

正準相関係数の分布の漸近展開

広大 総合科学部 杉浦成昭

§ 1. 単根の場合 $P (= P_1 + P_2)$ 次元正規分布において始めの $P_1 (\leq P_2)$ 成分に関する正準相関係数を $1 > \rho_1 > \dots > \rho_{P_1} > 0$ とし、大きさ $n+1$ の任意標本から作られる標本正準相関係数を $y_1 > \dots > y_{P_1} > 0$ とおく。さらに $P^2 = \text{diag}(\rho_1^2, \dots, \rho_{P_1}^2)$, $\Gamma = I - P^2$, $\Theta = \Gamma^{-1} - I$ とおく。標準形で考えれば $Y^2 = c h_{\alpha} S_A (S_e + S_A)^{-1}$ と表わされ、 S_e は Wishart 分布 $W_{P_1}(I, n - P_2)$ に従い、 S_A は非心 Wishart 分布 $W_P(I_{P_2}, \Omega)$ に従う。 Ω が与えられたとき S_e と S_A は独立、 Ω の分布は $W_P(\Theta, n)$ で与えられる。筆者および長尾(3)による次の結果は独立性の検定(正準相関係数の関数で与えられるもの)に対する漸近展開を行うためのものであるが、同時分布の漸近展開を求める今の場合にも役に立つ。ただし重根の場合も扱うため、 A, B が対角行列と仮定していたところを一般の対称行列 A, B で成り立つよう修正する必要が生じた。

補助定理 正値対称行列 $\Lambda_e = (\lambda_{ij}^{(e)})$, $\Lambda_h = (\lambda_{ij}^{(h)})$ に関して

Taylor 展開可能な関数 $f(\Lambda_e, \Lambda_h)$ について

$$(1) \quad \partial_e = \left(\frac{1}{2} (1 + \delta_{ij}) \frac{\partial}{\partial \lambda_{ij}^{(e)}} \right), \quad \partial_h = \left(\frac{1}{2} (1 + \delta_{ij}) \frac{\partial}{\partial \lambda_{ij}^{(h)}} \right)$$

とすれば, 任意の対称行列 A, B ($p \times p$) について

$$(2) \quad E \left[\exp i t \left\{ A \sqrt{m} \left(\frac{S_e}{m} - I \right) + B \sqrt{m} \left(\frac{S_h}{m} - \Theta \right) \right\} \cdot f \left(\frac{S_e}{m}, \frac{S_h}{m} \right) \right] = \exp t \left\{ A^2 + (\Gamma^{-1} B)^2 - B^2 \right\} \left\{ 1 + \frac{1}{\sqrt{m}} \sum_1^2 d_{2\alpha-1} (it)^{2\alpha-1} + \frac{1}{m} \sum_0^2 g_{2\alpha} (it)^{2\alpha} \right\} f(\Lambda_e, \Lambda_h) \Big|_{\Lambda_e=I, \Lambda_h=\Theta} + O\left(\frac{1}{m^{3/2}}\right)$$

ただし $m = n - 2\Delta$ (Δ は補正項)

$$d_1 = 2 \operatorname{tr} \left\{ A \partial_e + \Gamma^{-1} B \Gamma^{-1} \partial_h - B \partial_h + \left(\Delta - \frac{p}{2} \right) (A - B) + \Delta \Gamma^{-1} B \right\}$$

$$d_3 = \frac{4}{3} \operatorname{tr} \left\{ A^3 + (\Gamma^{-1} B)^3 - B^3 \right\}$$

$$g_0 = \operatorname{tr} \left\{ \partial_e^2 - \partial_h^2 + (\Gamma^{-1} \partial_h)^2 + (2\Delta - p)(\partial_e - \partial_h) + 2\Delta \Gamma^{-1} \partial_h \right\}$$

$$(3) \quad g_2 = \operatorname{tr} \left\{ 4A^2 \partial_e + 4(\Gamma^{-1} B)^2 \Gamma^{-1} \partial_h - 4B^2 \partial_h + (2\Delta - p)(A^2 - B^2) + 2\Delta(\Gamma^{-1} B)^2 + \frac{1}{2} d_1^2 \right\}$$

$$g_4 = 2 \operatorname{tr} \left\{ A^4 + (\Gamma^{-1} B)^4 - B^4 \right\} + d_1 d_3$$

