

Remarks on the paper “On the infinite dimensionality of the middle L^2 cohomology of complex domains”

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1. Preliminaries

In this note, we will give an outline of the proof of Theorem in [Oh] and remark on the paper. Then we will describe conditions and some properties for infinite dimensionality of the middle L^2 cohomology.

Let $D \subset \mathbf{C}^n$ be a domain with the smooth boundary in the n -dimensional complex Euclidean space.

Then ∂D denotes the boundary of D and $\bar{D}$ denotes the topological closure of D .

Let $\varphi: \mathbf{C}^n \rightarrow \mathbf{R}$ be a defining function of D satisfying $D = \{z \in \mathbf{C}^n \mid \varphi(z) < 0\}$. For a subset $A \subset \mathbf{C}^n$, let $C^{p,q}(A)$ be the restriction on A for the space of smooth (p, q) -forms on the complex Euclidean space.

For $x \in \partial D$, let $\{U, (\zeta_1, \dots, \zeta_n) \mid U \subset \mathbf{C}^n, (\zeta_1, \dots, \zeta_n) \in \mathbf{C}^n\}$ be a local coordinate of x . We assume that $d\varphi \neq 0$ holds on $\bar{U}$. Let $\{\tau_l\}_{l=1 \sim n}$ be a set of $C^{1,0}(\bar{D} \cap U)$ such that $\{\tau_k(y)\}_{k=1 \sim n}$ are an orthonormal basis with respect to the complex Euclidean metric at $y \in \bar{D} \cap U$ with $\tau_1 = \partial\varphi/|\partial\varphi|_E$. Here $|\cdot|_E$ denotes the pointwise norm with respect to the complex Euclidean metric. For a differential form θ , let $e(\theta)$ be the exterior product $e(\theta): u \mapsto \theta \wedge u$ and $e^*(\theta)$ be the adjoint of $e(\theta)$ with respect to the complex Euclidean metric.

For a smooth (p, q) -form $u \in C^{p,q}(\bar{D} \cap U)$, there are unique four differential forms $\{u_k\}_{k=1 \sim 4}$ satisfying

$$\begin{aligned} u &= u_1 + \partial\varphi \wedge u_2 + \bar{\partial}\varphi \wedge u_3 + \partial\varphi \wedge \bar{\partial}\varphi \wedge u_4, \\ e^*(\tau_1)u_k &= e^*(\bar{\tau}_1)u_k = 0 \quad (k = 1 \sim 4). \end{aligned}$$

Moreover, the following holds.

Proposition 1 (cf: [We])

For $p + q = n$, we assume that Levi form $\sqrt{-1} \partial \bar{\partial} \varphi$ is not non-degenerated on U . Then we have the following.

(1) For $w \in C^{p,q}(\bar{D} \cap U)$, there are unique four forms on $(\bar{D} \cap U)$ such that $v_1 \in C^{p-1,q-1}(\bar{D} \cap U)$, $v_2 \in C^{p-1,q}(\bar{D} \cap U)$, $v_3 \in C^{p,q-1}(\bar{D} \cap U)$, $v_4 \in C^{p,q}(\bar{D} \cap U)$ satisfies that

$$(i) \quad w = \partial \bar{\partial} \varphi \wedge v_1 + \partial\varphi \wedge v_2 + \bar{\partial}\varphi \wedge v_3 + \partial\varphi \wedge \bar{\partial}\varphi \wedge v_4,$$

$$(ii) \quad e^*(\tau_1)v_k = e^*(\bar{\tau}_1)v_k = 0 \quad (k = 1 \sim 4) \quad \text{hold.}$$

(2) For $v_f \in C^{0,1}(\bar{D} \cap U)$ and $u_1 \in C^{p,q-1}(\bar{D} \cap U)$, we put $w = v_f \wedge u_1 \in C^{p,q}(\bar{D} \cap U)$. assume that $e^*(\tau_1)w = e^*(\bar{\tau}_1)w = 0$ hold. Then there exist $u_{11} \in C^{p-1,q-1}(\bar{D} \cap U)$, $u_{12} \in C^{p-1,q-1}(\bar{D} \cap U)$

