0$ with $h = 0$ in Theorem 1.1, where k_0 is the unique solution of $2E(k) - K(k) = 0$ ($0 < k < 1$).

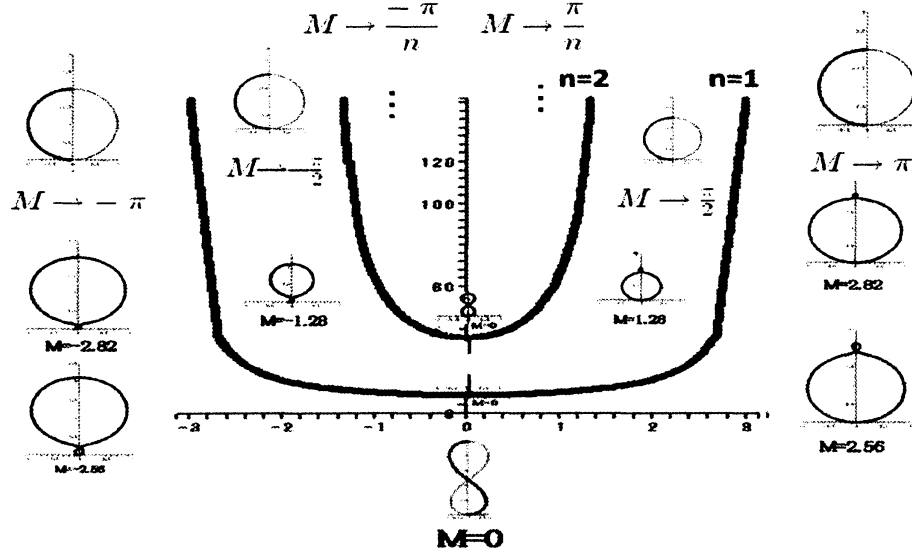


Figure 6: Energy curves of stationary solutions for $\omega = 0$

Figure 6 shows the energy curves of stationary solutions for $\omega = 0$ which are obtained from the equation (1.45) and Theorem 1.3.

Investigating the global structure, we obtain the following theorems.

Theorem 1.5 *Let $\omega = 0$ and $n \geq 1$. Then, there exists a unique n -mode solution $\kappa(s) = \kappa_n(s; M)$ of (P_n^0) for $-\frac{\pi}{n} < M < \frac{\pi}{n}$. Further there exists no solution for $M \leq -\frac{\pi}{n}$, $\frac{\pi}{n} \leq M$.*

Theorem 1.6 *Let $\omega = 0$ and $n \geq 1$. Then, there exists a unique minimizer $\kappa(s) = \kappa(s; M)$ for $-\pi < M < \pi$ with the normalizing condition $\kappa(0) := \max_{0 \leq s \leq 2\pi} \kappa(s)$. This minimizer is 1-mode solution.*

Theorem 1.7 *Let $\omega = 0$ and $n \geq 1$. Then, the n -mode solution $\kappa(s) = \kappa_n(s; M)$ of (P_n^0) with property $\kappa(0) := \max_{0 \leq s \leq \pi/n} \kappa(s)$ for $0 \leq s \leq \pi/n$ satisfies the following relations:*

$$(i) \quad \lim_{M \uparrow \frac{\pi}{n}} \kappa_n(s; M) = \begin{cases} n & \text{for } s \in \left[0, \frac{\pi}{n}\right), \quad \text{uniformly in } \left[0, \frac{\pi}{n}\right) \\ -\infty & \text{for } s = \frac{\pi}{n}. \end{cases}$$

$$(ii) \quad \lim_{M \downarrow -\frac{\pi}{n}} \kappa_n(s; M) = \begin{cases} \infty & s \in \left(0, \frac{\pi}{n}\right], \quad \text{uniformly in } \left(0, \frac{\pi}{n}\right] \\ -n & s = 0. \end{cases}$$

In this paper, we show the proof of Theorem 1.1. We need long calculation to obtain Theorem 1.2 ~ 1.7. The complete proofs of them will appear in a forthcoming papers.

