

Analytic construction of multi-brane solutions in cubic string field theory for any brane number

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We present an analytic construction of multi-brane solutions with any integer brane number in cubic open string field theory (CSFT) on the basis of the KBc algebra. Our solution is given in the pure-gauge form $\Psi = UQ_B U^{-1}$ by a unitary string field U , which we choose to satisfy two requirements. First, the energy density of the solution should reproduce that of the $(N + 1)$ -branes. Second, the equations of motion (EOM) of the solution should hold against the solution itself. In spite of the pure-gauge form of Ψ , these two conditions are non-trivial ones due to the singularity at $K = 0$. For the $(N + 1)$ -brane solution, our U is specified by $[N/2]$ independent real parameters α_k . For the 2-brane ($N = 1$), the solution is unique and reproduces the known one. We find that α_k satisfying the two conditions indeed exist as far as we have tested for various integer values of N ($= 2, 3, 4, 5, \dots$). Our multi-brane solutions consisting only of the elements of the KBc algebra have the problem that the EOM is not satisfied against the Fock states and therefore are not complete ones. However, our construction should be an important step toward understanding the topological nature of CSFT, which has similarities to the Chern–Simons theory in three dimensions.
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Subject Index B26, B28

1. Introduction

Since Schnabl's construction [1] of an analytic solution for tachyon condensation in cubic open string field theory (CSFT), there have appeared lots of studies on the analytic construction of solutions representing multiple D25-branes within the framework of the KBc algebra [2].¹ Among these, the construction presented in Ref. [4] by using the boundary condition changing operators, in addition to the elements of the KBc algebra, may be a satisfactory one. However, in this paper, we pursue the construction of multi-brane solutions consisting solely of (K, B, c) . Such types of solutions have been studied, e.g., in Refs. [5–8], where they considered candidate solutions of the pure-gauge type $\Psi = UQ_B U^{-1}$ given in terms of U and U^{-1} of the following form [2]:

$$U = \left(1 - \sqrt{1 - GBc}\sqrt{1 - G}\right) \frac{1}{\sqrt{G}}, \quad U^{-1} = U^\dagger = \frac{1}{\sqrt{G}} \left(G + \sqrt{1 - GBc}\sqrt{1 - G}\right), \quad (1.1)$$

where $G = G(K)$ is a function of K that should be suitably chosen. Explicitly, Ψ reads

$$\Psi = UQ_B U^{-1} = \sqrt{1 - G} cK \frac{1}{G} Bc \sqrt{1 - G}. \quad (1.2)$$

¹ For a recent numerical approach toward the construction of multi-brane solutions, see Ref. [3].

The KBc algebras that we need here and in the following are

$$[K, B] = 0, \quad \{B, c\} = 1, \quad B^2 = c^2 = 0, \quad (1.3)$$

and

$$Q_B B = K, \quad Q_B K = 0, \quad Q_B c = cKc. \quad (1.4)$$

The string field Ψ (as well as the elements (K, B, c) of the KBc algebra) is subject to the self-conjugateness condition $\Psi^\ddagger = \Psi$ with $\ddagger$ denoting the composition of the BPZ and the Hermitian conjugations. Therefore, U in Eq. (1.1) is chosen to be unitary in the sense that $U^\ddagger = U^{-1}$. In fact, U in Eq. (1.1) is the most generic form of unitary U , which is the sum of two terms, one containing Bc and the other without it.

Though the configuration (1.2) is a pure-gauge one and formally satisfies the equations of motion (EOM),

$$Q_B \Psi + \Psi^2 = 0, \quad (1.5)$$

this is in fact a subtle problem due to the singularity at $K = 0$. For the requirements on the pure-gauge configuration Ψ (1.2) as a solution, the number of D25-branes Ψ represents and the EOM test of Ψ against itself were examined for various $G(K)$ defining U [5–8]. For calculating these quantities, we have to regularize the singularity at $K = 0$. In Ref. [6], we adopted the K_ε -regularization of replacing K by $K_\varepsilon = K + \varepsilon$ with ε being an infinitesimal positive constant. In this paper, for any $\mathcal{O}(K, B, c)$, $\mathcal{O}_\varepsilon$ with subscript ε denotes the K_ε -regularized one:

$$\mathcal{O}_\varepsilon = \mathcal{O}|_{K \mapsto K_\varepsilon} = \mathcal{O}(K_\varepsilon, B, c). \quad (1.6)$$

Then, in terms of the K_ε -regularized pure-gauge configuration,

$$\Psi_\varepsilon = (U Q_B U^{-1})_\varepsilon, \quad (1.7)$$

the brane number is given by $\mathcal{N} + 1$ with $\mathcal{N}$ being

$$\mathcal{N} = \frac{\pi^2}{3} \int \Psi_\varepsilon^3, \quad (1.8)$$

while the EOM test $\mathcal{T}$ of Ψ_ε against itself is

$$\mathcal{T} = \int \Psi_\varepsilon (Q_B \Psi_\varepsilon + \Psi_\varepsilon^2). \quad (1.9)$$

In fact, $\mathcal{N}$ (1.8) is equal to the minus of the action of Ψ_ε , $-S = -\int (\frac{1}{2} \Psi_\varepsilon Q_B \Psi_\varepsilon + \frac{1}{3} \Psi_\varepsilon^3)$, divided by the D25-brane tension $1/(2\pi^2)$ only when the EOM test (1.9) vanishes, $\mathcal{T} = 0$.²

The tachyon vacuum with $\mathcal{N} = -1$ and the 2-brane with $\mathcal{N} = 1$ are realized by Eq. (1.2) by taking $G(K)$ with its small- K behavior given by $G(K) \sim K$ and $G(K) \sim 1/K$, respectively.³ Concrete choices for $G(K)$ are, e.g. [5–7,9],

$$G_{\text{tachyon vac.}}(K) = \frac{K}{1+K}, \quad G_{\text{2-brane}}(K) = \frac{1+K}{K}. \quad (1.10)$$

² We are taking both the open string coupling constant and the space-time volume equal to one.

³ $G(K)$ should not have zero nor pole at $K = \infty$ to avoid their additional contribution to $\mathcal{N}$ [8].

The EOM test is also passed; namely, $\mathcal{T} = 0$ in these two cases. It was shown that the origin of non-trivial $\mathcal{N}$ in these solutions is the singularity coming from the zero or pole of $G(K)$ at $K = 0$ [5–8].

However, the construction of multi-brane solutions with a larger $\mathcal{N}$ has been problematic. From the above two examples in Eq. (1.10), it may be guessed that a solution with $\mathcal{N} = N = 2, 3, \dots$ is obtained by taking $G(K)$ with a multiple pole at $K = 0$, $G(K) \sim 1/K^N$ ($K \sim 0$); e.g., $G(K) = ((1 + K)/K)^N$. However, it was found that $\mathcal{N}$ and $\mathcal{T}$ for this type of $G(K)$ are given by [6–8]

$$\mathcal{N} = N + A_N, \quad \mathcal{T} = B_N, \quad (1.11)$$

where the “anomalous terms” A_N and B_N are expressed in terms of the confluent hypergeometric function⁴ as

$$\begin{aligned} A_N &= -\frac{\pi^2}{3} N (N^2 - 1) \operatorname{Re} {}_1F_1(2 - N, 4; 2\pi i), \\ B_N &= \frac{N(N + 1)}{\pi} \operatorname{Im} {}_1F_1(1 - N, 2; 2\pi i). \end{aligned} \quad (1.12)$$

Examples are as follows:

$$A_N = \begin{cases} 0 & (N = 0, \pm 1) \\ -2\pi^2 & (N = 2) \\ -8\pi^2 & (N = 3) \\ -20\pi^2 + 4\pi^4 & (N = 4) \end{cases}, \quad B_N = \begin{cases} 0 & (N = 0, \pm 1) \\ -6 & (N = 2) \\ -24 & (N = 3) \\ -60 + (20/3)\pi^2 & (N = 4) \end{cases}. \quad (1.13)$$

Namely, $\mathcal{N}$ is not an integer and the EOM test is not passed ($\mathcal{T} \neq 0$) for the present type of solutions with $N \geq 2$.

In Ref. [8], we proposed that the 3-brane solution with $\mathcal{N} = 2$ and $\mathcal{T} = 0$ can be constructed in the form (1.2) by making use of the singularities both at $K = 0$ and $K = \infty$, and taking, e.g., $G(K) = (1 + K)^2/K$. However, multi-brane solutions with larger $\mathcal{N}$ ($= 3, 4, 5, \dots$) and $\mathcal{T} = 0$ seem not to exist in the form of Eq. (1.2).

In this paper, we present an analytic expression of multi-brane solutions carrying any integer $\mathcal{N}$ and satisfying the EOM test $\mathcal{T} = 0$. We start with the most generic form of unitary string field U consisting only of (K, B, c) and examine the pure-gauge configuration $\Psi = UQ_B U^{-1}$, which manifestly satisfies the self-conjugateness condition. For considering the most generic unitary U , we adopt a convenient notation for expressing a string field, which is given as the sum of products of (K, B, c) . Then, by referring to the successful examples of the tachyon vacuum and the 2-brane solutions given by Eqs. (1.1), (1.2), and (1.10), we make a natural ansatz on the functions of K defining U . As a result, U , which is expected to represent $(N + 1)$ -branes, is specified by $(N + 1)$

⁴ The confluent hypergeometric function is defined by

$${}_1F_1(a, b; z) = 1 + \sum_{k=1}^{\infty} \frac{a(a+1) \cdots (a+k-1)}{b(b+1) \cdots (b+k-1)} \frac{z^k}{k!}.$$

Note that ${}_1F_1(a, b; z)$ is a polynomial in z of degree $(-a)$ for a non-positive integer a .

real parameters $(\alpha_0, \alpha_1, \dots, \alpha_N)$, among which only $[N/2]$ are independent.⁵ We carry out the calculation of $\mathcal{N}$ (1.8) and $\mathcal{T}$ (1.9) for this type of solution, and find that these two quantities are again given in the form (1.11): the anomalous terms A_N and B_N are polynomials in $(2\pi i)^2$ of order $[N/2]$ and $[N/2] - 1$, respectively (A_N starts with the $(2\pi i)^2$ term). This is also the case for A_N and B_N of Eq. (1.12) for the solution (1.2). A different point in the present U is that the coefficients of the polynomials are not constants but are linear functions of α_k . Moreover, the coefficient $f_n(\alpha_k)$ multiplying $(2\pi i)^{2n}$ is common between A_N and $(2\pi i)^2 B_N$ up to a constant factor. Therefore, both $A_N = 0$ and $B_N = 0$, namely, $\mathcal{N} = N$ and $\mathcal{T} = 0$, are realized by choosing as $\{\alpha_k\}$ the solution to $f_n(\alpha_k) = 0$ ($n = 1, 2, \dots, [N/2]$). In fact, we find that α_k and hence the solution $\Psi = U Q_B U^{-1}$ are uniquely determined in this way for any integer values of N ($= 2, 3, 4, 5, \dots$) that we have tested. For example, the 3-brane solution is given by Eq. (5.5) with $G = (1 + K)/K$.

