

## Almost $n$ -dimensional spaces

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We consider only separable metric spaces. A space  $X$  is said to be almost  $n$ -dimensional if it has a basis  $\{U_i\}$  such that if  $\text{cl}U_i \cap \text{cl}U_j = \emptyset$  then  $X = G \cup H$  where  $G$  and  $H$  are closed sets,  $U_i \subset G \setminus H$ ,  $U_j \subset H \setminus G$  and  $\dim G \cap H \leq n-1$  and  $n$  is the smallest natural number such that such a basis exists for  $n$ . It is clear that  $n$ -dimensional spaces are at most almost  $n$ -dimensional.

Oversteegen and Tymchatyn [9] proved that almost 0-dimensional spaces are at most 1-dimensional. The Erdős space of irrational sequences in Hilbert space is known to be a universal almost 0-dimensional space [5]. Erdős space is 1-dimensional. Homeomorphism groups of positive dimensional Menger compacta are almost 0-dimensional [9] and at least 1-dimensional by classical results of Brechner [2] and Bestvina [1].

Almost 0-dimensional spaces are at most 1-dimensional and the 1-dimensionality cannot be improved. Our first result shows that this interesting behaviour does not occur in higher dimensions and the following one points out an interesting property of almost 0-dimensional spaces.

**Theorem 1** (Levin-Tymchatyn [7]) *If  $X$  is almost  $n$ -dimensional,  $n \geq 1$  then  $X$  is  $n$ -dimensional.*

**Theorem 2** (Levin-Tymchatyn [7]) *Let  $X = X_1 \cup X_2$  where  $X_1$  is almost 0-dimensional and  $X_2$  is 0-dimensional. Then  $\dim X \leq 1$ .*

The proof of these theorems employs so-called  $L$ -embeddings. A subset  $X$  of a compactum  $K$  is  $L$ -embedded in  $K$  if for every open cover  $\mathcal{U}$  of  $K$  there is a neighbourhood  $U$  of  $X$  in  $K$  such that the continua in  $U$  refine  $\mathcal{U}$ . An almost 0-dimensional space is  $L$ -embeddable in a compactum [6] and

**Theorem 3** (Levin-Pol [6]) *If a space  $X$  is  $L$ -embeddable in a compactum  $K$  then  $\dim X \leq 1$ .*

As an application of almost 1-dimensional spaces we will consider an old question of R. Duda about the dimension of a hereditarily locally connected,

non-degenerate space  $X$ . Nishiura and Tymchatyn [8] showed that each pair of disjoint, closed, connected subsets of  $X$  can be separated by a closed countable subset of  $X$ . Hence each basis for  $X$  of open connected sets witnesses the almost 1-dimensionality of  $X$ . Then Theorem 1 implies:

**Theorem 4** (Levin-Tymchatyn [7]) *If  $X$  is a hereditarily locally connected, non-degenerate space then  $\dim X = 1$ .*

A partial solution to the question of R. Duda was given in [9] where it was proved that hereditarily locally connected spaces are at most 2-dimensional.

Finally let us note that Theorem 2 does not hold if  $X_2$  is almost 0-dimensional. Indeed, let  $Y$  be 1-dimensional and almost 0-dimensional, let  $M$  be a 1-dimensional compactum and let  $M = M_1 \cup M_2$ ,  $\dim M_1 = \dim M_2 = 0$ . Then  $X_1 = Y \times M_1$  and  $X_2 = Y \times M_2$  are almost 0-dimensional, and by a theorem of Hurewicz [4] (see also [3], p. 78, 1.9.E(b))  $X = X_1 \cup X_2 = Y \times M$  is 2-dimensional.

## References

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