This kind of method was first proposed by Lou-Ni-Yotsutani[5]. Later, Ikeda-Kondo-Okamoto-Yotsutani [3] and Kosugi-Morita-Yotsutani [4] developed the method.

2 Proof of Theorem 1.1

We rewrite (E_n) as first order differential equation to find the solution $\kappa(s)$.

Let us set

$$\kappa(0) := \alpha, \quad \kappa\left(\frac{L}{2n}\right) := \beta \quad (\alpha > 0, \alpha > \beta).$$

Multiplying $2\kappa_s$ to (E_n) , we have

$$\frac{d}{ds} \left(\frac{d\kappa}{ds} \right)^2 = \frac{d}{ds} \left(-\frac{1}{4}\kappa^4 - \mu\kappa^2 + 2\nu\kappa \right).$$

Integrating above equation on $[0, s]$, we obtain

$$\frac{d\kappa}{ds} = -\sqrt{\tilde{g}(\kappa)}, \quad (2.1)$$

where

$$\tilde{g}(\kappa) = -\frac{1}{4}\kappa^4 - \mu\kappa^2 + 2\nu\kappa + \frac{1}{4}\alpha^4 + \mu\alpha^2 - 2\nu\alpha. \quad (2.2)$$

By the Neumann boundary condition of (E_n) , we can rewrite $\tilde{g}(\kappa)$ as

$$\tilde{g}(\kappa) = \frac{1}{4}(\alpha - \kappa)(\kappa - \beta) \left(\left(\kappa + \frac{\alpha + \beta}{2} \right)^2 + 4\delta \right), \quad (2.3)$$

where δ is some constant. Comparing the coefficients of (2.2) with that of (2.3), we obtain

$$\begin{aligned} \mu &= -\frac{1}{8}(3(\alpha + \beta)^2 - \frac{1}{2}(3\alpha + \beta)(\alpha + 3\beta) - 8\delta), \\ \nu &= \frac{1}{32}(\alpha + \beta)((\alpha - \beta)^2 + 16\delta). \end{aligned}$$

Let us set

$$A := \frac{3\alpha + \beta}{4}, \quad B := \frac{\alpha + 3\beta}{4}.$$

Then μ and ν are represented by

$$\begin{aligned}\mu &= \frac{-1}{8}(3(A+B)^2 - 8(AB + \delta)), \\ \nu &= \frac{1}{8}(A+B)((A-B)^2 + \delta).\end{aligned}\tag{2.4}$$

Further, let us set

$$\hat{\kappa} := \frac{1}{2} \left(\kappa + \frac{A+B}{2} \right).\tag{2.5}$$

Then (2.1) is represented by

$$\begin{aligned}\frac{d\hat{\kappa}}{ds} &= -\sqrt{\hat{g}(\hat{\kappa})}, \\ \hat{\kappa}(0) &= A, \quad \hat{\kappa}\left(\frac{L}{2n}\right) = B,\end{aligned}\tag{2.6}$$

where

$$\hat{g}(\hat{\kappa}) = (A - \hat{\kappa})(\hat{\kappa} - B)(\hat{\kappa}^2 + \delta).\tag{2.7}$$

We need to consider the following five cases in (2.6):

- (i) $A + B < 0, \delta \leq 0,$
- (ii) $A + B < 0, \delta > 0,$
- (iii) $A + B > 0, \delta > 0,$
- (iv) $A + B > 0, \delta \leq 0,$
- (v) $A + B = 0.$

After the proof of Theorem 1.1, we obtain the following five equivalent relations:

- (i) $A + B < 0, \delta \leq 0 \iff \Sigma_{S^*},$
- (ii) $A + B < 0, \delta > 0 \iff \Sigma_S,$
- (iii) $A + B > 0, \delta > 0 \iff \Sigma_R,$
- (iv) $A + B > 0, \delta \leq 0, \iff \Sigma_{R^*},$
- (v) $A + B = 0, \delta \geq 0 \iff \Sigma_0,$

where $\Sigma_{S^*}, \Sigma_S, \Sigma_{R^*}, \Sigma_R$ and Σ_0 are given by (1.8), (1.9), (1.22), (1.23) and (1.25), respectively.