However, we have not succeeded in determining α_k for a generic N . The reason is that the expressions of $\mathcal{N}$ and $\mathcal{T}$ that we will obtain in this paper are too complicated to get $f_n(\alpha_k)$ in a closed form for a generic N . Even more, the fact that $f_n(\alpha_k)$ are common between A_N and $(2\pi i)^2 B_N$ is merely an “experimental fact” obtained by the evaluation of A_N and B_N for various values of N . However, there is no doubt that we can determine α_k so that our solution can realize both $\mathcal{N} = N$ and $\mathcal{T} = 0$ for any integer N . The technical problem of giving $f_n(\alpha_k)$ for a generic N will be resolved by mathematical sophistication.

Even if the solution $\{\alpha_k\}$ to $f_n(\alpha_k) = 0$ is found for a generic integer N , there still is an important problem in our construction of solutions. In this paper, as the EOM test, we consider only $\mathcal{T}$ (1.9), namely, the EOM test against the candidate solution Ψ itself. However, it is known that the 2-brane solution given by U of Eq. (1.1) with $G = G_{2\text{-brane}}$ (1.10) does not pass the EOM test against the Fock states [7], and this property is inherited by the multi-brane solutions in this paper consisting solely of (K, B, c) (see Sect. 6). This problem of the failure of the EOM test against the Fock states might be resolved by some improvements of the solution, or by some consistent truncation of the space of fluctuations around multi-branes, which excludes the Fock states.

However, even if this problem persists, the construction in this paper should give an important hint to understanding the meaning of $\mathcal{N}$ (1.8) as the “winding number”. Namely, note the analogy of $\mathcal{N}$ (1.8) to the winding number,

$$\mathcal{W}[g] = \frac{1}{24\pi^2} \int_M \text{tr} (g dg^{-1})^3, \quad (1.14)$$

of the mapping $g(x)$ from a three-manifold M to a Lie group. This analogy was emphasized and examined in Ref. [6]. There, $\mathcal{N}$ was evaluated by making use of its topological nature, namely, the invariance of $\mathcal{N}$ under small deformations of U , to identify the zero or pole of $G(K)$ at $K = 0$ as the origin of non-trivial $\mathcal{N}$ (see Sect. 3 of this paper). For explaining the relevance of the present construction of U giving integer $\mathcal{N}$ to the identification of $\mathcal{N}$ as winding number, let us consider the simplest example of $\mathcal{W}$: $g(x) \in SU(2)$, $M = S^3$, and the hedgehog type $g(x) = \exp(i f(r) \mathbf{x} \cdot \boldsymbol{\tau} / r)$ with $r = |\mathbf{x}|$. In this case, $\mathcal{W}$ is given in terms of $f(r)$ at the origin and infinity by $\mathcal{W} = (f(\infty) - f(0)) / \pi$. This $\mathcal{W}$ becomes an integer by demanding the regularity of $g(x)$ at the two points, which implies that both $f(0)$ and $f(\infty)$ are integer multiples of π . The non-integer results (1.11) and (1.13) of $\mathcal{N}$ for U of the form (1.1) and our finding in this paper of a new type of U realizing integer $\mathcal{N}$ for larger N may give a clue to understanding the meaning of regularity of

⁵ $[x]$ denotes the greatest integer less than or equal to x .

U .⁶ Of course, we have to find answers to more basic questions: “What are the counterparts of the three-manifold M and the Lie group in CSFT? What is the meaning of winding represented by $\mathcal{N}$?” These considerations are further expected to lead to deeper understanding of the similarity of CSFT to the Chern–Simons theory in three dimensions, and topological aspects of CSFT.

The rest of this paper is organized as follows. In Sect. 2, first introducing our convenient notation for expressing the sum of products of (K, B, c) , we determine the form of the most generic unitary string field U , and present our assumption on the form of U that is specified by α_k . Then, in Sects. 3 and 4, we obtain $\mathcal{N}$ and $\mathcal{T}$, respectively, as functions of α_k . In particular, we calculate $\mathcal{N}$ not directly but in a way where the role of the singularity at $K = 0$ as the origin of non-trivial $\mathcal{N}$ is manifest. In Sect. 5, we examine the conditions $\mathcal{N} = N$ and $\mathcal{T} = 0$ on our solution to determine α_k for various values of N . We summarize the paper and discuss future problems in Sect. 6. In the appendices, we present technical details used in the text.

2. Assumptions on the solution

In this section, we first introduce our convenient notation for expressing string fields in the framework of the KBc algebra. Then, we obtain the form of the most generic unitary string field U for our candidate solution $\Psi = UQ_B U^{-1}$ of the pure-gauge type. After these preparations, we restrict U to a particular form that is specified by real parameters $(\alpha_0, \alpha_1, \dots, \alpha_N)$.

2.1. Convenient notation

To make our equations look simpler, we first introduce a convenient notation for expressing the sum of products of (K, B, c) . Let us consider, e.g., the following string field $\mathcal{O}$:

$$\mathcal{O} = \sum_{\{f_a\}} f_1(K) c f_2(K) c \cdots c f_n(K) B c f_{n+1}(K), \quad (2.1)$$

where $f_a(K)$ ($a = 1, 2, \dots, n+1$) are functions of K and the sum $\sum_{\{f_a\}}$ is the sum over various sets of f_a . In this particular example, there appear $(n-1)$ ghosts c and a single Bc . Our new notation also applies to the cases where some of the c are replaced with Bc (and Bc with c).

In our new notation, we first consider the product of c only in Eq. (2.1) to write it as $c_{12} c_{23} \cdots c_{n-1,n} c_{n,n+1}$ by attaching to each c a pair of numbers (a, b) . Namely, each of the numbers $(1, 2, \dots, n, n+1)$ specifies a position in the sequence of c . Then, we assign each of K and B in Eq. (2.1) (which are commutative with each other) a single number a specifying their position in the product of c to write K_a and B_a . Then, the string field $\mathcal{O}$ (2.1) now carries a pair of indices $(1, n+1)$ and is written as

$$\mathcal{O}_{1,n+1} = A_{1,2,\dots,n,n+1} c_{12} c_{23} \cdots c_{n-1,n} (Bc)_{n,n+1}, \quad (2.2)$$

where $A_{1,2,\dots,n,n+1}$, which depends only on K , is given by

$$A_{1,2,\dots,n,n+1} = \sum_{\{f_a\}} f_1(K_1) f_2(K_2) \cdots f_n(K_n) f_{n+1}(K_{n+1}). \quad (2.3)$$

⁶ Naively, it is guessed that the two points $r = 0, \infty$ in the example of hedgehog $g(x)$ correspond to $K = 0, \infty$ in CSFT.

In Eq. (2.2) and in the following, we use notations such as $(Bc)_{ab} (= B_a c_{ab})$, $(cB)_{ab} (= c_{ab} B_b)$, and $(cK)_{ab} (= c_{ab} K_b)$. The advantage of the present notation is that we can put the K dependences at any place without any ambiguity.

As examples, U in Eq. (1.1) and Ψ (1.2) are expressed in our notation as

$$U_{12} = \frac{1}{\sqrt{G_2}} \mathbb{I}_{12} - \frac{\sqrt{(1-G_1)(1-G_2)}}{\sqrt{G_2}} (Bc)_{12}, \quad (2.4)$$

and

$$\Psi_{13} = \sqrt{1-G_1} \frac{K_2}{G_2} \sqrt{1-G_3} c_{12} (Bc)_{23}, \quad (2.5)$$

with $G_a = G(K_a)$ and $\mathbb{I}_{ab}$ being the identity string field. Finally, the conjugate of $\mathcal{O}$ (2.2) for a self-conjugate $A_{1,2,\dots,n,n+1}$ is given by

$$(\mathcal{O}^\dagger)_{n+1,1} = A_{1,2,\dots,n,n+1} (cB)_{n+1,n} c_{n,n-1} \cdots c_{32} c_{21}. \quad (2.6)$$

2.2. The most generic unitary U

For constructing self-conjugate solutions in the pure-gauge form, $\Psi = UQ_B U^{-1}$, in terms of a unitary U satisfying $UU^\dagger = \mathbb{I}$, let us first establish the most general form of the string field U , which is unitary and carries the ghost number $N_{\text{gh}} = 0$. First, from $N_{\text{gh}} = 0$, U is expressed without losing generality as

$$U_{12} = \frac{1}{\Gamma_2} \mathbb{I}_{12} - \frac{\mathcal{F}_{12}}{\Gamma_2} (Bc)_{12}, \quad (2.7)$$

where Γ_2 and $\mathcal{F}_{12}$ on the RHS are given by $\Gamma_a = \Gamma(K_a)$ and $\mathcal{F}_{ab} = \mathcal{F}(K_a, K_b)$ in terms of two real functions $\Gamma(x)$ and $\mathcal{F}(x, y)$.⁷ Then, U is unitary if $\mathcal{F}_{aa}$ and Γ_a are related by

$$\mathcal{F}_{aa} = 1 - (\Gamma_a)^2, \quad (2.8)$$

and $\mathcal{F}_{ab}$ is symmetric:

$$\mathcal{F}_{ab} = \mathcal{F}_{ba}. \quad (2.9)$$

The derivation of these two conditions as well as those of some of the equations in this subsection are given in Appendix A.

When the two conditions (2.8) and (2.9) are met, U^{-1} is given by

$$(U^{-1})_{12} = (U^\dagger)_{12} = \Gamma_1 \mathbb{I}_{12} + \frac{\mathcal{F}_{12}}{\Gamma_1} (Bc)_{12}, \quad (2.10)$$

and the corresponding candidate solution of the pure-gauge type $UQ_B U^{-1}$ reads

$$(UQ_B U^{-1})_{13} = E_{123} (cK)_{12} (Bc)_{23}, \quad (2.11)$$

⁷ We are assuming that U is real, namely, that U does not contain any imaginary unit i . This reality assumption is only for the sake of simplicity.

where E_{abc} is defined by

$$E_{abc} = \mathcal{F}_{ac} + \mathcal{F}_{ab} \frac{1}{\Gamma_b^2} \mathcal{F}_{bc}. \quad (2.12)$$

We summarize three kinds of relations concerning E_{abc} (2.12):

$$E_{abc} = E_{cba}, \quad (2.13)$$

$$E_{aab} = \frac{1}{\Gamma_a^2} \mathcal{F}_{ab}, \quad E_{abb} = \mathcal{F}_{ab} \frac{1}{\Gamma_b^2}, \quad E_{aaa} = \frac{1}{\Gamma_a^2} - 1, \quad (2.14)$$

$$E_{abb}E_{bcd} - E_{abc}E_{ccd} = E_{abd} - E_{acd}. \quad (2.15)$$

2.3. Assumptions on Γ_a and $\mathcal{F}_{ab}$

It is impossible to evaluate $\mathcal{N}$ and $\mathcal{T}$ for $\Psi = UQ_B U^{-1}$ given by Eq. (2.11) without any assumptions on Γ_a and $\mathcal{F}_{ab}$. Here, on the basis of known facts, we would like to make plausible assumptions on the form of Γ_a and $\mathcal{F}_{ab}$, which is expected to realize $\mathcal{N} = N$ and $\mathcal{T} = 0$ for each positive integer N .

The first fact is the satisfactory example of the $\mathcal{N} = 1$ solution with U given by Eq. (1.1) or by Eq. (2.4) in the present notation. In this case, $G(K)$ should have a simple pole at $K = 0$ and no other zeros/poles in the complex half-plane $\text{Re } K \geq 0$ including $K = \infty$, but otherwise arbitrary. For definiteness, we take

$$G(K) = \frac{1+K}{K}. \quad (2.16)$$

Comparing Eq. (2.4) with the generic form (2.7), we see that Γ_a and $\mathcal{F}_{ab}$ in this example are given by

$$\Gamma_a = \sqrt{G_a} = \sqrt{G(K_a)}, \quad \mathcal{F}_{ab} = \sqrt{(1-G_a)(1-G_b)}, \quad (2.17)$$

which certainly satisfy Eqs. (2.8) and (2.9).