$$g_6 = \frac{1}{2} d_3^2$$

これにより $\frac{1}{m} S_e$ は I に $\frac{1}{m} S_h$ は Θ に確率収束する: とわかる

から筆者 (4), (5) を用いて, 如く攝動法により $\frac{1}{m} S_e$

$= I$, $\frac{S_h}{m} = \Theta$ の周りで γ_α^2 の Taylor 展開を求めれば, $\frac{1}{m} S_e$

$-I = (b_{ij})$, $\frac{1}{m} S_h - \Theta = (a_{ij})$ とおいて

$$(4) \quad \gamma_\alpha^2 = \rho_\alpha^2 + \frac{a_{\alpha\alpha} - \partial_\alpha b_{\alpha\alpha}}{(1 + \partial_\alpha)^2} - \frac{(a_{\alpha\alpha} - \partial_\alpha b_{\alpha\alpha})(a_{\alpha\alpha} + b_{\alpha\alpha})}{(1 + \partial_\alpha)^3} + \frac{1}{(1 + \partial_\alpha)^2} \sum_{j \neq \alpha} \frac{(a_{\alpha j} - \partial_\alpha b_{\alpha j})^2}{\partial_\alpha + \partial_j} \\ + \frac{(a_{\alpha\alpha} - \partial_\alpha b_{\alpha\alpha})(a_{\alpha\alpha} + b_{\alpha\alpha})^2}{(1 + \partial_\alpha)^4} - \frac{a_{\alpha\alpha} + b_{\alpha\alpha}}{(1 + \partial_\alpha)^3} \sum_{j \neq \alpha} \frac{(a_{\alpha j} - \partial_\alpha b_{\alpha j})^2}{\partial_\alpha - \partial_j} - \frac{(a_{\alpha\alpha} - \partial_\alpha b_{\alpha\alpha})}{(1 + \partial_\alpha)^3} \\ \sum_{j \neq \alpha} \frac{(a_{\alpha j} + b_{\alpha j})(a_{j\alpha} - \partial_\alpha b_{j\alpha})}{\partial_\alpha - \partial_j} - \frac{a_{\alpha\alpha} - \partial_\alpha b_{\alpha\alpha}}{(1 + \partial_\alpha)^2} \sum_{j \neq \alpha} \frac{(a_{\alpha j} - \partial_\alpha b_{\alpha j})(a_{j\alpha} - \partial_j b_{j\alpha})}{(\partial_\alpha - \partial_j)^2} \\ + \frac{1}{(1 + \partial_\alpha)^2} \sum_{\substack{j \neq \alpha \\ k \neq \alpha}} \frac{(a_{\alpha j} - \partial_\alpha b_{\alpha j})(a_{j\alpha} - \partial_\alpha b_{j\alpha})(a_{\alpha k} - \partial_\alpha b_{\alpha k})}{(\partial_\alpha - \partial_j)(\partial_\alpha - \partial_k)} + (4) \text{ 次の項}$$

これより正準相関係数をえ乘 (14) の $\gamma_1^2, \dots, \gamma_p^2$ の同時分布の特性関数が次のように展開される.

$$(5) \quad E\left\{\exp\left\{i\sqrt{m}\sum_1^p t_\alpha (\gamma_\alpha^2 - \rho_\alpha^2)\right\}\right\} = E\left\{\exp\left\{i\sqrt{m}\sum_1^p t_\alpha \frac{(a_{\alpha\alpha} - \partial_\alpha b_{\alpha\alpha})}{(1 + \partial_\alpha)^2}\right\}\right. \\ \left. \cdot \left\{1 + \frac{K_1}{\sqrt{m}} + \frac{1}{m}\left(K_2 + \frac{K_1^2}{2}\right) + O\left(\frac{1}{m\sqrt{m}}\right)\right\}\right\} \\ K_1 = i\sqrt{m}\sum_\alpha \frac{t_\alpha}{(1 + \partial_\alpha)^2} \left\{ - \frac{(a_{\alpha\alpha} - \partial_\alpha b_{\alpha\alpha})(a_{\alpha\alpha} + b_{\alpha\alpha})}{(1 + \partial_\alpha)} + \sum_{j \neq \alpha} \frac{(a_{\alpha j} - \partial_\alpha b_{\alpha j})^2}{\partial_\alpha - \partial_j} \right\} \\ (6) \quad K_2 = i\sqrt{m}\sum_\alpha \frac{t_\alpha}{(1 + \partial_\alpha)^2} \left\{ \frac{(a_{\alpha\alpha} - \partial_\alpha b_{\alpha\alpha})(a_{\alpha\alpha} + b_{\alpha\alpha})^2}{(1 + \partial_\alpha)^2} - \frac{a_{\alpha\alpha} + b_{\alpha\alpha}}{1 + \partial_\alpha} \sum_{j \neq \alpha} \frac{(a_{\alpha j} - \partial_\alpha b_{\alpha j})^2}{\partial_\alpha - \partial_j} \right. \\ \left. - \frac{a_{\alpha\alpha} - \partial_\alpha b_{\alpha\alpha}}{1 + \partial_\alpha} \sum_{j \neq \alpha} \frac{(a_{\alpha j} + b_{\alpha j})(a_{\alpha j} - \partial_\alpha b_{\alpha j})}{\partial_\alpha - \partial_j} - (a_{\alpha\alpha} - \partial_\alpha b_{\alpha\alpha}) \sum_{j \neq \alpha} \frac{(a_{\alpha j} - \partial_\alpha b_{\alpha j})(a_{\alpha j} - \partial_j b_{\alpha j})}{(\partial_\alpha - \partial_j)^2} \right. \\ \left. + \sum_{\substack{j \neq \alpha \\ k \neq \alpha}} \frac{(a_{\alpha j} - \partial_\alpha b_{\alpha j})(a_{j k} - \partial_\alpha b_{j k})(a_{k\alpha} - \partial_\alpha b_{k\alpha})}{(\partial_\alpha - \partial_j)(\partial_j - \partial_k)} \right\}$$