We note that there exists no solution for $A + B = 0, \delta < 0$. We also note that if $(\kappa(s), \mu, \nu)$ is a solution of (E_n) , then $(-\kappa(\pi/n - s), \mu, -\nu)$ is also the solution of (E_n) by (2.1) and (2.3). Hence if we have the solutions of (E_n) in the case of (i) and (ii), then we also obtain the solutions of (E_n) in the case (iv) and (iii), respectively. Thus we treat the case (i) and (ii).

2.1 Representation of solutions for $A + B < 0, \delta \leq 0$

Lemma 2.1 *Suppose that $A + B < 0$ and $\delta \leq 0$ in (2.6). Then the solution $\kappa(s)$ of (E_n) is represented by*

$$\kappa(s) = \kappa_n^{S^*}(s; A, B, \eta) := 2\hat{\kappa}_n^{S^*}(s; A, B, \eta) - \frac{A+B}{2} + 2\eta, \quad (2.8)$$

where $\eta := \sqrt{-\delta}$ and

$$\hat{\kappa}_n^{S^*}(s; A, B, \eta) := \frac{(A-\eta)(B-\eta)}{B-\eta + (A-B)\text{cn}^2\left(\frac{n}{\pi}K(k)\left(\frac{\pi}{n}-s\right), k\right)}. \quad (2.9)$$

Moreover it holds that

$$\sqrt{(A-\eta)(B+\eta)} = \frac{2n}{\pi}K(k) \quad (2.10)$$

with

$$k = -\sqrt{\frac{2\eta(A-B)}{(A-\eta)(B+\eta)}}. \quad (2.11)$$

Proof. Under the condition that $A + B < 0$ and $\delta = -\eta^2 \leq 0$ ($\eta \geq 0$), we have

$$B < A \leq -\eta \leq 0 \leq \eta,$$

since $A > B$ and (2.7) is positive on the interval (B, A) . Now we show $A \neq -\sqrt{-\delta}$. Assume that $A = -\sqrt{-\delta} < 0$. Then, substituting $\hat{\kappa} = A - 1/\xi$ into (2.6), we have

$$\begin{aligned} \frac{L}{2n} &= \int_B^A \frac{d\hat{\kappa}}{(A-\hat{\kappa})\sqrt{-(\hat{\kappa}-B)(\hat{\kappa}+A)}} \\ &= \frac{1}{\sqrt{-2A(A-B)}} \int_{1/(A-B)}^\infty \frac{d\xi}{\sqrt{\left(\xi - \frac{1}{A-B}\right)\left(\xi - \frac{1}{2A}\right)}} \\ &= \frac{1}{\sqrt{-2A(A-B)}} \left[2 \log \left| \sqrt{\xi - \frac{1}{A-B}} + \sqrt{\xi - \frac{1}{2A}} \right| \right]_{1/(A-B)}^\infty \\ &= \infty. \end{aligned}$$

This is a contradiction. In the same way, we obtain

$$\frac{L}{2n} = \int_B^0 \frac{d\hat{\kappa}}{-\hat{\kappa}\sqrt{-\hat{\kappa}(\hat{\kappa}-B)}} = \infty$$

in the case $A = -\sqrt{\eta} = 0$. This is also contradiction. Thus it holds that

$$B < A < -\eta \leq 0 \leq \eta. \quad (2.12)$$

Let us set

$$\hat{\kappa}(s) = \frac{1}{\tilde{\kappa}(s)} + \eta. \quad (2.13)$$

Then (2.6) becomes

$$\frac{d\tilde{\kappa}}{ds} = \sqrt{(A-\eta)(B-\eta)} \sqrt{\left(\tilde{\kappa} - \frac{1}{A-\eta}\right) \left(\frac{1}{B-\eta} - \tilde{\kappa}\right) (2\eta\tilde{\kappa} + 1)}.$$