Secondly, by replacing G_a in Eq. (2.17) with $G_a^N = ((1+K_a)/K_a)^N$, we get $\mathcal{N}$ and $\mathcal{T}$ given by Eqs. (1.11) and (1.12). As we saw there, $\mathcal{N}$ for $N \geq 2$ is a polynomial in $(2\pi i)^2$ starting with the zeroth term N . This seems to suggest that the replacement $G_a \mapsto (G_a)^N$ is, though not perfect, fairly close to the final answer realizing $\mathcal{N} = N$.

Taking these facts into account, let us take as our candidate Γ_a and $\mathcal{F}_{ab}$ for a generic N , which possibly realize $\mathcal{N} = N$ and $\mathcal{T} = 0$, the following ones given in terms of $G(K)$ of Eq. (2.16):

$$\Gamma_a = G_a^{N/2} = G(K_a)^{N/2}, \quad (2.18)$$

$$\mathcal{F}_{ab} = \prod_{k=0}^N \left(1 - G_a^k G_b^{N-k}\right)^{\alpha_k} = - \prod_{k=0}^N \left(G_a^k G_b^{N-k} - 1\right)^{\alpha_k}. \quad (2.19)$$

Here, α_k are numerical coefficients satisfying

$$\sum_{k=0}^N \alpha_k = 1, \quad (2.20)$$

and

$$\alpha_{N-k} = \alpha_k \quad (k = 0, 1, \dots, N). \quad (2.21)$$

These conditions (2.20) and (2.21) are necessary for Eqs. (2.8) and (2.9), respectively. Note that

$$\sum_{k=0}^N k\alpha_k = \frac{N}{2} \quad (2.22)$$

follows from Eqs. (2.20) and (2.21). The simple replacement $G_a \mapsto (G_a)^N$ in Eq. (2.17) corresponds to the following choice of α_k :

$$\alpha_0 = \alpha_N = \frac{1}{2}, \quad \text{other } \alpha_k = 0. \quad (2.23)$$

Though $\mathcal{F}_{ab}$ (2.19) itself is not of a factorized form with respect to the K_a and K_b dependences, it should be suitably expressed as a sum of factorized terms by, e.g., Taylor expansion, for calculating correlators containing $\mathcal{F}_{ab}$.

In the rest of this paper, we shall first obtain $\mathcal{N}$ and $\mathcal{T}$ for the present solution as functions of $\{\alpha_k\}$, and then examine whether there exists $\{\alpha_k\}$ satisfying both $\mathcal{N}[\alpha_k] = N$ and $\mathcal{T}[\alpha_k] = 0$ for each positive integer N .

3. Expression of $\mathcal{N}[\alpha_k]$

As preparation for examining $\mathcal{N}$ (1.8) for our candidate solution proposed above, we in this section obtain a calculable concrete expression of $\mathcal{N}[\alpha_k]$ for a given $\{\alpha_k\}$.

3.1. $\mathcal{N}$ in terms of Γ_a and $\mathcal{F}_{ab}$

Instead of calculating $\mathcal{N}$ (1.8) directly, we here use the method of Ref. [6] to evaluate $\mathcal{N}$ as a “topological” quantity. Concretely, we use the following formula for the variation of $\mathcal{N}$ under an arbitrary infinitesimal deformation δU^{-1} of U^{-1} :

$$\frac{1}{\pi^2} \delta \mathcal{N} = \varepsilon \int \left\{ T_B \left[(U Q_B U^{-1})^2 (U \delta U^{-1}) \right] \right\}_\varepsilon. \quad (3.1)$$

Here, T_B is the Grassmann-odd operation of replacing B with the identity $\mathbb{I}$ one by one:

$$\begin{aligned} T_B(f_1 B f_2 B f_3 \cdots f_n B f_{n+1}) &= (-1)^{|f_1|} f_1 \mathbb{I} f_2 B f_3 \cdots f_n B f_{n+1} + (-1)^{|f_1|+|f_2|+1} f_1 B f_2 \mathbb{I} f_3 \cdots f_n B f_{n+1} \\ &+ \cdots + (-1)^{\sum_{i=1}^n |f_i|+n-1} f_1 B f_2 B f_3 \cdots f_n \mathbb{I} f_{n+1}, \end{aligned} \quad (3.2)$$

where $f_i = f_i(c, K)$ is a product of K and c , and $|f| = 0$ ($= 1$) if f is Grassmann-even (-odd). The operation of T_B on a quantity without B is defined to be zero:

$$T_B f(c, K) = 0. \quad (3.3)$$

The derivation of Eq. (3.1) is given in Appendix B.

For calculating $\mathcal{N}$ of our solution with $G(K)$ given by Eq. (2.16), we introduce $G(K, u)$ with a parameter u ,

$$G(K, u) = \frac{1+K}{u+K}, \quad (3.4)$$

and regard the deformation δ as that of u : $\delta = \delta u (d/du)$. Then, since $G(K, u)$ with $u > 0$ corresponds to the trivial solution with $\mathcal{N} = 0$, $\mathcal{N}$ for $G(K)$ (2.16) is given by integrating Eq. (3.1) over u as

$$\mathcal{N} = \int_{u=u_0}^{u=0} \delta \mathcal{N}, \quad (3.5)$$

where $\delta\mathcal{N}$ on the RHS is that for $G(K, u)$ (3.4), and u_0 is positive but otherwise arbitrary (the integration (3.5) is independent of u_0 in the limit $\varepsilon \rightarrow +0$). Equation (3.5), which is multiplied by ε , can be non-vanishing in the limit $\varepsilon \rightarrow +0$ due to the $1/\varepsilon$ singularity arising from the u -integration near $u = 0$ as we saw in Ref. [6].

Let us express the integrand on the RHS of Eq. (3.1) in terms of Γ_a and $\mathcal{F}_{ab}$. The expression of $UQ_B U^{-1}$ is already given by Eqs. (2.11) and (2.12). Using this, $(UQ_B U^{-1})^2$ is calculated as follows:

$$\begin{aligned} [(UQ_B U^{-1})^2]_{15} &= (UQ_B U^{-1})_{13} (UQ_B U^{-1})_{35} = E_{123} E_{345} (cK)_{12} (Bc)_{23} (cK)_{34} (Bc)_{45} \\ &= (E_{122} E_{235} - E_{123} E_{335}) (cK)_{12} (cK)_{23} (Bc)_{35} \\ &= (E_{125} - E_{135}) (cK)_{12} (cK)_{23} (Bc)_{35}, \end{aligned} \quad (3.6)$$

where we have used Eq. (A.6), and the last equality is due to the relation (2.15). Next, let us consider $U\delta U^{-1}$. Taking the variation of U^{-1} (2.10), we obtain

$$(\delta U^{-1})_{12} = \Gamma_1 (\delta \ln \Gamma_1) \mathbb{I}_{12} + \frac{1}{\Gamma_1} (\delta \mathcal{F}_{12} - (\delta \ln \Gamma_1) \mathcal{F}_{12}) (Bc)_{12}. \quad (3.7)$$

Using this, $U\delta U^{-1}$ is calculated as follows:

$$\begin{aligned} (U\delta U^{-1})_{13} &= U_{12} (\delta U^{-1})_{23} \\ &= \left(\frac{1}{\Gamma_2} \mathbb{I}_{12} - \frac{\mathcal{F}_{12}}{\Gamma_2} (Bc)_{12} \right) \left[\Gamma_2 (\delta \ln \Gamma_2) \mathbb{I}_{23} + \frac{1}{\Gamma_2} (\delta \mathcal{F}_{23} - (\delta \ln \Gamma_2) \mathcal{F}_{23}) (Bc)_{23} \right] \\ &= (\delta \ln \Gamma_1) \mathbb{I}_{13} + [\delta \mathcal{F}_{13} - (\delta \ln \Gamma_1 + \delta \ln \Gamma_3) \mathcal{F}_{13}] (Bc)_{13}, \end{aligned} \quad (3.8)$$

where we have used Eqs. (A.3) and (2.8). Finally, multiplying Eqs. (3.6) and (3.8), we get

$$\begin{aligned} [(UQ_B U^{-1})^2 U\delta U^{-1}]_{15} &= [(UQ_B U^{-1})^2]_{14} (U\delta U^{-1})_{45} \\ &= \left\{ (E_{125} - E_{135}) \delta \ln \Gamma_5 + (E_{123} - E_{133}) [\delta \mathcal{F}_{35} - (\delta \ln \Gamma_3 + \delta \ln \Gamma_5) \mathcal{F}_{35}] \right\} \\ &\quad \times (cK)_{12} (cK)_{23} (Bc)_{35}. \end{aligned} \quad (3.9)$$

By the substitution of this into Eq. (3.1), $(Bc)_{35}$ is replaced with c_{35} by the T_B operation, and the last index 5 is identified with the first index 1. Then, we get the desired formula for calculating $\mathcal{N}$:

$$\frac{1}{\pi^2} \delta\mathcal{N} = \varepsilon \int (W_{123})_\varepsilon (cK_\varepsilon)_{12} (cK_\varepsilon)_{23} c_{31}, \quad (3.10)$$

with W_{123} given by

$$W_{123} = (E_{123} - E_{133}) [\delta \mathcal{F}_{31} - (\delta \ln \Gamma_3 + \delta \ln \Gamma_1) \mathcal{F}_{31}]. \quad (3.11)$$

Note that the $(E_{125} - E_{135}) \delta \ln \Gamma_5$ term in Eq. (3.9) does not contribute to Eq. (3.10) due to the L/R-reversing symmetry of the *ccc*-correlator:

$$\int A_{123} c_{12} c_{23} c_{31} = \int A_{132} c_{12} c_{23} c_{31}, \quad (3.12)$$

valid for any $A_{123}(K)$.

Though we do not use it in this paper, $\mathcal{N}$ itself is of course given in terms of E_{abc} :

$$\frac{1}{\pi^2} \mathcal{N} = \int (M_{1234})_\varepsilon (cK_\varepsilon)_{12} (cK_\varepsilon)_{23} (cK_\varepsilon)_{34} (Bc)_{41}, \quad (3.13)$$

where M_{1234} is

$$M_{1234} = (E_{123} - E_{133}) E_{341} - (E_{124} - E_{134}) E_{441}. \quad (3.14)$$

3.2. $\mathcal{N}$ for a given $\{\alpha_k\}$

The formula (3.10) is valid for any U (2.7) given in terms of $(\Gamma_a, \mathcal{F}_{ab})$. In this subsection, we use Eqs. (3.10) and (3.5) to calculate $\mathcal{N}$ for our particular choice of Γ_a and $\mathcal{F}_{ab}$, Eqs. (2.18) and (2.19), specified by $\{\alpha_k\}$. An important point in this calculation is that Eq. (3.10) is multiplied by ε , which should be taken to $+0$ in the end. This implies that we are allowed to keep only the most singular part of W_{123} (3.11) with respect to ε .