各項の平均は補助定理を使つて計算できる. 簡単のため $\psi = i\sqrt{m}\sum_1^p t_\alpha (a_{\alpha\alpha} - \partial_\alpha b_{\alpha\alpha}) / (1 + \partial_\alpha)^2$, $\varphi(t) = \exp\{-2\sum_1^p \rho_\alpha^2 (1 - \rho_\alpha^2)^2 t_\alpha^2\}$ とおけば

$$(7) \quad E\{\exp \psi\} = \varphi(t) \left[1 + \frac{1}{\sqrt{m}} (id_1 + i^3 d_3) + \frac{1}{m} \left\{ i^2 \left(\frac{1}{2} d_1^2 + \sum_\alpha \{ (4\Delta - 2\rho_\alpha^2) \rho_\alpha^2 + \rho_\alpha^2 \} \right) \right. \right. \\ \left. \left. \cdot (1 - \rho_\alpha^2)^2 t_\alpha^2 \right\} \right. \\ \left. + i^4 \left(d_1 d_3 + 4 \sum_\alpha \rho_\alpha^2 (1 - \rho_\alpha^2)^4 (2 - 3\rho_\alpha^2 + 2\rho_\alpha^4) t_\alpha^4 \right) + i^6 \frac{d_3^2}{2} \right] + O\left(\frac{1}{m\sqrt{m}}\right)$$

ただし $d_1 = 2\sum_1^p \rho_\alpha^2 (1 - \rho_\alpha^2) t_\alpha$, $d_3 = 4\sum_1^p \rho_\alpha^2 (1 - \rho_\alpha^2)^4 t_\alpha^3$ とする.

$$E\{K_1 \exp \psi\} = \varphi(t) \left\{ i \left\{ -2 \sum_\alpha \rho_\alpha^2 (1 - \rho_\alpha^2) t_\alpha + \sum_{j \neq \alpha} \frac{1 - \rho_\alpha^2}{\rho_\alpha^2 - \rho_j^2} (\rho_\alpha^2 + \rho_j^2 - 2\rho_\alpha^2 \rho_j^2) t_\alpha \right\} \right. \\ \left. - 8i^3 \sum_\alpha \rho_\alpha^4 (1 - \rho_\alpha^2)^3 t_\alpha^3 \right\} + \frac{1}{\sqrt{m}} \varphi(t) \left\{ i^2 \left\{ -8 \sum_\alpha \frac{\partial_\alpha (\partial_\alpha + 3)}{(1 + \partial_\alpha)^4} t_\alpha^2 \right. \right. \\ \left. \left. + 2 \sum_{\alpha \neq j} \frac{t_\alpha}{(1 + \partial_\alpha)^2 (\partial_\alpha - \partial_j)} \right\} - \frac{1 + \partial_\alpha^3}{(1 + \partial_\alpha)^2} t_\alpha - \frac{1 + \partial_j \partial_\alpha^2}{(1 + \partial_j)^2} t_j + t_\alpha (1 + \partial_j) + t_j (1 + \partial_\alpha) \right\} \\ (8) \quad + \rho_\alpha^2 \left(\sum_\alpha \frac{t_\alpha}{1 + \partial_\alpha} \right) \left\{ -2 \sum_\alpha \rho_\alpha^2 (1 - \rho_\alpha^2) t_\alpha + \sum_{\alpha \neq j} \frac{1 - \rho_\alpha^2}{\rho_\alpha^2 - \rho_j^2} (\rho_\alpha^2 + \rho_j^2 - 2\rho_\alpha^2 \rho_j^2) t_\alpha \right\} \\ - 2(4\Delta + \rho_2) \sum_\alpha \frac{\partial_\alpha t_\alpha^2}{(1 + \partial_\alpha)^3} \left\{ + i^4 \left\{ -8 \sum_\alpha \frac{\partial_\alpha^2 (2\partial_\alpha + 9)}{(1 + \partial_\alpha)^5} t_\alpha^4 \right. \right. \\ \left. \left. + 4 \sum_\alpha \frac{\partial_\alpha t_\alpha^3}{(1 + \partial_\alpha)^5} \right\} - 2 \sum_\alpha \rho_\alpha^2 (1 - \rho_\alpha^2) t_\alpha + \sum_{j \neq \alpha} \frac{1 - \rho_\alpha^2}{\rho_\alpha^2 - \rho_j^2} (\rho_\alpha^2 + \rho_j^2 - 2\rho_\alpha^2 \rho_j^2) t_\alpha \right\}$$