Further we introduce change of variable $\tilde{\kappa}$ to φ by

$$\tilde{\kappa}(s) = \frac{1}{B-\eta} - \left(\frac{1}{B-\eta} - \frac{1}{A-\eta}\right) \sin^2 \varphi(s) \quad \text{for } \varphi(s) \in \left[0, \frac{\pi}{2}\right]. \quad (2.14)$$

We obtain

$$\frac{d\varphi}{ds} = \frac{-1}{2} \sqrt{(A-\eta)(B+\eta)} \sqrt{1 - k^2 \sin^2 \varphi}.$$

Integrating the above equation on $[0, s]$, we have

$$\frac{\sqrt{(A-\eta)(B+\eta)}}{2} s = K(k) - \int_0^{\varphi(s)} \frac{d\varphi}{\sqrt{1 - k^2 \sin^2 \varphi}}, \quad (2.15)$$

since $\varphi(0) = \pi/2$. At $s = \pi/n$, we obtain (2.10) by $\varphi(\pi/n) = 0$.

Substituting (2.10) and $\xi = \sin \varphi$ into (2.15), we have

$$\sin(\varphi(s)) = \operatorname{sn} \left(\frac{n}{\pi} K(k) \left(\frac{\pi}{n} - s \right), k \right),$$

which implies that

$$\tilde{\kappa}(s) = \frac{1}{A-\eta} + \left(\frac{1}{B-\eta} - \frac{1}{A-\eta} \right) \operatorname{cn}^2 \left(\frac{n}{\pi} K(k) \left(\frac{\pi}{n} - s \right), k \right). \quad (2.16)$$

On the other hand, it follows from (2.5) and (2.13) that we have

$$\kappa(s) = 2\hat{\kappa}(s) - \frac{A+B}{2} = \frac{2}{\tilde{\kappa}(s)} - \frac{A+B}{2} + 2\eta.$$

Thus, substituting (2.16) into above relation, the lemma holds. $\square$

2.2 Representation of solutions for $A + B < 0, \delta > 0$

Lemma 2.2 *Suppose that $A + B < 0, \delta > 0$ in (2.6). Then the solution of $\kappa(s)$ of (E_n) is represented by*

$$\kappa(s) = \kappa_n^S(s; A, B, \delta) := 2\hat{\kappa}_n^S(s; A, B, \delta) - \frac{A+B}{2}, \quad (2.17)$$

where

$$\begin{aligned} \hat{\kappa}_n^S(s; A, B, \delta) := & \\ & \frac{AB_\delta + A_\delta B - (AB_\delta - A_\delta B)\text{cn}\left(\frac{2n}{\pi}K(k)\left(\frac{\pi}{n} - s\right), k\right)}{A_\delta + B_\delta + (A_\delta - B_\delta)\text{cn}\left(\frac{2n}{\pi}K(k)\left(\frac{\pi}{n} - s\right), k\right)}, \end{aligned} \quad (2.18)$$

$$A_\delta := \sqrt{A^2 + \delta}, \quad B_\delta := \sqrt{B^2 + \delta}. \quad (2.19)$$

Moreover it holds that

$$\sqrt{A_\delta B_\delta} = \frac{2n}{\pi}K(k) \quad (2.20)$$

with

$$k = \sqrt{\frac{1}{2} \left(1 - \frac{AB + \delta}{A_\delta B_\delta}\right)}. \quad (2.21)$$

Proof. Let us set

$$\hat{\kappa}(s) = \frac{1}{\tilde{\kappa}(s)} + B. \quad (2.22)$$

Then (2.6) becomes

$$\frac{d\tilde{\kappa}}{ds} = B_\delta \sqrt{A - B} \sqrt{\left(\tilde{\kappa} - \frac{1}{A - B}\right) \left(\tilde{\kappa}^2 + \frac{2B}{B_\delta^2} \tilde{\kappa} + \frac{1}{B_\delta^2}\right)}.$$