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uniquely satisfying the following:

- (i) $w = \partial \bar{\partial} \varphi \wedge u_{11} - \partial \varphi \wedge \bar{\partial} \varphi \wedge u_{12}$,
- (ii) $e^*(\tau_1)u_{11} = e^*(\bar{\tau}_1)u_{11} = 0$, $e^*(\tau_1)u_{12} = e^*(\bar{\tau}_1)u_{12} = 0$.

Especially, the form u_{11} may have a divisor v_f in general.

2. Theorem of Ohsawa

We will explain Theorem of Ohsawa and describe an outline of the proof [Oh] by using the argument of Ohsawa in view of Proposition 1.

Let ds^2 be a complete hermitian metric on a domain D with the smooth boundary and $L^{p,q}(D)$ be the space of square integrable (p, q) -forms.

From now on, we assume the following:

- (i) There exist positive constants C_0 and a, b satisfying

$$C_0^{-1} ds^2 < \left(-\frac{1}{\varphi}\right)^a \sum_{l=2}^n d\zeta_l \wedge d\bar{\zeta}_l + \left(-\frac{1}{\varphi}\right)^b \partial \varphi \wedge \bar{\partial} \varphi < C_0 ds^2 \text{ on } D \cap U,$$

- (ii) $1 \leq a, 1 \leq b, a < b + 1$ hold,
- (iii) $\partial \varphi, d\zeta_2, \dots, d\zeta_n$ are linearly independent on $\bar{D} \cap U$.

Then we have the following theorem about unreduced L^2 cohomology.

Theorem([Oh])

For $p + q = n$, we have that

$$\dim_C \frac{\{u \in L^{p,q}(D) \mid \bar{\partial} u = 0\}}{\{\bar{\partial} u \in L^{p,q}(D) \mid u \in L^{p,q-1}(D)\}} = +\infty$$

holds.

Proof of the theorem is as follows.

For $p + q = n$, we set $u = f d\bar{\zeta}_2 \wedge \dots \wedge d\bar{\zeta}_q \wedge d\zeta_2 \wedge \dots \wedge d\zeta_{p+1}$ which is a smooth (p, q) -form on $\bar{D} \cap U$.

Then there exist four forms $\{v_k\}_{k=1 \sim 4}$ uniquely from Proposition 1 satisfying

$$\bar{\partial} u = \partial \bar{\partial} \varphi \wedge v_1 + \partial \varphi \wedge v_2 + \bar{\partial} \varphi \wedge v_3 + \partial \varphi \wedge \bar{\partial} \varphi \wedge v_4.$$

We put

$$w = w(u) = u - \partial \varphi \wedge v_1. \quad (*)$$

Then we have $\bar{\partial} w = \partial \varphi \wedge (v_2 - \bar{\partial} v_1) + \bar{\partial} \varphi \wedge v_3 + \partial \varphi \wedge \bar{\partial} \varphi \wedge v_4$. By calculating, the following hold.

Proposition 2

- (1) $\partial \varphi \wedge v_2 \in L^{p,q}(D)$
- (2) $\bar{\partial} \varphi \wedge v_3 \in L^{p,q}(D)$
- (3) $\partial \varphi \wedge \bar{\partial} \varphi \wedge v_4 \in L^{p,q}(D)$ for $b - a > -1$
- (4) $\partial \varphi \wedge \bar{\partial} v_1 \in L^{p,q}(D)$ for $b - a > -1$

Then the following proposition holds from Proposition 2.

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Proposition 3 For $w \in C^{p,q-1}(D)$ in $(*)$, $\bar{\partial}w \in L^{p,q}(D)$ holds.