$$\begin{aligned}
& -8\rho_\alpha^2 \sum_{j \neq \alpha} (1-\rho_j^2) t_\alpha \cdot \sum_{\alpha} \rho_\alpha^4 (1-\rho_\alpha^2)^3 t_\alpha^3 \} - 32\lambda^6 \sum_{\alpha} \rho_\alpha^2 (1-\rho_\alpha^2)^4 t_\alpha^3 \cdot \sum_{\alpha} \rho_\alpha^4 (1-\rho_\alpha^2)^3 t_\alpha^3 \} + O(m^{-1}) \\
& E[K_2 \exp S] = \varphi(t) \left[\lambda^2 \left\{ 8 \sum_{\alpha} \frac{\partial_\alpha (2\partial_\alpha + 1)}{(1+\partial_\alpha)^4} t_\alpha^2 - 2 \sum_{j \neq \alpha} \frac{\partial_\alpha (\partial_\alpha + 3\partial_j)}{(1+\partial_\alpha)^3 (\partial_\alpha - \partial_j)} \frac{t_\alpha^2}{\partial_\alpha} \right. \right. \\
(9) \quad & \left. - 4 \sum_{j \neq \alpha} \frac{\partial_\alpha (\partial_\alpha + \partial_j + 2\partial_\alpha \partial_j)}{(1+\partial_\alpha)^3 (\partial_\alpha - \partial_j)^2} t_\alpha^2 + 2 \sum_{j \neq \alpha} \frac{\partial_j (\partial_\alpha + \partial_j) (\partial_\alpha + \partial_j + 2)}{(1+\partial_\alpha) (1+\partial_j)^2 (\partial_\alpha - \partial_j)^2} t_\alpha t_j \right. \\
& \left. + \lambda^4 \cdot 16 \sum_{\alpha} \frac{\partial_\alpha^3 t_\alpha^4}{(1+\partial_\alpha)^2} \right] \\
& E\left[\frac{K_1^2}{2} \exp S\right] = \varphi(t) \left[\lambda^2 \left\{ 2 \sum_{\alpha} \frac{\partial_\alpha (3\partial_\alpha + 2)}{(1+\partial_\alpha)^4} t_\alpha^2 + 2 \left\{ \sum_{\alpha} \frac{\partial_\alpha t_\alpha}{1+\partial_\alpha} - \frac{1}{2} \sum_{\alpha \neq j} \frac{(\partial_\alpha + \partial_j) t_\alpha}{(1+\partial_\alpha) (\partial_\alpha - \partial_j)} \right\} \right. \right. \\
& \left. + \frac{1}{2} \sum_{\alpha \neq \beta} \frac{1}{(\partial_\alpha - \partial_\beta)^2} \left\{ \frac{\partial_\alpha^2 t_\alpha}{(1+\partial_\alpha)^2} - \frac{\partial_\beta^2 t_\beta}{(1+\partial_\beta)^2} \right\} + \sum_{\alpha \neq \beta} \frac{\partial_\alpha + \partial_\beta + \partial_\alpha \partial_\beta}{(\partial_\alpha - \partial_\beta)^2} \left\{ \frac{\partial_\alpha t_\alpha}{(1+\partial_\alpha)^2} - \frac{\partial_\beta t_\beta}{(1+\partial_\beta)^2} \right\} \right. \\
(10) \quad & \left. + \frac{1}{2} \sum_{\alpha \neq \beta} \frac{(\partial_\alpha + \partial_\beta + \partial_\alpha \partial_\beta)^2}{(\partial_\alpha - \partial_\beta)^2} \left\{ \frac{t_\alpha}{(1+\partial_\alpha)^2} - \frac{t_\beta}{(1+\partial_\beta)^2} \right\}^2 \right\} \\
& + \lambda^4 \left\{ 8 \sum_{\alpha} \frac{\partial_\alpha^2 (5\partial_\alpha + 2)}{(1+\partial_\alpha)^2} t_\alpha^4 + 16 \sum_{\alpha} \frac{\partial_\alpha t_\alpha}{(1+\partial_\alpha)^2} \cdot \sum_{\alpha} \frac{\partial_\alpha^2 t_\alpha^3}{(1+\partial_\alpha)^5} \right. \\
& \left. - 8 \sum_{\alpha} \frac{\partial_\alpha^2 t_\alpha^3}{(1+\partial_\alpha)^5} \cdot \sum_{j \neq \alpha} \frac{(\partial_\alpha + \partial_j) t_\alpha}{(1+\partial_\alpha) (\partial_\alpha - \partial_j)} \right\} + \lambda^6 \cdot 32 \left(\sum_{\alpha} \frac{\partial_\alpha^2 t_\alpha^3}{(1+\partial_\alpha)^5} \right)^2 \}
\end{aligned}$$

(7) ~ (10) を加えて $(\gamma_1^2 - \rho_1^2, \dots, \gamma_n^2 - \rho_n^2) \sqrt{n}$ の特性関数加次のように評価される。