Further we introduce change of variable $\tilde{\kappa}$ to φ by

$$\tilde{\kappa}(s) = \frac{1}{A - B} \left(1 + \frac{A_\delta}{B_\delta} \tan^2 \frac{\varphi(s)}{2}\right), \quad (2.23)$$

where $\varphi(s) \in [0, \pi]$. We get

$$\begin{aligned} & \frac{A_\delta}{(A - B)B_\delta} \tan \frac{\varphi}{2} \left(1 + \tan^2 \frac{\varphi}{2}\right) \frac{d\varphi}{ds} \\ &= \sqrt{A_\delta B_\delta} \cdot \frac{A_\delta}{(A - B)B_\delta} \tan \frac{\varphi}{2} \sqrt{1 + \tan^4 \frac{\varphi}{2} + 2 \frac{AB + \delta}{A_\delta B_\delta} \tan^2 \frac{\varphi}{2}} \\ &= \sqrt{A_\delta B_\delta} \cdot \frac{A_\delta}{(A - B)B_\delta} \cdot \tan \frac{\varphi}{2} \sqrt{\left(1 + \tan^2 \frac{\varphi}{2}\right)^2 - 4k^2 \tan^2 \frac{\varphi}{2}} \\ &= \sqrt{A_\delta B_\delta} \cdot \frac{A_\delta}{(A - B)B_\delta} \cdot \tan \frac{\varphi}{2} \left(1 + \tan^2 \frac{\varphi}{2}\right) \sqrt{1 - k^2 \sin^2 \varphi}. \end{aligned}$$

Thus we obtain

$$\frac{d\varphi}{ds} = \sqrt{A_\delta B_\delta} \sqrt{1 - k^2 \sin^2 \varphi}.$$

Integrating the above equation on $[0, s]$, we have

$$\sqrt{A_\delta B_\delta} s = \int_0^{\varphi(s)} \frac{d\varphi}{\sqrt{1 - k^2 \sin^2 \varphi}} \quad (2.24)$$

by $\varphi(0) = 0$. At $s = \pi/n$, we have (2.20) by $\varphi(\pi/n) = \pi$.

Substituting (2.20) and $\xi = \sin \varphi$ into (2.24), we obtain

$$\sin \varphi(s) = \operatorname{sn} \left(\frac{2n}{\pi} K(k)s, k \right),$$

which implies that

$$\cos(\varphi(s)) = \operatorname{cn} \left(\frac{2n}{\pi} K(k)s, k \right).$$

Thus we have

$$\tan^2 \frac{\varphi(s)}{2} = \frac{1 - \cos \varphi(s)}{1 + \cos \varphi(s)} = \frac{1 - \operatorname{cn} \left(\frac{2n}{\pi} K(k)s, k \right)}{1 + \operatorname{cn} \left(\frac{2n}{\pi} K(k)s, k \right)}.$$

Substituting above relation into (2.23), $\tilde{\kappa}(s)$ becomes

$$\tilde{\kappa}(s) = \frac{A_\delta + B_\delta - (A_\delta - B_\delta) \operatorname{cn} \left(\frac{2n}{\pi} K(k)s, k \right)}{(A - B) B_\delta \left(1 + \operatorname{cn} \left(\frac{2n}{\pi} K(k)s, k \right) \right)}. \quad (2.25)$$

On the other hand, we obtain

$$\kappa(s) = 2 \hat{\kappa}(s) - \frac{A + B}{2} = 2 \left(\frac{1}{\tilde{\kappa}(s)} + B \right) - \frac{A + B}{2}$$

by (2.5) and (2.22). Thus, substituting (2.25) into above relation, we obtain (2.17) since

$$\operatorname{cn} \left(\frac{2n}{\pi} K(k)s, k \right) = -\operatorname{cn} \left(\frac{2n}{\pi} K(k) \left(\frac{\pi}{n} - s \right), k \right).$$

□

2.3 Change of parameters for $A + B < 0, \delta \leq 0$

Let us consider the case $A + B < 0, \delta \leq 0$. It is difficult for us to investigate the global structure by using the parameters A, B and $\eta := \sqrt{-\delta}$. A and B belong to semi-infinite interval and η is constrained by (2.20) and (2.21). Thus we change the parameter.

