

Mirror Symmetry for Weighted Projective Lines

代数幾何学シンポジウム記録

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0. INTRODUCTION

There are three constructions of Frobenius manifolds (Saito's flat structures) from completely different origins and purposes; the Gromov-Witten theory, the theory of primitive forms and the invariant theory of Weyl groups.

Our main purpose is to give isomorphisms among these Frobenius manifolds by studying the Gromov-Witten theory for weighted projective lines, the theory of primitive forms for cusp singularities and the invariant theory of extended affine Weyl groups. In particular, we simplify and generalize the result given by Milanov-Tseng [3] and Rossi [4].

1. THREE CONSTRUCTIONS

Let (a_1, a_2, a_3) be a triple of positive integers such that $\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} > 1$.

A. GROMOV-WITTEN THEORY

OBJECT

Counting numbers of stable maps from orbifold curves to $X := \mathbb{P}_{a_1, a_2, a_3}^1$, where $\mathbb{P}_{a_1, a_2, a_3}^1$ is the orbifold $\mathbb{P}^1$ with 3 isotropic points of orders a_1, a_2, a_3 .

GENUS g POTENTIAL

$$\mathcal{F}_g^X := \sum_{n, \beta} \frac{1}{n!} \langle t, \dots, t \rangle_{g, n, \beta}^X q^\beta, \quad t = \sum_i t_i \gamma_i$$

$$\langle \gamma_1, \dots, \gamma_n \rangle_{g, n, \beta}^X := \int_{[\overline{\mathcal{M}}_{g, n, \beta}(X)]^{\text{vir}}} ev_1^* \gamma_1 \wedge \dots \wedge ev_n^* \gamma_n, \quad \gamma_1, \dots, \gamma_n \in H_{\text{orb}}^*(X, \mathbb{Q}),$$

$$ev_i^* : H_{\text{orb}}^*(X, \mathbb{Q}) \rightarrow H^*(\overline{\mathcal{M}}_{g, n, \beta}(X, \mathbb{Q})).$$

Genus 0 potential $\mathcal{F}_0^X$ satisfies WDVV equations.

$\Rightarrow$ Frobenius manifold of rank $a_1 + a_2 + a_3 - 1$ and of dimension 1.

B. THEORY OF PRIMITIVE FORM

OBJECT

Universal unfolding $F(x, t) = f(x, e^t) + \sum_{i=1}^{\mu-1} s_i \phi_i(x)$ of the polynomial $f(x, e^t) = x_1^{a_1} + x_2^{a_2} + x_3^{a_3} - e^t x_1 x_2 x_3$, where $\{[\phi_i(x)]\}$ form $\mathbb{C}[e^{\pm t}]$ -linear basis of Jacobian ring $\mathbb{C}[e^{\pm t}][x_1, x_2, x_3]/(\partial f/\partial x_1, \partial f/\partial x_2, \partial f/\partial x_3)$ of f .

SAITO STRUCTURE

We can construct (by the use of Sabbah's formalism [1])

- Relative de Rham cohomology groups $\mathcal{H}_F$,
- Gauss-Manin connection ∇ on $\mathcal{H}_F$,
- Higher residue pairings K_F on $\mathcal{H}_F$.

Also, we can show that $\zeta = [e^{-t} dx_1 \wedge dx_2 \wedge x_3]$ is a primitive form.

$\Rightarrow$ Frobenius manifold of rank $a_1 + a_2 + a_3 - 1$ and of dimension 1.

Remark. The choice of primitive form is not unique.

C. WEYL GROUP INVARIANT THEORY

OBJECT

Extended affine Weyl group $\widetilde{W}$ of W acting on $\widetilde{\mathfrak{h}} := \mathfrak{h} \oplus \mathbb{C}$, where $W = W_{a_1, a_2, a_3}$ and $\mathfrak{h}$ are the Weyl group and the Cartan subalgebra associated to the Coxeter-Dynkin diagram T_{a_1, a_2, a_3} .

GEOMETRY OF ORBIT SPACE

Dubrovin-Zhang [2] construct a Frobenius structure on the orbit space $\widetilde{\mathfrak{h}}/\widetilde{W} := \text{Spec}(\mathcal{R})$ where $\mathcal{R}$ is the ring of $\widetilde{W}$ -invariant Fourier polynomials.

- Chevalley's theorem tells us that $\widetilde{\mathfrak{h}}/\widetilde{W} \simeq \mathbb{C}^{a_1+a_2+a_3-2} \times \mathbb{C}^*$.
- There exists a "nice" subring $\mathcal{A}$ of $\mathcal{R}$ isomorphic to a polynomial ring which defines a partial compactification $\widetilde{\mathfrak{h}}/\widetilde{W} \hookrightarrow \mathbb{C}^{a_1+a_2+a_3-2} \times \mathbb{C}$.
- Cartan matrix $(+\alpha)$ defines a pairing on the cotangent space of $\widetilde{\mathfrak{h}}/\widetilde{W}$.