$$\begin{aligned}
& \exp \left\{ -2 \sum_{\alpha} \rho_\alpha^2 (1-\rho_\alpha^2)^3 t_\alpha^2 \right\} \cdot \left\{ 1 + \frac{1}{\sqrt{n}} (i d_1 + i^3 d_3) + \frac{1}{n} (i^2 g_2 + i^4 g_4 + i^6 g_6) + O(m^{-3/2}) \right\} \\
& d_1 = \rho_2 \sum_{\alpha} t_\alpha (1-\rho_\alpha^2) - 2 \sum_{\alpha} t_\alpha (1-\rho_\alpha^2) \rho_\alpha^2 + \sum_{j \neq \alpha} \frac{1-\rho_\alpha^2}{\rho_\alpha^2 - \rho_j^2} (\rho_\alpha^2 + \rho_j^2 - 2\rho_\alpha^2 \rho_j^2) t_\alpha \\
& d_3 = 4 \sum_{\alpha} (1-\rho_\alpha^2) \rho_\alpha^2 (1-3\rho_\alpha^2) t_\alpha^3 \\
(11) \quad & g_2 = \frac{1}{2} d_1^2 + \sum_{\alpha} \left\{ -4(\Delta + \rho_2) \rho_\alpha^2 + \rho_2 \right\} (1-\rho_\alpha^2)^2 t_\alpha^2 + 2 \sum_{\alpha} \rho_\alpha^2 (1-\rho_\alpha^2) (1/\rho_\alpha^2 - 6) t_\alpha^2 \\
& + \sum_{j \neq \alpha} \frac{(\partial_\alpha^2 + \partial_\beta^2 - 6\partial_\alpha \partial_\beta) t_\alpha^2}{(1+\partial_\alpha)^2 (\partial_\alpha - \partial_\beta)^2} - \sum_{\alpha \neq \beta} \frac{(\partial_\alpha + \partial_\beta + 2\partial_\alpha \partial_\beta) (\partial_\alpha^2 + \partial_\beta^2 + \partial_\alpha \partial_\beta)}{(1+\partial_\alpha)^2 (1+\partial_\beta)^2 (\partial_\alpha - \partial_\beta)^2} t_\alpha t_\beta \\
& + \sum_{\alpha \neq \beta} \frac{\{-4\partial_\alpha (\partial_\alpha + \partial_\beta) + 2\partial_\beta\} t_\alpha^2}{(1+\partial_\alpha)^2 (\partial_\alpha - \partial_\beta)} - \sum_{\alpha \neq \beta} \frac{\partial_\alpha \partial_\beta t_\alpha t_\beta}{(1+\partial_\alpha)^2 (1+\partial_\beta)^2} - \sum_{\alpha \neq \beta} \frac{t_\alpha t_\beta}{(1+\partial_\alpha) (1+\partial_\beta)} \\
& g_4 = d_1 d_3 + 4 \sum_{\alpha} \rho_\alpha^2 (1-\rho_\alpha^2)^4 (1-2\rho_\alpha^2) (2-1/\rho_\alpha^2) t_\alpha^4 \\
& g_6 = \frac{1}{2} d_3^2
\end{aligned}$$

これを反転し 2 次の漸近展開を得る。

定理 1. 至 (x) を標準正規分布の分布関数 $\varphi(x) = \frac{\Phi^{(1)}(x)}{\sigma \sqrt{2\pi}}$,

$\zeta_\alpha^2 = 4\rho_\alpha^2(1-\rho_\alpha^2)$ とおく. $1 > \rho_1 > \dots > \rho_n > 0$ のとき

$$P\left(\frac{P_1}{\sqrt{m}} \sqrt{m}(Y_\alpha^2 - \rho_\alpha^2) / \zeta_\alpha < x_\alpha\right) = \left\{ \prod_{j=1}^n \Phi(x_{\alpha j}) \right\} \left[1 - \frac{1}{\sqrt{m}}(d_1 + d_3) + \frac{1}{m}(g_2 + g_4 + g_6) + O\left(\frac{1}{m\sqrt{m}}\right) \right]$$

$$d_1 = P_2 \sum_{j \neq \alpha} (1-\rho_\alpha^2) \chi_j(x_\alpha) - 2 \sum_{j \neq \alpha} \rho_\alpha^2 (1-\rho_\alpha^2) \chi_j(x_\alpha) + \sum_{j \neq \alpha} \frac{1-\rho_\alpha^2}{\rho_\alpha^2 - \rho_j^2} (\rho_\alpha^2 + \rho_j^2 - 2\rho_\alpha^2 \rho_j^2) \chi_j(x_\alpha)$$

$$d_3 = 4 \sum_{j \neq \alpha} (1-\rho_\alpha^2)^3 \rho_\alpha^2 (1-3\rho_\alpha^2) \chi_j(x_\alpha)$$

$$g_2 = \tilde{g}_2 + \frac{1}{2} \sum_{j \neq \alpha} (P_2 - 2\rho_\alpha^2)^2 (1-\rho_\alpha^2)^2 (\chi_2(x_\alpha) - \chi_1(x_\alpha)^2)$$

$$(12) \quad + \sum_{j \neq \alpha} \frac{(P_2 - 2\rho_\alpha^2)(1-\rho_\alpha^2)^2 (\rho_\alpha^2 + \rho_j^2 - 2\rho_\alpha^2 \rho_j^2)}{\rho_\alpha^2 - \rho_j^2} (\chi_2(x_\alpha) - \chi_1(x_\alpha)^2) \\ + \frac{1}{2} \sum_{\substack{j \neq \alpha \\ j \neq k}} \frac{(1-\rho_\alpha^2)^2 (\rho_\alpha^2 + \rho_j^2 - 2\rho_\alpha^2 \rho_j^2) (\rho_\alpha^2 + \rho_k^2 - 2\rho_\alpha^2 \rho_k^2)}{(\rho_\alpha^2 - \rho_j^2)(\rho_\alpha^2 - \rho_k^2)} (\chi_2(x_\alpha) - \chi_1(x_\alpha)^2)$$