Recall that G_a in W_{123} (3.11) is given by Eq. (3.4) with the parameter u , and the variation δ is that with respect to u . The K_ε -regularized G_a in $(W_{123})_\varepsilon$ is taken as

$$G(K_\varepsilon, u) = \frac{1}{u + K_\varepsilon}, \quad (3.15)$$

where K_ε in the numerator of the original $G(K_\varepsilon, u)$ corresponding to Eq. (3.4) has been omitted since it is irrelevant (i.e., higher order in ε) in the present calculation. We regard this $G(K_\varepsilon, u)$ as an $O(1/\varepsilon)$ quantity⁸ and expand W_{123} in inverse powers of G_a . In the following calculations, the properties of α_k , Eqs. (2.20), (2.21), and (2.22), are repeatedly used without mentioning it. In addition, we omit the subscript ε in Eq. (1.6) for the replacement $K \mapsto K_\varepsilon$ for the sake of notational simplicity. Everything in this subsection should be regarded as the K_ε -regularized one.

First, we obtain without approximation

$$\delta \mathcal{F}_{ab} - (\delta \ln \Gamma_a + \delta \ln \Gamma_b) \mathcal{F}_{ab} = \sum_{k=0}^N \alpha_k \frac{k \delta \ln G_a + (N-k) \delta \ln G_b}{G_a^k G_b^{N-k} - 1} \mathcal{F}_{ab}, \quad (3.16)$$

where we have used $\delta \ln \Gamma_a = (N/2) \delta \ln G_a$. Next, $\mathcal{F}_{ab}$ is expanded in inverse powers of G_a as

$$\mathcal{F}_{ab} = -(G_a G_b)^{N/2} \left[1 - \sum_{k=0}^N \frac{\alpha_k}{G_a^k G_b^{N-k}} + O\left(\frac{1}{G^{2N}}\right) \right], \quad (3.17)$$

and, using this, E_{abc} (2.12) is expanded as

$$E_{abc} = (G_a G_c)^{N/2} \sum_{k=0}^N \alpha_k \left[\frac{1}{G_a^k G_c^{N-k}} - \frac{1}{G_a^k G_b^{N-k}} - \frac{1}{G_b^k G_c^{N-k}} \right] + O\left(\frac{1}{G^N}\right). \quad (3.18)$$

⁸ u can also be regarded as $O(\varepsilon)$ since only the part $0 \leq u < O(\varepsilon)$ of the u -integration region contributes to Eq. (3.5).

From Eqs. (3.16), (3.17), and (3.18), we obtain the following expansion of W_{123} :

$$W_{123} = - \sum_{k=0}^N \alpha_k \left[\left(\frac{G_3}{G_1} \right)^N \left(\frac{G_1}{G_2} \right)^k + \left(\frac{G_3}{G_2} \right)^k - \left(\frac{G_3}{G_1} \right)^k - 1 \right] \\ \times \sum_{\ell=0}^N \alpha_\ell [\ell G_3 + (N - \ell) G_1] \left(\frac{G_1}{G_3} \right)^\ell \times \delta u + O\left(\frac{\delta \ln G}{G^N} \right). \quad (3.19)$$

In Eq. (3.19), we have used that $\delta \ln G$ for G of Eq. (3.15) is given by

$$\delta \ln G = - \frac{\delta u}{u + K_\varepsilon} = -G \delta u. \quad (3.20)$$

As seen from Eq. (3.19), the leading part of W_{123} is the sum of terms of the form $G_1^{n_1} G_2^{n_2} G_3^{n_3} \delta u$ with integers n_a satisfying $n_1 + n_2 + n_3 = 1$. As we shall see, this leading part makes a finite $O(\varepsilon^0)$ contribution to $\mathcal{N}$, while the contribution of the subleading part is of $O(\varepsilon^N)$. Therefore, we keep only the leading part of W_{123} in the rest of this subsection. Then, defining $\mathcal{N}_{n_1, n_2, n_3}$ by

$$\frac{1}{\pi^2} \mathcal{N}_{n_1, n_2, n_3} = \varepsilon \int_{u_0}^0 du \int G_1^{n_1} G_2^{n_2} G_3^{n_3} (cK_\varepsilon)_{12} (cK_\varepsilon)_{23} c_{31} \left(\sum_{a=1}^3 n_a = 1 \right), \quad (3.21)$$

we see that $\mathcal{N}[\alpha_k]$ is given by

$$\mathcal{N}[\alpha_k] = - \sum_{k, \ell=0}^N \alpha_k \alpha_\ell \left[\ell \left(\mathcal{N}_{k-\ell+1, -k, \ell} + \mathcal{N}_{\ell, -k, k-\ell+1} - \mathcal{N}_{\ell-k, 0, k-\ell+1} - \mathcal{N}_{\ell, 0, -\ell+1} \right) \right. \\ \left. + (N - \ell) \left(\mathcal{N}_{k-\ell, -k, \ell+1} + \mathcal{N}_{\ell+1, -k, k-\ell} - \mathcal{N}_{\ell-k+1, 0, k-\ell} - \mathcal{N}_{\ell+1, 0, -\ell} \right) \right], \quad (3.22)$$

where we have made the replacement $\ell \rightarrow N - \ell$ for a number of terms to eliminate N from their indices.

Next, using

$$K_\varepsilon = \frac{1}{G} - u \quad (3.23)$$

for cK_ε in Eq. (3.21) and defining S_{m_1, m_2, m_3} by

$$\frac{1}{\pi^2} S_{m_1, m_2, m_3} = \varepsilon \int_{u_0}^0 du u^{1 + \sum_{a=1}^3 m_a} \int c_{12} c_{23} c_{31} G_1^{m_1} G_2^{m_2} G_3^{m_3} \left(\sum_{a=1}^3 m_a = -1, 0, 1 \right), \quad (3.24)$$

$\mathcal{N}_{n_1, n_2, n_3}$ (3.21) is given as

$$\mathcal{N}_{n_1, n_2, n_3} = S_{n_1, n_2, n_3} - S_{n_1, n_2-1, n_3} - S_{n_1, n_2, n_3-1} + S_{n_1, n_2-1, n_3-1}. \quad (3.25)$$

Note that S_{m_1, m_2, m_3} is totally symmetric with respect to its indices and vanishes if at least one of the three m_a is equal to zero. We calculate S_{m_1, m_2, m_3} (3.24) in Appendix C by using the (s, z) -integration method of Refs. [5, 7].⁹ The results are as follows. First, we introduce a function $F_{P, Q}(z)$ defined by

⁹ The (s, z) -integration method has an ambiguity when the poles of the z -integration are located on the imaginary axis ($\text{Re } z = 0$). This ambiguity is avoided in the present case due to the K_ε -regularization.

a pair of integers (P, Q) with $P + Q = 0, 1$ or 2 :

$$F_{P,Q}(z) = \theta(P \geq 1) \theta(Q \neq 0) \frac{(P+Q)!}{4} \sum_{k=0}^{P-1} \binom{-Q}{P-1-k} \frac{\sum_{\pm} (\pm z)^{k-P-Q}}{k!} \\ - (\text{the same series with } P \leftrightarrow Q) \quad (P+Q = 0, 1, 2), \quad (3.26)$$

where $\theta(P \geq 1)$ and $\theta(Q \neq 0)$ are defined by

$$\theta(\text{condition}) = \begin{cases} 1 & \text{if the condition is satisfied} \\ 0 & \text{otherwise} \end{cases}. \quad (3.27)$$

Then, S_{m_1, m_2, m_3} is given as

$$S_{m_1, m_2, m_3} = m_1 f_{m_1+1, m_2+m_3} + m_2 f_{m_2+1, m_3+m_1} + m_3 f_{m_3+1, m_1+m_2} \\ - (m_2 + m_3) f_{m_2+m_3+1, m_1} - (m_3 + m_1) f_{m_3+m_1+1, m_2} - (m_1 + m_2) f_{m_1+m_2+1, m_3}, \quad (3.28)$$

where $f_{P,Q}$ is

$$f_{P,Q} = F_{P,Q}(2\pi i). \quad (3.29)$$

Note that $F_{P,Q}(z)$ and hence $f_{P,Q}$ are anti-symmetric with respect to (P, Q) .

In summary, we have shown that $\mathcal{N}[\alpha_k]$ is given by a series of equations, Eqs. (3.22), (3.25), (3.28), (3.29), and (3.26). It is a polynomial in $z^2 = (2\pi i)^2$ and, as shown in Appendix D, the z^0 term is equal to N :

$$\mathcal{N}[\alpha_k] = N + O(z^2). \quad (3.30)$$

The terms of non-trivial power of z^2 are the ‘‘anomalous’’ part. We present the analysis of the anomalous part as well as that of $\mathcal{T}[\alpha_k]$ in Sect. 5 after obtaining a calculable expression of $\mathcal{T}[\alpha_k]$ in the next section.

4. Expression of $\mathcal{T}[\alpha_k]$

First, let us express the EOM test $\mathcal{T}$ (1.9) in terms of E_{abc} . From the K_ε -regularized version of Eqs. (2.11) and (3.6),

$$(\Psi_\varepsilon)_{14} = (E_{124})_\varepsilon (cK_\varepsilon)_{12} (Bc)_{24}, \quad (4.1)$$

$$(\Psi_\varepsilon^2)_{14} = (E_{124} - E_{134})_\varepsilon (cK_\varepsilon)_{12} (cK_\varepsilon)_{23} (Bc)_{34}, \quad (4.2)$$

we find that the EOM is violated (apparently) by $O(\varepsilon)$:

$$(Q_B \Psi_\varepsilon + \Psi_\varepsilon^2)_{14} = (E_{124})_\varepsilon [(cKcK_\varepsilon)_{12} (Bc)_{24} - (cK_\varepsilon)_{12} (cKBc)_{24}] \\ + (E_{124} - E_{134})_\varepsilon (cK_\varepsilon)_{12} (cK_\varepsilon)_{23} (Bc)_{34} \\ = \varepsilon \times (E_{124})_\varepsilon (cK_\varepsilon)_{12} c_{24}. \quad (4.3)$$

From this and Eq. (4.1), $\mathcal{T}$ is given by

$$\mathcal{T} = \varepsilon \int (T_{1234})_\varepsilon (cK_\varepsilon)_{12} (Bc)_{23} (cK_\varepsilon)_{34} c_{41}, \quad (4.4)$$

with T_{1234} defined by

$$T_{1234} = E_{123}E_{341}. \quad (4.5)$$

As in the previous section, all the quantities in the rest of this section should be regarded as K_ε -regularized ones, and we omit the corresponding subscript ε . For example, T_{1234} means $(T_{1234})_\varepsilon$.