Also, Dubrovin-Zhang show the existence of a special point in $\widetilde{\mathfrak{h}}$.

$\Rightarrow$ Frobenius manifold of rank $a_1 + a_2 + a_3 - 1$ and of dimension 1.

2. STATEMENTS

The following theorem on the uniqueness is motivated by the one in [7].

Theorem 1. There exists a unique Frobenius manifold M with flat coordinates $t_0, t_{1,1}, \dots, t_{1,a_1-1}, t_{2,1}, \dots, t_{2,a_2-1}, t_{3,1}, \dots, t_{3,a_3-1}, t$ satisfying

- (i) There exists an isomorphism of complex manifolds:

$$M \simeq \mathbb{C} \times \mathbb{C}^{(a_1-1)+(a_2-1)+(a_3-1)} \times \mathbb{C}^* \\ s \mapsto (t_0(s), t_{1,1}(s), \dots, t_{3,a_3-1}(s), e^{t(s)}).$$

- (ii) The unit vector field e and the Euler vector field E are given by

$$e = \frac{\partial}{\partial t_0}, \quad E = t_0 \frac{\partial}{\partial t_0} + \sum_{i=1}^3 \sum_{j=1}^{a_i-1} \frac{a_i-j}{a_i} \frac{\partial}{\partial t_{i,j}} + \left(\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} - 1 \right) \frac{\partial}{\partial t}.$$

- (iii) Non-degenerate symmetric bilinear form η satisfies

$$\eta \left(\frac{\partial}{\partial t_0}, \frac{\partial}{\partial t} \right) = \eta \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t} \right) = 1, \\ \eta \left(\frac{\partial}{\partial t_{i_1, j_1}}, \frac{\partial}{\partial t_{i_2, j_2}} \right) = \begin{cases} 1/a_{i_1}, & \text{if } i_1 = i_2 \text{ and } j_2 = a_{i_1} - j_1. \\ 0, & \text{otherwise.} \end{cases}$$

- (iv) The Frobenius potential $\mathcal{F}_0$ satisfies $E\mathcal{F}_0 = 2\mathcal{F}_0$,

$$\mathcal{F}_0|_{t_0=0} \in \mathbb{C}[t_{1,1}, \dots, t_{1,a_1-1}, t_{2,1}, \dots, t_{2,a_2-1}, t_{3,1}, \dots, t_{3,a_3-1}, e^t]$$

- (v) Assume the condition (v). In the frame $\frac{\partial}{\partial t_0}, \frac{\partial}{\partial t_{1,1}}, \dots, \frac{\partial}{\partial t_{3,a_3-1}}, \frac{\partial}{\partial t}$ of TM , the product $\circ$ can be extended to the limit $t_0 = t_{1,1} = \dots = t_{3,a_3-1} = e^t = 0$. The $\mathbb{C}$ -algebra obtained in this limit is isomorphic to

$$\mathbb{C}[x_1, x_2, x_3] / (x_1 x_2, x_2 x_3, x_3 x_1, x_1 x_1^{a_1} - a_2 x_2^{a_2} - a_3 x_3^{a_3} - a_3 x_3^{a_3} - a_1 x_1^{a_1}),$$

where $\partial/\partial t_{1,1}, \partial/\partial t_{2,1}, \partial/\partial t_{3,1}$ are mapped to x_1, x_2, x_3 , respectively.

- (vi) The term $t_{1,1} t_{2,1} t_{3,1} e^t$ occurs with the coefficient 1 in $\mathcal{F}_0$.

We have another theorem on the uniqueness (see also [2] and [6]):

Theorem 2. A Frobenius manifold of rank $a_1 + a_2 + a_3 - 1$ and of dimension 1 with the following e and E is uniquely determined by the intersection form:

$$e = \frac{\partial}{\partial t_0}, \quad E = t_0 \frac{\partial}{\partial t_0} + \sum_{i=1}^3 \sum_{j=1}^{a_i-1} \frac{a_i-j}{a_i} \frac{\partial}{\partial t_{i,j}} + \left(\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} - 1 \right) \frac{\partial}{\partial t}.$$

3. ISOMORPHISMS

A $\iff$ B: Use Theorem 1.

- Calculate the orbifold cohomology ring of X and count the number of certain stable maps of degree one.
- Calculate the "limit" of Jacobian ring in B.

B $\iff$ C: Use Theorem 2.

- Calculate the intersection form in (B) by the period mapping of the primitive form (see [5]) and by a classical geometric approach.

REFERENCE

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