Let us see $(k, \tilde{h})$ be known and A, B and η be the solutions of the system of

$$\begin{cases} k^2 = \frac{2\eta(A - B)}{(A - \eta)(B + \eta)}, \end{cases} \quad (2.26)$$

$$\begin{cases} \sqrt{(A - \eta)(B + \eta)} = \frac{2n}{\pi} K(k), \end{cases} \quad (2.27)$$

$$\begin{cases} A = (1 - \tilde{h})B \quad (0 < \tilde{h} < 2). \end{cases} \quad (2.28)$$

Then we obtain the following lemma:

Lemma 2.3 *Suppose that $A + B < 0$ and $\delta \leq 0$. Then A, B and η are represented by*

$$\begin{aligned} A &= -\frac{((1 - k^2)v + 1)\sqrt{1 - v}}{\sqrt{2}\sqrt{v + 1}\sqrt{(1 - k^2)v + k^2 + 1}} \cdot \left(\frac{2n}{\pi} K(k)\right), \\ B &= -\frac{(2 - k^2)v + k^2 + 2}{\sqrt{2}\sqrt{1 - v^2}\sqrt{(1 - k^2)v + k^2 + 1}} \cdot \left(\frac{2n}{\pi} K(k)\right), \\ \eta &= \frac{k^2\sqrt{1 - v}}{\sqrt{2}\sqrt{1 + v}\sqrt{(1 - k^2)v + k^2 + 1}} \cdot \left(\frac{2n}{\pi} K(k)\right) \end{aligned} \quad (2.29)$$

and $v = v(k, h)$ for $(k, h) \in \Sigma_{S^*}$, where Σ_{S^*} and $v(k, h)$ are defined by (1.8) and (1.16), respectively.

Proof. It follows from (2.26), (2.27) and (2.28) that we obtain

$$\eta = \frac{-k^2}{2\tilde{h}B} \left(\frac{2n}{\pi} K(k)\right)^2. \quad (2.30)$$

Substituting (2.28) and (2.30) into (2.27), we have

$$(1 - \tilde{h})B^4 - \frac{1}{2}(2 - k^2) \left(\frac{2n}{\pi} K(k)\right)^2 B^2 - \frac{k^4}{4h^2} \left(\frac{2n}{\pi} K(k)\right)^4 = 0.$$

If $\tilde{h} \geq 1$, then the left hand side of above equation is negative. Thus, we may consider $0 < \tilde{h} < 1$. Solving the above equation with respect to B , we obtain

$$\begin{aligned} A &= -(1 - \tilde{h})\xi \left(\frac{2n}{\pi} K(k)\right), \quad B = -\xi \left(\frac{2n}{\pi} K(k)\right), \\ \eta &= \frac{k^2}{2\tilde{h}\xi} \left(\frac{2n}{\pi} K(k)\right) \end{aligned} \quad (2.31)$$

since $B < 0$, where

$$\xi := \frac{\sqrt{(2-k^2)\tilde{h}} + \sqrt{(2-k^2)^2\tilde{h}^2 + 4k^4(1-\tilde{h})}}{2\sqrt{\tilde{h}}\sqrt{1-\tilde{h}}}$$

for $(k, \tilde{h}) \in \{(k, \tilde{h}); 0 < k < 1, 0 < \tilde{h} < 1\}$.

To simplify the representation, we set

$$v = \frac{-2 + (2-k^2)\tilde{h} + \sqrt{(2-k^2)^2\tilde{h}^2 + 4k^4(1-\tilde{h})}}{2(1-k^2)},$$

which implies that

$$2(1-k^2)v + 2 - (2-k^2)\tilde{h} = \sqrt{(2-k^2)^2\tilde{h}^2 + 4k^4(1-\tilde{h})}. \quad (2.32)$$

Solving (2.32) with respect to $\tilde{h}$ yields

$$\tilde{h} = \frac{(v+1)((1-k^2)v + k^2 + 1)}{(2-k^2)v + k^2 + 2}.$$

Hence we have

$$(2-k^2)\tilde{h} + \sqrt{(2-k^2)^2\tilde{h}^2 + 4k^4(1-\tilde{h})} = 2(1-k^2)v + 2,$$

$$1 - \tilde{h} = \frac{(1-v)((1-k^2)v + 1)}{(2-k^2)v + k^2 + 2}$$

by (2.32). Thus ξ becomes

$$\xi = \frac{(2-k^2)v + k^2 + 2}{\sqrt{2}\sqrt{1-v^2}\sqrt{(1-k^2)v + k^2 + 1}}.$$