For evaluating $\mathcal{T}$, which is multiplied by ε , it is sufficient to take the leading part of the expansion (3.18) of E_{123} in inverse powers of G_a . In the present calculation, G_a is simply

$$G(K_\varepsilon) = \frac{1}{K_\varepsilon}. \quad (4.6)$$

Using Eq. (3.18) and keeping only the leading part, we see that T_{1234} (4.5) is given by

$$\begin{aligned} T_{1234} &= \sum_{k=0}^N \alpha_k \left[\left(\frac{G_1}{G_3} \right)^k - \left(\frac{G_1}{G_2} \right)^k - \left(\frac{G_1}{G_2} \right)^N \left(\frac{G_2}{G_3} \right)^k \right] \\ &\quad \times \sum_{\ell=0}^N \alpha_\ell \left[\left(\frac{G_3}{G_1} \right)^\ell - \left(\frac{G_3}{G_4} \right)^\ell - \left(\frac{G_3}{G_4} \right)^N \left(\frac{G_4}{G_1} \right)^\ell \right]. \end{aligned} \quad (4.7)$$

Substituting this into Eq. (4.4), we find that $\mathcal{T}$ is given in terms of $\mathcal{T}_{n_1, n_2, n_3, n_4}$ defined by

$$\begin{aligned} \mathcal{T}_{n_1, n_2, n_3, n_4} &= \varepsilon \int G_1^{n_1} G_2^{n_2} G_3^{n_3} G_4^{n_4} (cK_\varepsilon)_{12} (Bc)_{23} (cK_\varepsilon)_{34} c_{41} \\ &= \varepsilon \int Bc \frac{1}{K_\varepsilon^{n_3}} c \frac{1}{K_\varepsilon^{n_4-1}} c \frac{1}{K_\varepsilon^{n_1}} c \frac{1}{K_\varepsilon^{n_2-1}}, \end{aligned} \quad (4.8)$$

as

$$\begin{aligned} \mathcal{T}[\alpha_k] &= \sum_{k, \ell=0}^N \alpha_k \alpha_\ell \left[\mathcal{T}_{k-\ell, 0, \ell-k, 0} - \mathcal{T}_{k, 0, \ell-k, -\ell} - \mathcal{T}_{\ell-k, 0, k, -\ell} - \mathcal{T}_{k-\ell, -k, \ell, 0} + \mathcal{T}_{k, -k, \ell, -\ell} \right. \\ &\quad \left. + \mathcal{T}_{k-\ell, -k, N, \ell-N} - \mathcal{T}_{\ell, -k, k-\ell, 0} + \mathcal{T}_{N, k-N, \ell-k, -\ell} + \mathcal{T}_{\ell, -k, k, -\ell} \right], \end{aligned} \quad (4.9)$$

where we have made the replacement of the summation indices $(k, \ell) \rightarrow (N-k, N-\ell)$ for the third, the seventh, and the last terms on the RHS to eliminate N from the indices. Therefore, the calculation of $\mathcal{T}$ is reduced to that of $\mathcal{T}_{n_1, n_2, n_3, n_4}$. Note that the indices of $\mathcal{T}_{n_1, n_2, n_3, n_4}$ appearing in Eq. (4.9) satisfy

$$n_1 + n_2 + n_3 + n_4 = 0. \quad (4.10)$$

In Appendix E, we show that $\mathcal{T}_{n_1, n_2, n_3, n_4}$ is given as

$$\begin{aligned} \mathcal{T}_{n_1, n_2, n_3, n_4} &= n_1 (h_{n_3+n_4-1} - h_{n_3} - h_{n_4-1}) + n_3 (h_{n_4+n_1-1} - h_{n_4-1} - h_{n_1}) \\ &\quad + (n_4 - 1) (h_{n_3+n_4} + h_{n_4+n_1} - h_{n_4} - h_{n_3+n_4+n_1}), \end{aligned} \quad (4.11)$$

where h_Q for an integer Q is given by

$$h_Q = H_Q(2\pi i), \quad (4.12)$$

with $H_Q(z)$ (which is not the Hermite polynomial) defined by

$$H_Q(z) = \sum_{\pm} \frac{1}{\pm z} \left[\theta(Q \leq -2) \sum_{k=0}^{-Q-2} \binom{-Q}{k+2} \frac{(\pm z)^k}{k!} - \theta(Q \geq 1) \sum_{k=0}^{Q-1} \binom{Q+1}{k+2} \frac{(\pm z)^k}{k!} \right]. \quad (4.13)$$

Note that $H_Q(z)$ is a polynomial in z^2 .

5. Solutions with $\mathcal{N} = N$ and $\mathcal{T} = 0$

Having obtained calculable expressions of $\mathcal{N}[\alpha_k]$ and $\mathcal{T}[\alpha_k]$ in Sects. 3 and 4, respectively, we in this section examine whether CSFT solutions satisfying both of

$$\mathcal{N}[\alpha_k] = N, \quad \mathcal{T}[\alpha_k] = 0 \quad (5.1)$$

exist, namely, whether there exists $\{\alpha_k\}$ satisfying the two conditions of Eq. (5.1) for each N . Of course, it is desirable to present a general argument applicable to any N . However, we have not yet succeeded in keeping the complicated expressions of $\mathcal{N}[\alpha_k]$ and $\mathcal{T}[\alpha_k]$ under full control sufficient for general arguments. Postponing complete analysis to future studies, we here present arguments for various values of N .

As independent elements among α_k ($k = 0, 1, \dots, N$) subject to the constraints (2.20) and (2.21), we take the first $[N/2]$ ones, $(\alpha_0, \alpha_1, \dots, \alpha_{[N/2]-1})$. Since our solution for $N = 1$ is unique, $(\alpha_0, \alpha_1) = (1/2, 1/2)$, and agrees with that of Ref. [9] satisfying $\mathcal{N} = 1$ and $\mathcal{T} = 0$, let us start with the $N = 2$ case. In the following, z^2 implies $(2\pi i)^2$.

5.1. α_k for $N = 2, 3, 4, 5$

$N = 2$ For $N = 2$, $\mathcal{N}$ and $\mathcal{T}$ are given by

$$\mathcal{N} = 2 + \alpha_0 z^2, \quad \mathcal{T} = -12\alpha_0. \quad (5.2)$$

Therefore, $\mathcal{N} = 2$ and $\mathcal{T} = 0$ are simultaneously realized by taking $\alpha_0 = 0$:

$$(\alpha_0, \alpha_1, \alpha_2) = (0, 1, 0). \quad (5.3)$$

In this case, $\mathcal{F}_{ab}$ (2.19) and E_{abc} (2.12) for the 3-brane solution are

$$\mathcal{F}_{ab} = 1 - G_a G_b, \quad E_{abc} = 1 - \frac{G_a + G_c}{G_b} + \frac{1}{G_b^2}. \quad (5.4)$$

Explicitly, the solution is given by

$$\Psi_{3\text{-brane}} = cK \left(1 + \frac{1}{G^2} \right) Bc - GcK \frac{1}{G} Bc - cK \frac{1}{G} Bc G, \quad (5.5)$$

with $G(K)$ of Eq. (2.16).

$N = 3$ For $N = 3$, we obtain

$$\mathcal{N} = 3 + 3 \left(\alpha_0 + \frac{1}{6} \right) z^2, \quad \mathcal{T} = -36 \left(\alpha_0 + \frac{1}{6} \right). \quad (5.6)$$

$\mathcal{N} = 3$ and $\mathcal{T} = 0$ are simultaneously realized by taking $\alpha_0 = -1/6$:

$$(\alpha_0, \alpha_1, \alpha_2, \alpha_3) = \left(-\frac{1}{6}, \frac{2}{3}, \frac{2}{3}, -\frac{1}{6} \right). \quad (5.7)$$

$N = 4$ For $N = 4$, we obtain

$$\begin{aligned}\mathcal{N} &= 4 + (1 + 8\alpha_0 + 2\alpha_1)z^2 + \frac{1}{2}\alpha_0 z^4, \\ \mathcal{T} &= -12(1 + 8\alpha_0 + 2\alpha_1) - \frac{10}{3}\alpha_0 z^2.\end{aligned}\quad (5.8)$$

Demanding $\mathcal{N} = 4$ and $\mathcal{T} = 0$, α_k are uniquely determined by two equations, $1 + 8\alpha_0 + 2\alpha_1 = 0$ and $\alpha_0 = 0$, as

$$(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) = \left(0, -\frac{1}{2}, 2, -\frac{1}{2}, 0\right). \quad (5.9)$$

$N = 5$ For $N = 5$, we have

$$\begin{aligned}\mathcal{N} &= 5 + \frac{5}{2}(1 + 6\alpha_0 + 2\alpha_1)z^2 + \frac{1}{2}(6\alpha_0 + \alpha_1)z^4, \\ \mathcal{T} &= -30(1 + 6\alpha_0 + 2\alpha_1) - \frac{10}{3}(6\alpha_0 + \alpha_1)z^2,\end{aligned}\quad (5.10)$$

and the conditions $\mathcal{N} = 5$ and $\mathcal{T} = 0$ uniquely determine α_k as

$$(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) = \left(\frac{1}{6}, -1, \frac{4}{3}, \frac{4}{3}, -1, \frac{1}{6}\right). \quad (5.11)$$

5.2. α_k for $N \geq 6$

As seen above, $\mathcal{N}$ and $\mathcal{T}$ for $N \leq 5$ are polynomials in z^2 and take the following form:

$$\begin{aligned}\mathcal{N} &= N + \sum_{n=1}^{[N/2]} f_n(\alpha_k) z^{2n}, \\ \mathcal{T} &= - \sum_{n=1}^{[N/2]} t_n f_n(\alpha_k) z^{2(n-1)},\end{aligned}\quad (5.12)$$

where $f_n(\alpha_k)$ are linear functions of α_k ($k = 0, 1, \dots, [N/2] - 1$), and t_n in $\mathcal{T}$ are numerical coefficients. In particular, $f_n(\alpha_k)$ are common between $\mathcal{N}$ and $\mathcal{T}$. As we see below, the form (5.12) is valid for larger N that we will test. It must be possible to prove Eq. (5.12) for a generic N by using the expressions of $\mathcal{N}$ and $\mathcal{T}$ given in Sects. 3 and 4.

Then, a problem with the $N \geq 6$ cases is that, while the number of conditions is only two of Eq. (5.1), the number of independent α_k is $[N/2]$, which is greater than 2 for $N \geq 6$. A general solution $\{\alpha_k\}$ to Eq. (5.1) contains powers of π^2 and is generically irrational. In order to fix α_k uniquely, we here adopt a special (and probably a “natural”) solution to Eq. (5.1) by demanding $[N/2]$ conditions:

$$f_n(\alpha_k) = 0 \quad (n = 1, \dots, [N/2]). \quad (5.13)$$

In fact, $\{\alpha_k\}$ given above for $N = 2, 3, 4, 5$ have been determined by Eq. (5.13). For larger N , the conditions (5.13) provide us with sufficient conditions to uniquely determine $\{\alpha_k\}$, and the resultant α_k is a rational number, as far as we have checked. Here, we present $\mathcal{N}$ and $\mathcal{T}$ and the solution to Eq. (5.13) in the cases $N = 6, 7$, and 11 , as examples.