Next step is to find conditions such that $\bar{\partial}w$ is L^2 -cohomologous to 0. We assume that there exists $w_{p,q-1} \in L^{p,q-1}(D)$ satisfying the following:

$$\begin{aligned}
 (i) \quad w_{p,q-1} &= \sum_{|I|=p, |J|=q-1} w_{1I J} d\zeta_I \wedge d\bar{\zeta}_J \\
 &+ \bar{\partial}\varphi \wedge \sum_{|K|=p-1, |L|=q-1} w_{2K L} d\zeta_K \wedge d\bar{\zeta}_L \\
 &+ \bar{\partial}\varphi \wedge \left(\bar{\partial}\varphi \wedge \sum_{|M|=p-1, |N|=q-2} w_{3M N} d\zeta_M \wedge d\bar{\zeta}_N \right) \\
 &\quad + \bar{\partial}\varphi \wedge \left(\sum_{|P|=p, |Q|=q-1} w_{4P Q} d\zeta_P \wedge d\bar{\zeta}_Q \right) \\
 (ii) \quad \bar{\partial}w &= \bar{\partial}w_{p,q-1}
 \end{aligned}$$

For a defining function φ of the domain D and $t < 0$, we put $D_t := \{ \varphi < t \}$ and dS_t denotes the volume element of D_t with respect to the complex Euclidean metric.

Then we have the following by Fubini's theorem.

Proposition 4

$$\begin{aligned}
 (1) \quad \liminf_{t \nearrow 0} \int_{D_t \cap U} |w_{1IJ}|^2 dS_t &= 0 \text{ for } b \geq 1 \\
 (2) \quad \liminf_{t \nearrow 0} \int_{D_t \cap U} |w_{2KL}|^2 dS_t &= 0 \text{ for } a \geq 1
 \end{aligned}$$

By comparing $\bar{\partial}w$ with $\bar{\partial}w_{p,q-1}$, we have the following from Proposition 4.

Proposition 5

We assume that $\bar{\partial}w$ is L^2 -cohomologous to 0 with respect to the complete metric ds^2 . Then we have the following:

$$\begin{aligned}
 (1) \quad \bar{\partial}\varphi \wedge (v_2 - \bar{\partial}v_1) &= 0 \text{ holds on } \partial D \cap U. \\
 (2) \quad \bar{\partial}\varphi \wedge \bar{\partial}w &= 0 \text{ holds on } \partial D \cap U.
 \end{aligned}$$

Here we will prove Theorem of Ohsawa by using previous propositions.

Proof of Theorem of Ohsawa

Let ρ be a real-valued smooth function which has the support in a neighborhood U of $x \in \partial D$ and

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$\rho \equiv 1$ on a neighborhood $V \subset U$. We put

$$u_k := \rho \overline{\zeta}_2^k d\overline{\zeta}_2 \wedge \cdots \wedge d\overline{\zeta}_q \wedge d\overline{\zeta}_2 \wedge \cdots \wedge d\overline{\zeta}_{p+1} \text{ for } k \in \mathbf{N}$$

and $w_k := w(u_k)$ in accordance with (*). Then we see that $\overline{\partial}w_k - \overline{\partial}w_l$ is not to L^2 -cohomologous to 0 if $k \neq l$ from Proposition 5. Hence our claim holds. $\square$

3. Remarks

For $p + q = n$, we put $u = f d\overline{\zeta}_2 \wedge \cdots \wedge d\overline{\zeta}_q \wedge d\overline{\zeta}_2 \wedge \cdots \wedge d\overline{\zeta}_{p+1}$ which is a smooth $(p, q - 1)$ -form with the support in $\overline{D} \cap U$. Then we see that $\overline{\partial}u = \overline{\partial}f \wedge d\overline{\zeta}_2 \wedge \cdots \wedge d\overline{\zeta}_q \wedge d\overline{\zeta}_2 \wedge \cdots \wedge d\overline{\zeta}_{p+1}$ is the smooth (p, q) -form.