Substituting $h = \tilde{h} + 2$ and above relation into (2.31), the lemma holds. $\square$

2.4 Change of parameters for $A + B < 0, \delta > 0$

Let us consider the case $A + B < 0, \delta > 0$. It is also difficult for us to investigate the global structure by using the parameters A, B and δ . Thus we change the parameters.

Let $(k, \tilde{h})$ be known and A, B and δ be the solutions of the system of

$$\begin{cases} k^2 = \frac{1}{2} \left(1 - \frac{AB + \delta}{\sqrt{(A^2 + \delta)(B^2 + \delta)}} \right), \end{cases} \quad (2.33)$$

$$\begin{cases} \sqrt[4]{(A^2 + \delta)(B^2 + \delta)} = \frac{2n}{\pi} K(k), \end{cases} \quad (2.34)$$

$$\begin{cases} A = (1 - \tilde{h})B \quad (0 < \tilde{h} < 2). \end{cases} \quad (2.35)$$

Then we obtain the following lemma:

Lemma 2.4 *Suppose that $A + B < 0$, $\delta > 0$. Then A, B and δ are represented by*

$$\begin{aligned} A &= -\frac{\sqrt{u}(1 - 2k^2 - 2k(1 - k^2)u)}{\sqrt{(1 - k^2)u^2 + 1}\sqrt{(1 - 2k^2)u + 2k}} \cdot \left(\frac{2n}{\pi} K(k) \right), \\ B &= -\frac{\sqrt{(1 - 2k^2)u + 2k}}{\sqrt{u}\sqrt{(1 - k^2)u^2 + 1}} \cdot \left(\frac{2n}{\pi} K(k) \right), \\ \delta &= \frac{(1 - k^2)u((1 - 2k^2)u + 2k)}{(1 - k^2)u^2 + 1} \cdot \left(\frac{2n}{\pi} K(k) \right)^2 \end{aligned} \quad (2.36)$$

and $u = u(k, h)$ for $(k, h) \in \Sigma_S$, where $u(k, h)$ and Σ_S are defined by (1.9) and (1.20), respectively.

Proof. It follows from (2.34) and (2.33) that we obtain

$$\delta = (1 - 2k^2) \left(\frac{2n}{\pi} K(k) \right)^2 - (1 - \tilde{h})B^2. \quad (2.37)$$

Substituting (2.35) and (2.37) into (2.34), we have

$$-\tilde{h}^2(1 - \tilde{h})B^4 + (1 - 2k^2)\tilde{h}^2 \left(\frac{2n}{\pi} K(k) \right)^2 B^2 - 4k^2(1 - k^2) \left(\frac{2n}{\pi} K(k) \right)^4 = 0.$$

Solving above equation with respect to B , we obtain the following two solutions (i) and (ii):

$$\begin{aligned} (i) \quad B &= \frac{-2\sqrt{2}k\sqrt{1 - k^2}}{\sqrt{\tilde{h}}\sqrt{(1 - 2k^2)\tilde{h} + \sqrt{\tilde{h}^2 - 4k^2(1 - k^2)(2 - \tilde{h})^2}}} \cdot \left(\frac{2n}{\pi} K(k) \right) \\ &\text{for } (k, \tilde{h}) \in \{(k, \tilde{h}); 0 < k \leq 1/\sqrt{2}, H(k) < \tilde{h} < 2\} \\ &\quad \cup \{(k, \tilde{h}); 1/\sqrt{2} < k \leq 1, 1 < \tilde{h} < 2\}, \end{aligned}$$