$N = 6$ In this case, $\mathcal{N}$ and $\mathcal{T}$ are certainly of the form of Eq. (5.12):

$$\begin{aligned}\mathcal{N} &= 6 + (4 + 27\alpha_0 + 12\alpha_1 + 3\alpha_2)z^2 + \frac{1}{2}(21\alpha_0 + 6\alpha_1 + \alpha_2)z^4 + \frac{1}{24}\alpha_0 z^6, \\ \mathcal{T} &= -12(4 + 27\alpha_0 + 12\alpha_1 + 3\alpha_2) - \frac{10}{3}(21\alpha_0 + 6\alpha_1 + \alpha_2)z^2 - \frac{7}{30}\alpha_0 z^4.\end{aligned}\quad (5.14)$$

The solution to Eq. (5.13) is given by:

$$(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6) = \left(0, \frac{2}{3}, -4, \frac{23}{3}, -4, \frac{2}{3}, 0\right). \quad (5.15)$$

$N = 7$ In this case, $\mathcal{N}$ and $\mathcal{T}$ are again of the form of Eq. (5.12):

$$\begin{aligned}\mathcal{N} &= 7 + 7(1 + 6\alpha_0 + 3\alpha_1 + \alpha_2)z^2 + \frac{1}{4}(1 + 110\alpha_0 + 40\alpha_1 + 10\alpha_2)z^4 + \frac{1}{24}(8\alpha_0 + \alpha_1)z^6, \\ \mathcal{T} &= -84(1 + 6\alpha_0 + 3\alpha_1 + \alpha_2) - \frac{5}{3}(1 + 110\alpha_0 + 40\alpha_1 + 10\alpha_2)z^2 - \frac{7}{30}(8\alpha_0 + \alpha_1)z^4.\end{aligned}\quad (5.16)$$

The solution to Eq. (5.13) is

$$(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7) = \left(-\frac{3}{10}, \frac{12}{5}, -\frac{32}{5}, \frac{24}{5}, \frac{24}{5}, -\frac{32}{5}, \frac{12}{5}, -\frac{3}{10}\right). \quad (5.17)$$

$N = 11$ In the case of $N = 11$, $\mathcal{N}$ and $\mathcal{T}$ are of the form (5.12) with $f_n(\alpha_k)$ and t_n given by

$$\begin{aligned}f_1 &= 11\left(\frac{5}{2} + 15\alpha_0 + 10\alpha_1 + 6\alpha_2 + 3\alpha_3 + \alpha_4\right), \\ f_2 &= \frac{3}{4}(9 + 510\alpha_0 + 290\alpha_1 + 150\alpha_2 + 66\alpha_3 + 20\alpha_4), \\ f_3 &= \frac{7}{24}\left(\frac{1}{14} + 113\alpha_0 + 47\alpha_1 + 17\alpha_2 + 5\alpha_3 + \alpha_4\right), \\ f_4 &= \frac{1}{720}(220\alpha_0 + 55\alpha_1 + 10\alpha_2 + \alpha_3), \\ f_5 &= \frac{1}{40320}(12\alpha_0 + \alpha_1),\end{aligned}\quad (5.18)$$

and

$$(t_1, t_2, t_3, t_4, t_5) = \left(12, \frac{20}{3}, \frac{28}{5}, \frac{36}{7}\right). \quad (5.19)$$

The solution to Eq. (5.13) is

$$(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) = \left(-\frac{691}{210}, \frac{1382}{35}, -\frac{20528}{105}, \frac{10652}{21}, -\frac{24384}{35}\right). \quad (5.20)$$

Summarizing this section, as far as we have checked for various positive integer N , $\mathcal{N}$ and $\mathcal{T}$ take the form of Eq. (5.12) in terms of common linear functions $f_n(\alpha_k)$, and the condition (5.13) uniquely determines $\{\alpha_k\}$.¹⁰ Of course, there are many questions to be answered and subjects to be studied, which we shall discuss in the next section.

¹⁰ We have checked this for N up to 35 by using Mathematica.

6. Summary and discussions

In this paper, we have presented an analytic expression of the multi-brane solutions of CSFT for arbitrary (positive integer) brane numbers. We started with the most generic unitary and real string field U (2.7) with Γ_a and $\mathcal{F}_{ab}$ satisfying Eqs. (2.8) and (2.9), and considered as a candidate solution the pure-gauge string field $UQ_B U^{-1}$. As Γ_a and $\mathcal{F}_{ab}$ for multi-brane solutions, we adopted the ansatz of Eqs. (2.18) and (2.19) using $G(K)$ with a simple pole at $K = 0$. For the $(N + 1)$ -brane solution, we in this paper demanded the following: First, the energy density of the solution calculated from the action should be that of the $(N + 1)$ -brane. Concretely, $\mathcal{N}$ (1.8) should be equal to the integer N . Second, the EOM test against the solution itself given by $\mathcal{T}$ (1.9) should vanish. In the previous constructions of multi-brane solutions based on the singularity at $K = 0$, these two conditions were hard to be realized in the cases of $N \geq 2$. In the present construction, our solution is specified by real parameters α_k subject to Eqs. (2.20) and (2.21), and the problem is whether there exists $\{\alpha_k\}$ that realizes $\mathcal{N} = N$ and $\mathcal{T} = 0$. We calculated $\mathcal{N}[\alpha_k]$ and $\mathcal{T}[\alpha_k]$ in the K_ε -regularization to find that there indeed exists $\{\alpha_k\}$ satisfying the two conditions for any $N = 2, 3, 4, 5, \dots$ as far as we have tested. For $N \geq 6$, the two conditions, $\mathcal{N} = N$ and $\mathcal{T} = 0$, cannot uniquely fix α_k , and we proposed to demand stronger conditions (5.13) on α_k , which give a sufficient number of equations to uniquely determine α_k as rational numbers.

Here, we add a remark for preventing a possible misunderstanding of the reader about our construction of solutions. One might think that our construction is almost trivial and meaningless since we are imposing only the two conditions (5.1) on the solutions, and this is always possible if the candidate solution has enough parameters (α_k in our case). However, we should recall that our candidate solution is “almost a solution” since it is of the pure-gauge form $\Psi = UQ_B U^{-1}$, which automatically satisfies the EOM if there is no subtlety at $K = 0$. The non-integer nature of $\mathcal{N}$ and, possibly, the failure of the EOM test against itself, for a generic $\{\alpha_k\}$ would be manifestations of the non-regularity of U at $K = 0$, as we explained in the introduction. The two conditions that we impose should be regarded as conditions necessary for making the pure-gauge configuration a more regular one.

We have certainly succeeded in constructing $(N + 1)$ -brane solutions satisfying the two conditions for $N = 2, 3, 4, 5, \dots$. However, our analysis in this paper is still at an “experimental” level. Namely, we have confirmed the existence of the “natural” choice of $\{\alpha_k\}$ determined by Eq. (5.13) only for sample values of N . Although there is no doubt that such $\{\alpha_k\}$ giving a desired multi-brane solution exists for any integer N , we should present a general proof for our expectation. For this, we have to show that the expressions of $\mathcal{N}$ and $\mathcal{T}$ given in Eq. (5.12) in terms of common functions $f_n(\alpha_k)$ are valid for any N . It is of course desirable that the solution α_k to Eq. (5.13) is explicitly given for a generic N .

Even if these technical problems are resolved, there still remain important questions on our construction of multi-brane solutions:

- What is the meaning of the stronger conditions (5.13) on α_k ? Possibly, these conditions could be derived by considering other natural requirements on the solution, e.g., the requirement that the energy density of the solution evaluated from the gravitational coupling [10–13] be equal to that of the $(N + 1)$ -brane. Besides, since the number of conditions (5.13) depends on N (and is equal to $[N/2]$), requirements related to the fluctuation modes on the solution might be the origins of the conditions.

- Is there any profound mathematical meaning in $\mathcal{F}_{ab}$ given by Eq. (2.19) in terms of α_k satisfying the condition (5.13)? Recalling that $\mathcal{N}$ (1.8) for the present pure-gauge type solution $\Psi = UQ_B U^{-1}$ has an analogy to the winding number $\mathcal{W}[g]$ (1.14) of the mapping $g(x)$ from a three-manifold M to a Lie group, it would be interesting if the present construction realizing arbitrary integer $\mathcal{N}$ gives some hint for uncovering the meaning of $\mathcal{N}$ as a “winding number”, as we explained in the introduction.
- In this paper, as the EOM tests, we considered only that against the solution itself given by $\mathcal{T}$ (1.9). Let us define the EOM test against a generic string field $\mathcal{O}$ with $N_{\text{gh}} = 1$ by

$$\mathcal{T}[\mathcal{O}] = \int \mathcal{O} * (Q_B \Psi_\varepsilon + \Psi_\varepsilon^2). \quad (6.1)$$

It is known that the $N = 1$ (2-brane) solution does not pass the EOM test against the Fock vacuum; $\mathcal{T}[(e^{-\frac{\pi}{4}K} c e^{-\frac{\pi}{4}K})_\varepsilon] = O(1/\varepsilon) \neq 0$ [7]. This property also persists in our $N \geq 2$ solutions irrespective of the choice of α_k , as we have already mentioned in the introduction. Instead, our solutions pass the EOM test against the unitary transformed Fock vacuum: $\mathcal{T}[(U e^{-\frac{\pi}{4}K} c e^{-\frac{\pi}{4}K} U^{-1})_\varepsilon] = 0$. On the other hand, the tachyon vacuum solution ($N = -1$) passes all the EOM tests. For full understanding of the problem of the EOM test, it would be necessary to solve the problem of the fluctuation modes around the solution (see Ref. [14]).

Among the above three questions/problems, the last one is the most serious one from the viewpoint of constructing complete solutions. However, we expect that, even if the third problem remains unresolved, our finding in this paper gives a useful hint in considering the topological aspects of CSFT, as we stated in the introduction and in the above second question. We finish this paper by giving some comments concerning our solution:

- In the particular case of $N = 2$, our U with α_k of Eq. (5.3) has the following manifestly unitary expression:

$$U = \exp \left(\frac{1}{2} \{ [B, c], g(K) \} \right), \quad (6.2)$$

where $g(K)$ is defined by

$$e^{g(K)} = G(K) = \frac{1+K}{K}. \quad (6.3)$$

In relation to this, the following U is also unitary for any self-conjugate $f(K)$:

$$U = \exp(f(K) [B, c] f(K)). \quad (6.4)$$

This U is rewritten into the standard form (2.7) and the corresponding Γ_a and $\mathcal{F}_{ab}$ are

$$\Gamma_a = e^{f(K_a)^2}, \quad \mathcal{F}_{ab} = \frac{2 (\ln \Gamma_a \ln \Gamma_b)^{1/2}}{\ln \Gamma_a + \ln \Gamma_b} (1 - \Gamma_a \Gamma_b). \quad (6.5)$$

Note that this $\mathcal{F}_{ab}$ is equal to $\mathcal{F}_{ab} = 1 - \Gamma_a \Gamma_b$ in Eq. (5.4) for $N = 2$ (recall that $\Gamma = G$ when $N = 2$) multiplied by the front term consisting of $\ln \Gamma$. However, we find that, due to the

presence of the $\ln \Gamma$ term in $\mathcal{F}_{ab}$, both $\mathcal{N}$ and $\mathcal{T}$ are *divergent* in the limit $\varepsilon \rightarrow +0$. In fact, if we take $\Gamma = G(K) = (1 + K)/K$, $\mathcal{N}$ diverges as

$$\mathcal{N} = O\left(\frac{1}{\varepsilon^2 \ln^2(1/\varepsilon)}\right). \quad (6.6)$$

In this respect also, our $\mathcal{F}_{ab}$ given by Eq. (2.19) is a good choice.

- The product $U^{(3)} = U^{(1)}U^{(2)}$ of two unitary $U^{(1)}$ and $U^{(2)}$ is of course unitary and is written in the form (2.7) with Γ_a and $\mathcal{F}_{ab}$ satisfying Eqs. (2.8) and (2.9). In fact, $(\Gamma_a, \mathcal{F}_{ab})$ of $U^{(3)}$ is given in terms of those of $U^{(1)}$ and $U^{(2)}$ by

$$\Gamma_a^{(3)} = \Gamma_a^{(1)}\Gamma_a^{(2)}, \quad \mathcal{F}_{ab}^{(3)} = \mathcal{F}_{ab}^{(1)} + \Gamma_a^{(1)}\mathcal{F}_{ab}^{(2)}\Gamma_b^{(1)}. \quad (6.7)$$

This relation implies that, even if $(\Gamma_a^{(1,2)}, \mathcal{F}_{ab}^{(1,2)})$ are of the form of Eqs. (2.18) and (2.19), $(\Gamma_a^{(3)}, \mathcal{F}_{ab}^{(3)})$ is no longer so and cannot realize integer $\mathcal{N} = N^{(3)} = N^{(1)} + N^{(2)}$ and $\mathcal{T} = 0$ in general. We have already seen this phenomenon of the violation of the additivity of $\mathcal{N}$ in the case of $N^{(1)} = N^{(2)} = 1$ in Ref. [6].