On the other hand, by using four forms $\{u_k\}$ ($k = 1 \sim 4$) with $e^*(\tau_1)u_k = e^*(\overline{\tau}_1)u_k = 0$, we can uniquely describe

$$d\overline{\zeta}_2 \wedge \cdots \wedge d\overline{\zeta}_q \wedge d\overline{\zeta}_2 \wedge \cdots \wedge d\overline{\zeta}_{p+1} = u_1 + \partial\varphi \wedge u_2 + \overline{\partial}\varphi \wedge u_3 + \partial\varphi \wedge \overline{\partial}\varphi \wedge u_4$$

from Proposition 1 (1). From now on, we will describe

$$\overline{\partial}f = f_{\overline{\partial}\varphi} \overline{\partial}\varphi + v_f$$

for any smooth function f by using a $(0, 1)$ -form v_f with $e^*(\tau_1)v_f = e^*(\overline{\tau}_1)v_f = 0$. Then we have

$$\overline{\partial}u = v_f \wedge u_1 + \partial\varphi \wedge (-v_f \wedge u_2) + \overline{\partial}\varphi \wedge f_{\overline{\partial}\varphi} u_1 + \partial\varphi \wedge \overline{\partial}\varphi \wedge (f_{\overline{\partial}\varphi} u_2 + v_f \wedge u_4)$$

On the other hand, there exist $u_{11} \in C^{p-1, q-1}(\overline{D} \cap U)$ and $u_{12} \in C^{p-1, q-1}(\overline{D} \cap U)$ uniquely such that

$$v_f \wedge u_1 = \partial \overline{\partial}\varphi \wedge u_{11} - \partial\varphi \wedge \overline{\partial}\varphi \wedge u_{12},$$

satisfying $e^*(\tau_1)u_{11} = e^*(\overline{\tau}_1)u_{11} = 0$ and $e^*(\tau_1)u_{12} = e^*(\overline{\tau}_1)u_{12} = 0$ from Proposition 1 (2).

Especially, u_{11} may have a divisor v_f .

Then we have

$$\overline{\partial}u = \partial \overline{\partial}\varphi \wedge u_{11} + \partial\varphi \wedge (-v_f \wedge u_2) + \overline{\partial}\varphi \wedge f_{\overline{\partial}\varphi} u_1 + \partial\varphi \wedge \overline{\partial}\varphi \wedge (f_{\overline{\partial}\varphi} u_2 + v_f \wedge u_4 - u_{12}).$$

We put

$$v_1 = u_{11}, v_2 = -v_f \wedge u_2, v_3 = f_{\overline{\partial}\varphi} u_1, v_4 = f_{\overline{\partial}\varphi} u_2 + v_f \wedge u_4 - u_{12}.$$

Then we have

$$\overline{\partial}u = \partial \overline{\partial}\varphi \wedge v_1 + \partial\varphi \wedge v_2 + \overline{\partial}\varphi \wedge v_3 + \partial\varphi \wedge \overline{\partial}\varphi \wedge v_4.$$

Remark 6

In our situation, both v_1 and v_2 may have a divisor v_f . Here we should claim that

$$\overline{\partial}\varphi \wedge (v_2 - \overline{\partial}v_1) = 0$$

holds if $\overline{\partial}w$ is L^2 -cohomologous to 0 from Proposition 5 (1).

Remark 7

In [Oh], assumptions about the claim for complete metrics is the following: " $1 \leq a \leq b < a + 3$ ". The condition $b < a + 3$ is needed for L^2 -integrability of φv_4 in [Oh] p.107. In our argument, this term does not appear.