$$\begin{aligned} (ii) \quad B &= \frac{-2\sqrt{2}k\sqrt{1 - k^2}}{\sqrt{\tilde{h}}\sqrt{(1 - 2k^2)\tilde{h} - \sqrt{\tilde{h}^2 - 4k^2(1 - k^2)(2 - \tilde{h})^2}}} \cdot \left(\frac{2n}{\pi} K(k) \right) \\ &\text{for } (k, \tilde{h}) \in \{(k, \tilde{h}); 0 < k \leq 1/\sqrt{2}, H(k) < \tilde{h} < 1\}, \end{aligned}$$

since $B < 0$, where

$$H(k) := \frac{4k\sqrt{1-k^2}}{1+2k\sqrt{1-k^2}}.$$

Further changing the parameter $(k, \tilde{h})$ to (k, h) by

$$h = \begin{cases} 2 - \sqrt{\tilde{h}^2 - 4k^2(1-k^2)(2-\tilde{h})^2} & \text{for case (i),} \\ 2 + \sqrt{\tilde{h}^2 - 4k^2(1-k^2)(2-\tilde{h})^2} & \text{for case (ii),} \end{cases}$$

$\tilde{h}$ becomes

$$\tilde{h} = \frac{(2-h)^2 + 16k^2(1-k^2)}{8k^2(1-k^2) + \sqrt{(1-2k^2)^2(2-h)^2 + 16k^2(1-k^2)}} \quad (2.38)$$

$for (k, h) \in \Sigma_S,$

where Σ_S is defined by (1.9).

To simplify the representation, we set

$$\begin{aligned} u &= \frac{1}{4k(1-k^2)} \cdot \left(2-h + \frac{(1-2k^2)((2-h)^2 + 16k^2(1-k^2))}{8k^2(1-k^2) + \sqrt{(1-2k^2)^2(2-h)^2 + 16k^2(1-k^2)}} \right) \\ &= \frac{1}{4k(1-k^2)} \cdot \left(2-h + \frac{-8k^2(1-k^2) + \sqrt{(1-2k^2)^2(2-h)^2 + 16k^2(1-k^2)}}{1-2k^2} \right), \end{aligned} \quad (2.39)$$

which implies that

$$(1-2k^2)(4k(1-k^2)u-2+h)+8k^2(1-k^2) = \sqrt{(1-2k^2)^2(2-h)^2 + 16k^2(1-k^2)}.$$

Solving the above equation with respect to h yields

$$h = \frac{2(1-ku)((1-k^2)(1-2k^2)u + k(3-2k^2))}{(1-2k^2)u + 2k}.$$

Substituting the above relation into (2.38) gives

$$\begin{aligned} \tilde{h} &= \frac{4k(1-k^2)u - (2-h)}{(1-2k^2)} \\ &= \frac{2k((1-k^2)u^2 + 1)}{(1-2k^2)u + 2k} \end{aligned}$$

by (2.39). Hence we have

$$1 - \tilde{h} = \frac{u(1 - 2k^2 - 2k(1 - k^2)u)}{(1 - 2k^2)u + 2k}.$$

Further we obtain

$$(1 - 2k^2)\tilde{h} + \sqrt{\tilde{h}^2 - 4k^2(1 - k^2)(2 - \tilde{h})^2} = 4k(1 - k^2)u$$

by (2.38) and (2.39) in the case (i). We also obtain

$$(1 - 2k^2)\tilde{h} - \sqrt{\tilde{h}^2 - 4k^2(1 - k^2)(2 - \tilde{h})^2} = 4k(1 - k^2)u.$$

in the case (ii). Using above relations, the lemma holds. $\square$

Proof of Theorem 1.1. Substituting (2.29) and (2.36) into (2.4), (2.8) and (2.17), we obtain (i) of Theorem 1.1.

We obtain (ii) of Theorem 1.1 since if $(\kappa(s), \mu, \nu)$ is a solution of (E_n) , then $(-\kappa(\pi/n - s), \mu, -\nu)$ is also the solution of (E_n) by (2.1) and (2.3).

It follows from Lemma 2.4 that $A + B = 0, \delta \geq 0$ is equivalent to Σ_0 . Thus we obtain (iii) of Theorem 1.1 since $u(k, 0) = 1/k$. $\square$

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