- In this paper, we considered explicitly only $(N + 1)$ -brane solutions with positive integer N . However, “ghost brane” solutions with $N \leq -2$ can also be constructed in the same manner.

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Appendix A. Calculations for Sect. 2.2

In this appendix, we present the derivations of some of the equations in Sect. 2.2, in particular, the conditions (2.8) and (2.9) for the unitarity of U (2.7). Though the calculations are straightforward, they may be helpful as examples of the convenient notation of this paper.

First, the conjugate of U (2.7) is

$$(U^\dagger)_{12} = \frac{1}{\Gamma_1} \mathbb{I}_{12} - \frac{\mathcal{F}_{21}}{\Gamma_1} (cB)_{12} = \frac{1 - \mathcal{F}_{11}}{\Gamma_1} \mathbb{I}_{12} + \frac{\mathcal{F}_{21}}{\Gamma_1} (Bc)_{12}, \quad (A.1)$$

and $UU^\dagger$ is given by

$$(UU^\dagger)_{13} = U_{12}(U^\dagger)_{23} = \frac{1 - \mathcal{F}_{11}}{(\Gamma_1)^2} \mathbb{I}_{13} + \left[\frac{(1 - \mathcal{F}_{11}) \mathcal{F}_{31}}{(\Gamma_1)^2} - \frac{\mathcal{F}_{13} (1 - \mathcal{F}_{33})}{(\Gamma_3)^2} \right] (Bc)_{13}. \quad (A.2)$$

In deriving Eq. (A.2), we have used

$$f(K_2) (Bc)_{12} (Bc)_{23} = f(K_2) \mathbb{I}_{12} (Bc)_{23} = f(K_1) (Bc)_{13}, \quad (A.3)$$

valid for any $f(K)$. From Eq. (A.2), we find that U is unitary if the two conditions (2.8) and (2.9) are satisfied. Equation (2.10) for U^{-1} follows immediately from Eq. (A.1) and the two conditions.

Next, let us evaluate $UQ_B U^{-1}$ for U (2.7) and U^{-1} (2.10):

$$(UQ_B U^{-1})_{13} = U_{12} (Q_B U^{-1})_{23} = \left[\frac{1}{\Gamma_2} \mathbb{I}_{12} - \frac{\mathcal{F}_{12}}{\Gamma_2} (Bc)_{12} \right] \frac{\mathcal{F}_{23}}{\Gamma_2} (cKBc)_{23} = E_{123} (cK)_{12} (Bc)_{23}, \quad (A.4)$$

where E_{abc} is given by Eq. (2.12). In the calculation of Eq. (A.4), we have used $Q_B(Bc) = cKBc$, the identity

$$(Bc)_{12} (cKBc)_{23} = \mathbb{I}_{12} (cKBc)_{23} - (cK)_{12} (Bc)_{23}, \quad (\text{A.5})$$

or, more generally,

$$(Bc)_{12} (cK)_{23} (Bc)_{34} = \left(\mathbb{I}_{12} (cK)_{23} - (cK)_{12} \mathbb{I}_{23} \right) (Bc)_{34}, \quad (\text{A.6})$$

and the condition (2.8).

Appendix B. The formula (3.1)

For an arbitrary infinitesimal deformation δU^{-1} , we have

$$\delta(UQ_B U^{-1}) = Q_B(U\delta U^{-1}) + [UQ_B U^{-1}, U\delta U^{-1}]. \quad (\text{B.1})$$

Using this, we obtain

$$\begin{aligned} \frac{1}{\pi^2} \delta \mathcal{N} &= \int (UQ_B U^{-1})_\varepsilon^2 \delta(UQ_B U^{-1})_\varepsilon = \int (UQ_B U^{-1})_\varepsilon^2 \{Q_B(U\delta U^{-1}) + [UQ_B U^{-1}, U\delta U^{-1}]\}_\varepsilon \\ &= \int \left\{ (UQ_B U^{-1})^2 Q_B(U\delta U^{-1}) \right\}_\varepsilon = \int \left\{ Q_B \left[(UQ_B U^{-1})^2 (U\delta U^{-1}) \right] \right\}_\varepsilon, \end{aligned} \quad (\text{B.2})$$

where we have used $Q_B(UQ_B U^{-1})^2 = 0$ in obtaining the last expression. Then, noticing that $(Q_B f(K))_\varepsilon = 0 = Q_B f(K_\varepsilon)$, $(Q_B c)_\varepsilon = cK_\varepsilon c = cKc = Q_B c$, and $(Q_B B)_\varepsilon = K_\varepsilon = Q_B B + \varepsilon$, we see that the following relation holds for any $\mathcal{O}(K, B, c)$:

$$(Q_B \mathcal{O}(K, B, c))_\varepsilon = Q_B \mathcal{O}(K_\varepsilon, B, c) + \varepsilon \times T_B \mathcal{O}(K_\varepsilon, B, c), \quad (\text{B.3})$$

where T_B is the operation (3.2). Since $\int Q_B \mathcal{O}(K_\varepsilon, B, c)$ vanishes without ambiguity, we obtain Eq. (3.1).

Appendix C. Derivation of Eq. (3.28)

In this appendix, we derive Eq. (3.28) for S_{m_1, m_2, m_3} (3.24) by using the (s, z) -integration formula for the $Bcccc$ -correlators [5, 7]. This formula is given by

$$\int BcF_1(K)cF_2(K)cF_3(K)cF_4(K) = \int_0^\infty ds \frac{s^2}{(2\pi)^3 i} \int_{-i\infty}^{i\infty} \frac{dz}{2\pi i} e^{sz} \mathcal{G}(z), \quad (\text{C.1})$$

where $\mathcal{G}(z)$ is defined in our convention by

$$\begin{aligned} \mathcal{G}(z) &= \left[(\Delta_s F_1) F_2 F_3' + F_1' F_2 (\Delta_s F_3) + (F_1 (\Delta_s F_2) F_3)' - \Delta_s (F_1 F_2') F_3 \right. \\ &\quad \left. - \Delta_s (F_1 F_2) F_3' - F_1' \Delta_s (F_2 F_3) - F_1 \Delta_s (F_2' F_3) + \Delta_s (F_1 F_2' F_3) \right] F_4, \end{aligned} \quad (\text{C.2})$$

with $F_i = F_i(z)$, $F_i' = (d/dz)F_i(z)$, and

$$(\Delta_s F_i)(z) \equiv F_i \left(z - \frac{2\pi i}{s} \right) - F_i \left(z + \frac{2\pi i}{s} \right). \quad (\text{C.3})$$

In the application of Eq. (C.1) to the *ccc*-correlator in S_{m_1, m_2, m_3} (3.24), $F_i(z)$ are (note that $cBc = c$)

$$F_a(z) = \frac{1}{(z + u + \varepsilon)^{m_a}} \quad (a = 1, 2, 3), \quad F_4(z) = 1. \quad (\text{C.4})$$

In this case, the contour of z -integration (C.1) can be closed by adding the infinite semi-circle in the left-half plane $\text{Re } z < 0$ due to the presence of e^{sz} . In addition, we find that the infinitesimal positive constant ε in S_{m_1, m_2, m_3} (3.24) is totally absorbed into the following replacements (rescaling) of the three integration variables (u, s, z) :

$$(u, s, z) \rightarrow \left(\varepsilon u, \frac{s}{\varepsilon}, \varepsilon z \right). \quad (\text{C.5})$$

Then, we obtain

$$\frac{1}{\pi^2} S_{m_1, m_2, m_3} = \int_0^\infty \frac{du}{(2\pi i)^3} u^{1+\sum_a m_a} \int_0^\infty ds s^2 \sum_{\substack{\text{poles in} \\ \text{Re } z < 0}} \text{Res } e^{sz} \mathcal{G}(z), \quad (\text{C.6})$$

where $F_i(z)$ for the present $\mathcal{G}(z)$ is given, instead of Eq. (C.4), by

$$F_a(z) = \frac{1}{(z + u + 1)^{m_a}} \quad (a = 1, 2, 3), \quad F_4(z) = 1. \quad (\text{C.7})$$

Explicitly, $\mathcal{G}(z)$ in Eq. (C.6) is given by

$$\begin{aligned} \mathcal{G}(z) = \sum_{\pm} (\pm) \left\{ -\frac{m_3}{\widehat{z}^{m_2+m_3+1} (\widehat{z} \mp \frac{2\pi i}{s})^{m_1}} - \frac{m_1}{\widehat{z}^{m_1+m_2+1} (\widehat{z} \mp \frac{2\pi i}{s})^{m_3}} - \frac{m_3 + m_1}{\widehat{z}^{m_3+m_1+1} (\widehat{z} \mp \frac{2\pi i}{s})^{m_2}} \right. \\ \left. - \frac{m_2}{\widehat{z}^{m_3+m_1} (\widehat{z} \mp \frac{2\pi i}{s})^{m_2+1}} + \frac{m_2}{\widehat{z}^{m_3} (\widehat{z} \mp \frac{2\pi i}{s})^{m_1+m_2+1}} + \frac{m_3}{\widehat{z}^{m_3+1} (\widehat{z} \mp \frac{2\pi i}{s})^{m_1+m_2}} \right. \\ \left. + \frac{m_1}{\widehat{z}^{m_1+1} (\widehat{z} \mp \frac{2\pi i}{s})^{m_2+m_3}} + \frac{m_2}{\widehat{z}^{m_1} (\widehat{z} \mp \frac{2\pi i}{s})^{m_2+m_3+1}} - \frac{m_2}{(\widehat{z} \mp \frac{2\pi i}{s})^{m_1+m_2+m_3+1}} \right\}, \quad (\text{C.8}) \end{aligned}$$

where $\widehat{z}$ is defined by $\widehat{z} \equiv z + u + 1$. Note that the contribution of each term in Eq. (C.8) to S_{m_1, m_2, m_3} (C.6) is given by the following $f_{P,Q}$:

$$\frac{1}{\pi^2} f_{P,Q} = \sum_{\pm} (\pm) \int_0^\infty \frac{du}{(2\pi i)^3} u^{P+Q} \int_0^\infty ds s^2 \sum_{\substack{\text{poles in} \\ \text{Re } z < 0}} \text{Res} \frac{e^{sz}}{(z + u + 1)^P (z + u + 1 \mp \frac{2\pi i}{s})^Q} \quad (\text{C.9})$$

where a pair of integers (P, Q) satisfy

$$P + Q = \sum_{a=1}^3 m_a + 1 = 0, 1, 2. \quad (\text{C.10})$$

Calculating the sum of residues in Eq. (C.9) at $z = -u - 1$ and $-u - 1 \pm (2\pi i/s)$ by using the formulas,

$$\text{Res}_{z=0} \frac{e^{sz}}{z^m (z + a)^n} = \theta(m \geq 1) \sum_{k=0}^{m-1} \frac{1}{k!} \binom{-n}{m-1-k} s^k a^{k-n-m+1} \quad (n \neq 0), \quad (\text{C.11})$$

$$\text{Res}_{z=0} \frac{e^{sz}}{z^m} = \theta(m \geq 1) \frac{s^{m-1}}{(m-1)!}, \quad (\text{C.12})$$

and carrying out the u - and s -integrations, we find that $f_{P,Q}$ is given by Eq. (3.29) by using $F_{P,Q}(z)$ (3.26), and that S_{m_1,m_2,m_3} is given by Eq. (3.28) in terms of $f_{P,Q}$ (by using the anti-symmetry of $f_{P,Q}$ for a number of terms). In particular, the last term of Eq. (C.8) does not contribute to S_{m_1,m_2,m_3} since we have $f_{P,0} = f_{0,Q} = 0$ (the residues cancel after the summation $\sum_{\pm}(\pm)$).