4. Infinite dimensionality of reduced L^2 cohomology

Let X be an n -dimensional non-compact complex manifold and (X, ds^2) be a complete hermitian manifolds. Let $(\cdot, \cdot)$ (resp. $\|\cdot\|$) be the inner product (resp. norm) of (p, q) -forms on X . Let $|\cdot|$ be the pointwise norm of (p, q) -forms on X . Let $C_0^{p,q}(X)$ be the space of (p, q) -forms on X with compact supports and $L^{p,q}(X)$ be the space of L^2 -integrable (p, q) -forms on X with respect to ds^2 . Let $\bar{\partial}: L^{p,q}(X) \rightarrow L^{p,q+1}(X)$ be the $\bar{\partial}$ operator and $\bar{\partial}^*: L^{p,q+1}(X) \rightarrow L^{p,q}(X)$ be the adjoint operator. Then

$$\mathcal{H}_{(2)}^{p,q}(X) := \{\varphi \in L^{p,q}(X) \mid \bar{\partial}\varphi = 0, \bar{\partial}^*\varphi = 0\}$$

denotes the L^2 harmonic space with degree (p, q) . We put

$$N_{\bar{\partial}}^{p,q}(X) := \{u \in L^{p,q}(X) \mid \bar{\partial}u = 0\}, R_{\bar{\partial}}^{p,q}(X) := \{\bar{\partial}u \in L^{p,q}(X) \mid u \in L^{p,q-1}(X)\}$$

and

$$H^{p,q}(X) := N_{\bar{\partial}}^{p,q}(X) / \overline{R_{\bar{\partial}}^{p,q}(X)}$$

, where $\overline{R_{\bar{\partial}}^{p,q}(X)}$ denotes the topological closure of $R_{\bar{\partial}}^{p,q}(X)$ in $L^{p,q}(X)$. Let $H^{p,q}(X)$ denotes the reduced L^2 cohomology with degree (p, q) . These are isomorphic to each other.

In general, it is well known that the following holds for the closedness of $R_{\bar{\partial}}^{p,q}(X)$.

Proposition 8 ([H]) For any $\varphi \in C_0^{p,q}(X)$, there is a constant $C > 0$ such that the following holds:

$$\|\bar{\partial}\varphi\|^2 + \|\bar{\partial}^*\varphi\|^2 > C\|\varphi\|^2.$$

Then $R_{\bar{\partial}}^{p,q} \subset L^{p,q}(X)$ and $R_{\bar{\partial}}^{p,q+1} \subset L^{p,q+1}(X)$ are closed subsets.

Infinite dimensionality of the middle L^2 cohomology has been investigated by many articles. It is expected that this property will play important roles in various situations. If an n -dimensional complex manifold satisfies Kähler hyperbolicity([Gr]) or strictly Kähler convexity([Mc]), it is well known that Proposition 8 holds for $p + q \neq n$.

In 2006, B. Y. Chen had claimed that infinite dimensionality of the middle L^2 cohomology holds for any non-compact Kähler hyperbolic manifolds([Ch]). However, unfortunately, the claim has counterexamples in the case of finite volumes and bounded curvatures([Ye]).

5. Localization

For nonvanishing L^2 cohomology, it is known that the following holds.

Proposition 9 (cf. [G-T] Proposition 8.4)

Let X be a complete hermitian manifold. Let α be an element of $N_{\bar{\partial}}^{p,q}(X)$. We assume that there exist a $\gamma \in N_{\bar{\partial}}^{n-p,n-q}(X)$ such that $\int_X \alpha \wedge \gamma \neq 0$ holds. Then $\alpha \notin N_{\bar{\partial}}^{p,q}(X)$ holds. Especially, the L^2 cohomology

$H^{p,q}(X)$ is nonvanishing.

For the unit ball $B := \{(z_1, z_2) \in \mathbb{C}^2 \mid |z| < 1\}$, we will induce a complete hermitian metric ds_φ^2 satisfying the following conditions by using a differential automorphism φ :

- (i) For $p = (1, 0) \in \partial B$ and a sufficiently small neighborhood $U \subset B \subset \mathbb{C}^2$, we put $ds_\varphi^2 = ds_{hyp}^2$ on U , where ds_{hyp}^2 denotes the Poincaré metric of B . This metric satisfies conditions in Theorem of Ohsawa.
- (ii) We put $ds_\varphi^2 = ds_E^2$ on the hemisphere $B_- := \{z \in B \mid \operatorname{Re} z_1 < 0\}$, where ds_E^2 denotes the complex Euclidean metric of $\mathbb{C}^2$.