$$g_4 = \tilde{g}_4 + 4 \sum_{j \neq \alpha} (1-\rho_\alpha^2)^4 (P_2 - 2\rho_\alpha^2) \rho_\alpha^2 (1-3\rho_\alpha^2) (\chi_4(x_\alpha) - \chi_1(x_\alpha) \chi_3(x_\alpha)) \\ + 4 \sum_{j \neq \alpha} \frac{(1-\rho_\alpha^2)^4 \rho_\alpha^2 (1-3\rho_\alpha^2) (\rho_\alpha^2 + \rho_j^2 - 2\rho_\alpha^2 \rho_j^2)}{\rho_\alpha^2 - \rho_j^2} (\chi_4(x_\alpha) - \chi_1(x_\alpha) \chi_3(x_\alpha))$$

$$g_6 = \tilde{g}_6 + 8 \sum_{j \neq \alpha} (1-\rho_\alpha^2)^6 \rho_\alpha^4 (1-3\rho_\alpha^2)^2 (\chi_6(x_\alpha) - \chi_3(x_\alpha)^2)$$

ただし $\tilde{g}_2, \tilde{g}_4, \tilde{g}_6$ は (11) 式の g_2, g_4, g_6 において t_α^j のところ

を $\chi_j(x_\alpha)$ で置きかえたものとする.

系 ρ_α が単根のとき ($1 > \rho_\alpha > 0$)

$$(13) \quad P(\sqrt{m}(Y_\alpha^2 - \rho_\alpha^2) / \zeta_\alpha < x) = \Phi(x) - \frac{1}{\sqrt{m}} \left(\frac{d_1}{\zeta_\alpha} \Phi^{(1)}(x) + \frac{d_3}{\zeta_\alpha^3} \Phi^{(3)}(x) \right) \\ + \frac{1}{m} \left(\frac{g_2}{\zeta_\alpha^2} \Phi^{(2)}(x) + \frac{g_4}{\zeta_\alpha^4} \Phi^{(4)}(x) + \frac{g_6}{\zeta_\alpha^6} \Phi^{(6)}(x) \right) + O\left(\frac{1}{m\sqrt{m}}\right)$$

ただし

$$(14) \quad d_1 = (P_2 - 2\rho_\alpha^2)(1-\rho_\alpha^2) + \sum_{j \neq \alpha} \frac{1-\rho_\alpha^2}{\rho_\alpha^2 - \rho_j^2} (\rho_\alpha^2 + \rho_j^2 - 2\rho_\alpha^2 \rho_j^2) \\ d_3 = 4(1-\rho_\alpha^2)^3 (1-3\rho_\alpha^2) \rho_\alpha^2 \\ g_2 = \frac{1}{2} d_1^2 + (1-\rho_\alpha^2)^2 \{ P_2 + 2\rho_\alpha^2 (13\rho_\alpha^2 - 6 - 2\Delta - 2P_2) \} \\ + \sum_{j \neq \alpha} \frac{(1-\rho_\alpha^2)^2}{(\rho_\alpha^2 - \rho_j^2)^2} \{ (\rho_\alpha^2 + \rho_j^2 - 2\rho_\alpha^2 \rho_j^2)^2 - 8\rho_\alpha^2 \rho_j^2 (1-\rho_\alpha^2)(1-\rho_j^2) \}$$

$$+ \sum_{j \neq d} \frac{(1-\rho_d^2)^3}{\rho_d^2 - \rho_j^2} \left\{ -4\rho_d^2(\rho_d^2 + \rho_j^2 - 2\rho_d^2\rho_j^2) + 2\rho_j^2(1-\rho_d^2)^2 \right\}$$

$$g_4 = d \cdot d_3 + 4\rho_d^2(1-\rho_d^2)^2(1-2\rho_d^2)(2-\rho_d^2)$$

$$g_6 = \frac{1}{2} d_3^2$$

こゝで特に $x = 0$ とすれば

$$P(\gamma_d^2 < \rho_d^2) = \frac{1}{2} - \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{m}} \frac{1}{2\rho_d} \left(\rho_d - 1 + \rho_d^2 \sum_{j \neq d} \frac{\rho_d^2 + \rho_j^2 - 2\rho_d^2\rho_j^2}{\rho_d^2 - \rho_j^2} \right) + O(m^{-3/2})$$

となり最大正準相関係数について $P(\gamma_d^2 < \rho_d^2) < \frac{1}{2} + O(m^{-3/2})$

となることがわかる。

§ 2. 重根の場合 母正準相関係数に重根がある場合, その重複度を

$$(15) \quad 1 > \rho_1, \dots, \rho_1 > \rho_2, \dots, \rho_2 > \dots > \rho_\pi, \dots, \rho_\pi > 0$$

k_1 個 k_2 個 k_π 個

これに通し番号をつけて $g_0 = 1, g_1 = k_1 + 1, g_2 = k_1 + k_2 + 1, \dots, g_\pi = \rho_1 + 1$