The series $F_{P,Q}(z)$ (3.26) has the following expression in terms of the confluent hypergeometric functions:

$$F_{P,Q}(z) = \theta(P \geq 1) \theta(Q \neq 0) \times \left(-\frac{1}{4}\right) \sum_{\pm} \begin{cases} Q {}_1F_1(Q+1, 2; \pm z) & (P+Q=0) \\ \frac{1}{(\pm z)} {}_1F_1(Q, 1; \pm z) & (P+Q=1) \\ \frac{2}{(\pm z)} {}_1F_1(Q, 2; \pm z) & (P+Q=2) \end{cases} - (P \rightleftharpoons Q). \quad (\text{C.13})$$

Appendix D. Proof of Eq. (3.30)

As we saw in Sect. 3.2, $\mathcal{N}$ for our solution is given as a polynomial in $z^2 = (2\pi i)^2$. In this appendix, we show Eq. (3.30); namely, that the z^0 term of $\mathcal{N}$ is equal to N .

Let us start with the expression (3.26) of $F_{P,Q}(z)$. Here, we repeatedly use the fact that the integers P and Q for $F_{P,Q}(z)$ are restricted to those of the three cases:

$$P+Q=0, 1, 2. \quad (\text{D.1})$$

First, note that the negative power terms of z in Eq. (3.26) are actually non-existent. Next, since the z^0 part of Eq. (3.26) comes from the $k = P+Q$ term, and this term is in the range of the k -summation only when $P+Q \leq P-1$, namely, $Q \leq -1$, we obtain

$$\begin{aligned} F_{P,Q}(z=0) &= \theta(P \geq 1) \theta(Q \leq -1) \frac{1}{4} \left(\frac{-Q}{-Q-1} \right) \sum_{\pm} 1 - (P \rightleftharpoons Q) \\ &= -\theta(P \geq 1) \theta(Q \leq -1) \frac{Q}{2} + \theta(Q \geq 1) \theta(P \leq -1) \frac{P}{2}. \end{aligned} \quad (\text{D.2})$$

Then, by taking into account Eq. (D.1), we find that $F_{P,Q}(0)$ is rewritten as

$$F_{P,Q}(0) = -\theta(P \geq 1) \frac{Q}{2} + \theta(Q \geq 1) \frac{P}{2} = \theta(Q \geq 1) \frac{P+Q}{2} - \frac{Q}{2} - \frac{1}{2} \theta(P=1) \theta(Q=1), \quad (\text{D.3})$$

where we need Eq. (D.1) also at the second equality. Plugging this into S_{m_1,m_2,m_3} (3.28) given by $f_{P,Q}$, we find that the contribution of the $-Q/2$ term of Eq. (D.3) cancels, while the $-(1/2)\theta(P=1)\theta(Q=1)$ term does not contribute since $f_{P,Q}$ is multiplied by $P-1$ in Eq. (3.28). Therefore, we get

$$\begin{aligned} S_{m_1,m_2,m_3}|_{z=0} &= \frac{\sum_a m_a + 1}{2} \left[\theta(m_2 + m_3 \geq 1) m_1 + \theta(m_3 + m_1 \geq 1) m_2 + \theta(m_1 + m_2 \geq 1) m_3 \right. \\ &\quad \left. - \theta(m_1 \geq 1) (m_2 + m_3) - \theta(m_2 \geq 1) (m_3 + m_1) - \theta(m_3 \geq 1) (m_1 + m_2) \right]. \end{aligned} \quad (\text{D.4})$$

Plugging this into $\mathcal{N}_{n_1, n_2, n_3}$ (3.25) given by S_{m_1, m_2, m_3} , and using that $\sum_{a=1}^3 n_a = 1$, we obtain

$$\begin{aligned}\mathcal{N}_{n_1, n_2, n_3}|_{z=0} &= -\theta(n_1 \geq 1) - \theta(n_2 \geq 1) - \theta(n_3 \geq 1) + 1 \\ &= -\theta(n_1 \geq 1)\theta(n_2 \geq 1) - \theta(n_2 \geq 1)\theta(n_3 \geq 1) - \theta(n_3 \geq 1)\theta(n_1 \geq 1),\end{aligned}\quad (\text{D.5})$$

where, in the derivation of the first expression, we have used, e.g.,

$$\theta(n_2 + n_3 \geq 1) - \theta(n_2 + n_3 \geq 2) = \theta(n_2 + n_3 = 1) = \theta(n_1 = 0). \quad (\text{D.6})$$

Finally, substituting Eq. (D.5) into $\mathcal{N}$ (3.22) and using $\mathcal{N}_{n_1, 0, n_3}|_{z=0} = 0$, we get

$$\mathcal{N}|_{z=0} = 2 \sum_{k, \ell=0}^N \alpha_k \alpha_\ell \left[\ell \theta(k \geq \ell) + (N - \ell) \theta(k \geq \ell + 1) \right]. \quad (\text{D.7})$$

Making the replacement of the summation indices $(k, \ell) \rightarrow (N - k, N - \ell)$ for the second term, we obtain

$$\mathcal{N}|_{z=0} = 2 \sum_{k, \ell=0}^N \alpha_k \alpha_\ell \ell [\theta(k \geq \ell) + \theta(k \leq \ell - 1)] = 2 \sum_{k=0}^N \alpha_k \sum_{\ell=0}^N \ell \alpha_\ell = N, \quad (\text{D.8})$$

where we have used Eqs. (2.20) and (2.22).

Appendix E. Derivation of Eqs. (4.11)–(4.13)

The *Bcccc*-correlator $\mathcal{T}_{n_1, n_2, n_3, n_4}$ (4.8) is evaluated by using Eq. (C.1). In this case, the four functions $F_i(z)$ are

$$F_1(z) = \frac{1}{(z + \varepsilon)^{n_3}}, \quad F_2(z) = \frac{1}{(z + \varepsilon)^{n_4-1}}, \quad F_3(z) = \frac{1}{(z + \varepsilon)^{n_1}}, \quad F_4(z) = \frac{1}{(z + \varepsilon)^{n_2-1}}. \quad (\text{E.1})$$

Taking into account Eq. (4.10), we see that the positive infinitesimal constant ε in $\mathcal{T}_{n_1, n_2, n_3, n_4}$ can be absorbed into the rescaling of two integration variables (s, z) ,

$$(s, z) \rightarrow \left(\frac{s}{\varepsilon}, \varepsilon z \right), \quad (\text{E.2})$$

to obtain

$$\mathcal{T}_{n_1, n_2, n_3, n_4} = -\frac{1}{(2\pi i)^3} \int_0^\infty ds s^2 \sum_{\substack{\text{poles in} \\ \text{Re } z < 0}} \text{Res } e^{sz} \mathcal{G}(z), \quad (\text{E.3})$$

where $\mathcal{G}(z)$ is given by

$$\begin{aligned}\mathcal{G}(z) = \sum_{\pm} (\pm) \Bigg\{ & -\frac{n_1}{\widehat{z}^{n_4+n_1+n_2-1} \left(\widehat{z} \mp \frac{2\pi i}{s} \right)^{n_3}} - \frac{n_3}{\widehat{z}^{n_2+n_3+n_4-1} \left(\widehat{z} \mp \frac{2\pi i}{s} \right)^{n_1}} \\ & - \frac{n_1 + n_3}{\widehat{z}^{n_1+n_2+n_3} \left(\widehat{z} \mp \frac{2\pi i}{s} \right)^{n_4-1}} - \frac{n_4 - 1}{\widehat{z}^{n_1+n_2+n_3-1} \left(\widehat{z} \mp \frac{2\pi i}{s} \right)^{n_4}} + \frac{n_4 - 1}{\widehat{z}^{n_1+n_2-1} \left(\widehat{z} \mp \frac{2\pi i}{s} \right)^{n_3+n_4}} \Bigg\}\end{aligned}$$

$$\begin{aligned}
& + \frac{n_1}{\widehat{z}^{n_1+n_2} \left(\widehat{z} \mp \frac{2\pi i}{s}\right)^{n_3+n_4-1}} + \frac{n_3}{\widehat{z}^{n_2+n_3} \left(\widehat{z} \mp \frac{2\pi i}{s}\right)^{n_4+n_1-1}} + \frac{n_4-1}{\widehat{z}^{n_2+n_3-1} \left(\widehat{z} \mp \frac{2\pi i}{s}\right)^{n_4+n_1}} \\
& - \frac{n_4-1}{\widehat{z}^{n_2-1} \left(\widehat{z} \mp \frac{2\pi i}{s}\right)^{n_3+n_4+n_1}} \Bigg\}, \tag{E.4}
\end{aligned}$$

with $\widehat{z} \equiv z + 1$. Note that each term in Eq. (E.4) is of the form $1/(\widehat{z}^P (\widehat{z} \mp 2\pi i/s)^Q)$ with $P + Q = \sum_{a=1}^4 n_a - 1 = -1$. Therefore, defining h_Q by

$$h_Q = -\frac{1}{(2\pi i)^3} \sum_{\pm} (\pm) \int_0^\infty ds s^2 \sum_{\substack{\text{poles in} \\ \text{Re } z < 0}} \text{Res} \frac{e^{sz}}{(z+1)^{-Q-1} (z+1 \mp \frac{2\pi i}{s})^Q}, \tag{E.5}$$

we see that $\mathcal{T}_{n_1, n_2, n_3, n_4}$ is given in terms of h_Q by Eq. (4.11). For h_Q (E.5), calculating the sum of residues at $z = -1$ and $-1 \pm (2\pi i/s)$ by using Eq. (C.11) and carrying out the s -integration, we obtain

$$\begin{aligned}
h_Q = -\frac{1}{2\pi i} \sum_{\pm} (\pm) \Bigg[\theta(-Q-1 \geq 1) \sum_{k=0}^{-Q-2} \binom{-Q}{k+2} \frac{(\mp 2\pi i)^k}{k!} \\
+ \theta(Q \geq 1) \sum_{k=0}^{Q-1} \binom{Q+1}{k+2} \frac{(\pm 2\pi i)^k}{k!} \Bigg]. \tag{E.6}
\end{aligned}$$

This leads to the expression of h_Q given by Eqs. (4.12) and (4.13). The series $H_Q(z)$ (4.13) is expressed by the confluent hypergeometric functions as

$$H_Q(z) = -\frac{Q(Q+1)}{2} \frac{1}{z} \sum_{\pm} (\pm) \Bigg[\theta(Q \leq -2) {}_1F_1(2+Q, 3; \pm z) - \theta(Q \geq 1) {}_1F_1(1-Q, 3; \pm z) \Bigg]. \tag{E.7}$$

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