For the complex Euclidean metric $(\mathbb{C}^2, ds_E^2)$, it is well-known that $H^{p,q}(\mathbb{C}^n, ds_E^2) = \{0\}$ holds for $0 \leq p \leq n$, $0 \leq q \leq n$ ([Lo], p345). Therefore $H^{1,1}(B, ds_\varphi^2) = \{0\}$ holds. Then the following claim holds by using Proposition 9. It is a Serre duality-like proposition.

Proposition 10

Let $\alpha \in N_{\bar{\partial}}^{1,1}(B, ds_{hyp}^2)$ be a representative of non-zero L^2 cohomology with $\operatorname{supp} \alpha \subset \bar{U}$ and γ be an element of $N_{\bar{\partial}}^{1,1}(B, ds_{hy}^2)$ such that $\int_X \alpha \wedge \gamma \neq 0$. Then $\operatorname{supp} \gamma := \overline{\{z \in \bar{B} \mid \gamma(z) \neq 0\}} \not\subset \bar{U}$ holds.

6. Examples

We will give examples which have infinite dimensionality of the middle L^2 cohomology by Theorem of Ohsawa.

Example 11 Let $D = \{z \in \mathbb{C}^n \mid |z| < 1\}$ be the unit ball and ds_D^2 be the Poincaré metric. Then (D, ds_D^2) satisfies conditions for Proposition 8 and any point of ∂D has a neighborhood satisfying conditions for Theorem of Ohsawa. Therefore $\dim_{\mathbb{C}} H^{p,q}(D) = +\infty$ holds.

Example 12 Let $D = \left\{ Z = \begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix} \mid I_2 - Z {}^t \bar{Z} > 0 \right\} \subset \mathbb{C}^3$ be the bounded symmetric domain and ds_D^2 be the Bergman metric of D . Then (D, ds_D^2) satisfies conditions for Proposition 8 ([Do]) and almost everywhere point of ∂D has a neighborhood satisfying conditions for Theorem of Ohsawa. Therefore $\dim_{\mathbb{C}} H^{p,q}(D) = +\infty$ holds.

References

- [Ch] B. Y. Chen, Infinite Dimensionality of the Middle L^2 -cohomology on Non-compact Kähler Hyperbolic Manifolds, Publ. RIMS, Kyoto Univ., 42 (2006), 683–689.
- [Do] H. Donnelly, L_2 cohomology of the Bergman metric for weakly pseudoconvex domains, Illinois J. Math., 41 (1997), 151–160.
- [Gr] M. Gromov, Kähler hyperbolicity and L^2 -Hodge theory, J. Differential Geom. 33 (1991), 263–292.

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- [G-T] V. Gol'dshtein, and M. Troyanov, Sobolev Inequalities for Differential Forms and $L_{q,p}$ -Cohomology, The Journal of Geometric Analysis Volume 16, Number 4, 2006.
- [H] L. Hörmander, An introduction to complex analysis in several variables, North Holland 1973.
- [Lo] J. Lott, The zero-in-the spectrum question, 42 (1996), 341-376.
- [Mc] J.D. McNeal, L^2 harmonic forms on some complete Kähler manifolds. "Mathematische Annalen" 323.2 (2002), 319-34
- [Oh] T. Ohsawa, On the infinite dimensionality of the middle L^2 cohomology of complex domains, Publ. RIMS, Kyoto Univ., 25 (1989), 499-502.
- [Ye] N. Yeganehfar, L^2 -cohomology of negatively curved Kaehler manifolds of finite volume, Geometric and Functional Analysis GAFA, Volume 15, pages 1128-1143, (2005)
- [We] A. Weil, Introduction à l'étude des variétés kählériennes, Hermann, 1958.