とおく。Lawley [2] 又は藤越 [1] により攝動法を用いて

(4) に対応する展開を求めれば

$$(16) \quad W_d = \rho_d^2 I + \frac{1}{\sqrt{m}} \left\{ (1-\rho_d^2)^2 A_{dd} - \rho_d^2 (1-\rho_d^2) B_{dd} \right\} \\ + \frac{1}{m} \left[\sum_{j \neq d} \frac{(1-\rho_d^2)(1-\rho_j^2)}{\rho_d^2 - \rho_j^2} \left\{ (1-\rho_d^2) A_{dj} - \rho_d^2 B_{dj} \right\} \left\{ (1-\rho_j^2) A_{jd} - \rho_j^2 B_{jd} \right\} \right. \\ \left. + \sum_j (1-\rho_j^2)(1-\rho_d^2) \left\{ -(1-\rho_j^2) A_{dj} + \rho_d^2 B_{dj} \right\} (A_{jd} + B_{jd}) \right] + O(m^{-3/2})$$

となり $\gamma_1^2, \dots, \gamma_{\rho_1}^2$ の分布は $\text{diag}(W_1, \dots, W_n)$ の固有根の分布と一致する。こゝで B_{dj} は ρ_d の重複度 $(k_1, \dots, k_\pi)$ に応じて分割したとき第 (α, β) 番目に展われる小行列から作られるも

の, すなわち $B_{\alpha j} = \sqrt{m} [S_e/m - I]_{\alpha j}$. 同様に $A_{\alpha j} = \sqrt{m} [S_h/m - \Theta]_{\alpha j}$,
 $\Theta = \text{diag}(\rho_1^2/(1-\rho_1^2), \dots, \rho_1^2/(1-\rho_1^2), \dots, \rho_n^2/(1-\rho_n^2), \dots, \rho_n^2/(1-\rho_n^2))$ と
 $A_{\alpha j}$, $B_{\alpha j}$ は共に $k_\alpha \times k_j$ の行列である. これより補助定理
を用いて $\sqrt{m}(W_\alpha - \rho_\alpha^2 I)_{1 \leq \alpha \leq n}$ の特性関数を評価すれば, $\tau_\alpha = 2\rho_\alpha(1-\rho_\alpha^2)$
と 12

$$(17) \quad \text{etr}(-\frac{1}{2} \sum_{\alpha} \tau_\alpha^2 T_\alpha^2) \cdot \left[1 + \frac{1}{\sqrt{m}} \left\{ i \sum_{\alpha} (1-\rho_\alpha^2) (\rho_\alpha - \rho_\alpha^2 - k_\alpha \rho_\alpha^2) \text{tr} T_\alpha \right. \right. \\
\left. \left. + i \sum_{j \neq \alpha} \frac{(1-\rho_j^2)}{\rho_j^2 - \rho_\alpha^2} (\rho_\alpha^2 + \rho_j^2 - 2\rho_\alpha^2 \rho_j^2) k_j \text{tr} T_\alpha + 4i \sum_{\alpha} \rho_\alpha^2 (1-\rho_\alpha^2)^3 (1-3\rho_\alpha^2) \text{tr} T_\alpha^3 \right\} + o(m^{-1}) \right]$$

藤越 (1) に従ってこれを反転するには $\varphi(W_\alpha) = \text{etr}(i T_\alpha W_\alpha - \frac{1}{2} W_\alpha^2)$
とある正規分布に関する積分

$$(18) \quad \frac{1}{2^{\frac{1}{2}k_\alpha} \pi^{k_\alpha(k_\alpha+1)/4}} \int \varphi(W_\alpha) dW_\alpha = \text{etr}(-\frac{1}{2} T_\alpha^2)$$

$$\quad \quad \quad \int \text{tr} W_\alpha \cdot \varphi(W_\alpha) dW_\alpha = i \text{tr} T_\alpha \text{etr}(-\frac{1}{2} T_\alpha^2)$$

$$\quad \quad \quad \int \left\{ \text{tr} W_\alpha^3 - \frac{3}{2} (k_\alpha+1) \text{tr} W_\alpha \right\} \varphi(W_\alpha) dT_\alpha = i^3 \text{tr} T_\alpha^3 \text{etr}(-\frac{1}{2} T_\alpha^2)$$

に注意すればよい. 藤越 (1) には更にくわしい公式が述べ
られている. これより $V_\alpha = \sqrt{m}(W_\alpha - \rho_\alpha^2 I)/\tau_\alpha$ $1 \leq \alpha \leq n$ の同時確率
密度関数は次のように展開される.

$$(19) \quad \prod_{\alpha=1}^n \frac{1}{2^{k_\alpha/2} \pi^{k_\alpha(k_\alpha+1)/4}} \text{etr}(-\frac{1}{2} V_\alpha^2) \cdot \left[1 + \frac{1}{\sqrt{m}} \left\{ \sum_{\alpha} (1-\rho_\alpha^2) (\rho_\alpha - \rho_\alpha^2 - k_\alpha \rho_\alpha^2) \frac{\text{tr} V_\alpha}{\tau_\alpha} \right. \right. \\
\left. \left. + \sum_{j \neq \alpha} \frac{(1-\rho_j^2)}{\rho_j^2 - \rho_\alpha^2} (\rho_\alpha^2 + \rho_j^2 - 2\rho_\alpha^2 \rho_j^2) k_j \frac{\text{tr} V_\alpha}{\tau_\alpha} + 4 \sum_{\alpha} \rho_\alpha^2 (1-\rho_\alpha^2)^3 (1-3\rho_\alpha^2) \frac{1}{\tau_\alpha^3} \right. \right. \\
\left. \left. \cdot (\text{tr} V_\alpha^3 - \frac{3}{2} (k_\alpha+1) \text{tr} V_\alpha) \right\} + o(m^{-1}) \right]$$

$\text{diag}(V_1, \dots, V_n)$ の固有根の分布を求めればそれが $t_j = \sqrt{m}(\rho_j^2 - \rho_\alpha^2)/\tau_\alpha$, $j = \rho_{\alpha-1}, \rho_{\alpha-1}+1, \dots, \rho_\alpha-1$, $\alpha = 1, 2, \dots, \alpha$ の同時分布を与え
ることになる. それには V_α をその固有根と直交行列とに分

解する変換 $V_\alpha = H_\alpha D_\alpha H'_\alpha$ を行い, その Jacobian 是 $\prod_{\alpha=1}^n \{ \pi^{k_\alpha/2} / \Gamma_{k_\alpha}(k_\alpha/2) \} \prod_{\alpha=1}^n \prod_{g_{\alpha-1} \leq i < j < g_\alpha} (\pi_i - \pi_j) d\pi_1 \cdots d\pi_{p_i} dH_1 \cdots dH_{p_i}$ となることおよび (19) の H_α を含まぬことから直交行列上の Haar 測度 $dH_1 \cdots dH_{p_i}$ で積分すればよい. ただし $g_{\alpha-1} \leq i < j < g_\alpha$ のとき $\pi_i > \pi_j$ である. これより正準相関係数の確率密度について次の展開を得る.

定理2 母正準相関係数 ρ_α が重複度 (15) を持つとする

. 標本正準相関係数 $\gamma_1, \dots, \gamma_{p_i}$ について $t_j = (\gamma_j^2 - \rho_\alpha^2) \sqrt{m} / c_\alpha$

$c_\alpha = 2\rho_\alpha(1-\rho_\alpha^2)$ $j = g_{\alpha-1}, g_{\alpha-1}+1, \dots, g_\alpha-1$ とおく.

$t_1, t_2, \dots, t_{p_i}$ の確率密度関数は次の展開をもつ

$$\prod_{\alpha=1}^n \frac{\pi^{k_\alpha(k_\alpha-1)/4}}{2^{k_\alpha/2} \Gamma_{k_\alpha}(\frac{1}{2}k_\alpha)} \cdot \exp\left(-\frac{1}{2} \sum_{j=1}^{p_i} t_j^2\right) \cdot \prod_{\alpha=1}^n \prod_{g_{\alpha-1} \leq i < j < g_\alpha} (\pi_i - \pi_j) \\ \cdot \left[1 + \frac{1}{\sqrt{m}} \left\{ \sum_{\alpha=1}^n \frac{\rho_\alpha^2 - \rho_\alpha^2 - k_\alpha \rho_\alpha^2}{2\rho_\alpha} \sum_{\ell=g_{\alpha-1}}^{g_\alpha-1} \pi_\ell + \sum_{\alpha \neq \beta} \frac{\rho_\alpha^2 + \rho_\beta^2 - 2\rho_\alpha^2 \rho_\beta^2}{2\rho_\alpha(\rho_\alpha^2 - \rho_\beta^2)} k_\beta \sum_{\ell=g_{\alpha-1}}^{g_\alpha-1} \pi_\ell \right. \right. \\ \left. \left. + \frac{1}{2} \sum_{\alpha=1}^n \frac{1 - \rho_\alpha^2}{\rho_\alpha} \sum_{\ell=g_{\alpha-1}}^{g_\alpha-1} \left\{ \pi_\ell^2 - \frac{\rho_\alpha^2}{2} (k_\alpha + 1) \pi_\ell \right\} \right\} + O\left(\frac{1}{m}\right) \right]$$

特に $k_1 = \dots = k_n = 1$ とすれば単根の場合となり分布関数に直せば定理1の $O(m^{-1/2})$ の項迄と一致する. 重根のある場合分布関数の形で展開を陽に書くことがむづかしいことかわかる.

引用文献

(1) 藤越康徳 (1975). Wishart 行列および多変量 F 行列の固有根の漸近展開—重根がある場合. 「多次元データの

情報解析」研究集会報告。(元大理)

(2) Lawley, D. N. (1956). Tests for significance for the latent roots of covariance and correlation matrices. *Biometrika* 43 125-136

(3) Sugiura, N. and Nagao, H. (1973). Asymptotic formulas for the distributions of two trace statistics for testing the independence under fixed alternative. *Comm. Statist.* 1 285-295.

(4) 杉浦成昭 (1974). Wishart 行列の固有根の漸近分布. 数学会特別講演.

(5) 杉浦成昭 (1974). Wishart 行列の固有ベクトルの分布の漸近展開. 京大数解研「多変量統計解析」研究集会報告.