

RATIONALITY OF BERSHADSKY-POLYAKOV VERTEX ALGEBRAS

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ABSTRACT. We prove the conjecture of Kac-Wakimoto on the rationality of exceptional W -algebras for the first non-trivial series, namely, for the Bershadsky-Polyakov vertex algebras $W_3^{(2)}$ at level $k = p/2 - 3$ with $p = 3, 5, 7, 9, \dots$. This gives new examples of rational conformal field theories.

1. INTRODUCTION

Recently, a remarkable family of W -algebras associated with simple Lie algebras and their *non-principal* nilpotent elements, called *exceptional W -algebras*, has been discovered by Kac and Wakimoto [10]. In [10] it was conjectured that with an exceptional W -algebra one can associate a rational conformal field theory.

As a first step to resolve the Kac-Wakimoto conjecture we have proved in the previous article [3] that exceptional W -algebras are *lisse*, or equivalently [2], C_2 -cofinite. Therefore it remains [15, 6] to show that exceptional W -algebras are *rational*, i.e., that the representations are completely reducible, in order to prove the conjecture. In this article we prove the rationality of the first non-trivial series of exceptional W -algebras, that is, the *Bershadsky-Polyakov (vertex) algebras* $W_3^{(2)}$ [13, 4] at level $k = p/2 - 3$ with $p = 3, 5, 7, 9, \dots$. The vertex algebra $W_3^{(2)}$ is the W -algebra associated with $\mathfrak{g} = \mathfrak{sl}_3$ and its minimal nilpotent element.

Let us state our main result more precisely: Let $\mathcal{W}_k$ denote the unique simple quotient of $W_3^{(2)}$ at level $k \neq -3$.

Main Theorem (Conjectured by Kac and Wakimoto [10]). *Let p be an odd integer equal or greater than 3, $k = p/2 - 3$. Then the vertex algebra $\mathcal{W}_k$ is rational. The simple $\mathcal{W}_k$ -modules are parameterized by the set of integral dominant weights of $\widehat{\mathfrak{sl}}_3$ of level $p - 3$. These simple modules can be obtained by the quantum BRST reduction from irreducible admissible representations of $\widehat{\mathfrak{sl}}_3$ of level k .*

For $p = 3$, $\mathcal{W}_{3/2-3}$ is one-dimensional. In the remaining cases $\mathcal{W}_{p/2-3}$ are conformal with negative central charges.

We note that Zhu's algebra of $W_3^{(2)}$ is closely related with Smith's algebra [14] which is a deformation of the universal enveloping algebra $U(\mathfrak{sl}_2(\mathbb{C}))$ of $\mathfrak{sl}_2(\mathbb{C})$, and that the rational quotient $\mathcal{W}_{p/2-3}$ has features in common with the $\widehat{\mathfrak{sl}}_2$ -integrable affine vertex algebras in the sense that the following relations hold:

$$:G^+(z)^{p-2}:= :G^-(z)^{p-2}:= 0,$$

where $G^+(z)$ and $G^-(z)$ are the standard generating fields of $\mathcal{W}_{p/2-3}$, see below.

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2. BERSHADSKY-POLYAKOV ALGEBRAS AT EXCEPTIONAL LEVELS.

Let $\mathcal{W}^k$ denote the Bershadsky-Polyakov (vertex) algebra $W_3^{(2)}$ at level $k \neq -3$, which is the vertex algebra freely generated by the fields $J(z), G^\pm(z), T(z)$ with the following OPE's:

$$\begin{aligned}
J(z)J(w) &\sim \frac{2k+3}{3(z-w)^2}, \quad G^\pm(z)G^\pm(w) \sim 0, \\
J(z)G^\pm(w) &\sim \pm \frac{1}{z-w}G^\pm(w), \\
T(z)T(w) &\sim -\frac{(2k+3)(3k+1)}{2(k+3)(z-w)^4} + \frac{2}{(z-w)^2}T(w) + \frac{1}{z-w}\partial T(w), \\
T(z)G^\pm(w) &\sim \frac{3}{2(z-w)^2}G^\pm(w) + \frac{1}{z-w}\partial G^\pm(w), \\
T(z)J(w) &\sim \frac{1}{(z-w)^2}J(w) + \frac{1}{z-w}\partial J(w), \\
G^+(z)G^-(w) &\sim \frac{(k+1)(2k+3)}{(z-w)^3} + \frac{3(k+1)}{(z-w)^2}J(w) \\
&\quad + \frac{1}{z-w} \left(3 : J(w)^2 : + \frac{3(k+1)}{2}\partial J(w) - (k+3)T(w) \right).
\end{aligned}$$

As in introduction we denote by $\mathcal{W}_k$ the unique simple quotient of $\mathcal{W}^k$.

Theorem 2.1 ([3]). *Let k, p be as in Main Theorem. Then $\mathcal{W}_k$ is lisse, or equivalently, C_2 -cofinite.*

Set

$$L(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2} = T(z) + \frac{1}{2}\partial J(w).$$

This defines a conformal vector of $\mathcal{W}^k$ with central charge

$$c(k) = -\frac{4(k+1)(2k+3)}{k+3} = -\frac{4(p-4)(p-3)}{p},$$

which gives J, G^+, G^- conformal weights 1, 1, and 2, respectively. Hence $\mathcal{W}^k$ is $\mathbb{Z}_{\geq 0}$ -graded with respect to the Hamiltonian L_0 . We expand the corresponding fields accordingly:

$$J(z) = \sum_{n \in \mathbb{Z}} J_n z^{-n-1}, \quad G^+(z) = \sum_{n \in \mathbb{Z}} G_n^+ z^{-n-1}, \quad G^-(z) = \sum_{n \in \mathbb{Z}} G_n^- z^{-n-2}.$$

We have

$$\begin{aligned}
[J_m, J_n] &= \frac{2k+3}{3}m\delta_{m+n,0}, \quad [J_m, G_n] = G_{m+n}, \quad [J_m, F_n] = -F_{m+n}, \\
[L_m, J_n] &= -nJ_{m+n} - \frac{(2k+3)(m+1)m}{6}\delta_{m+n,0}, \\
[L_m, G_n^+] &= -nG_{m+n}^+, \quad [L_m, G_n^-] = (m-n)G_{m+n}^-, \\
[G_m^+, G_n^-] &= 3(J^2)_{m+n} + (3(k+1)m - (2k+3)(m+n+1))J_{m+n} - (k+3)L_{m+n} \\
&\quad + \frac{(k+1)(2k+3)m(m+1)}{2}\delta_{m+n,0},
\end{aligned}$$

where $\sum_{n \in \mathbb{Z}} (J^2)_n z^{-n-2} \stackrel{\text{def}}{=} J(z)^2$.

For $(\xi, \chi) \in \mathbb{C}^2$, let $L(\xi, \chi)$ be the irreducible representation of $\mathcal{W}^k$ generated by the vector $|\xi, \chi\rangle$ such that

$$\begin{aligned} J_0 |\xi, \chi\rangle &= \xi |\xi, \chi\rangle, & J_n |\xi, \chi\rangle &= 0 \quad \text{for } n > 0, \\ L_0 |\xi, \chi\rangle &= \chi |\xi, \chi\rangle, & L_n |\xi, \chi\rangle &= 0 \quad \text{for } n > 0, \\ G_n^- |\xi, \chi\rangle &= 0 \quad \text{for } n \geq 0, & G_n^+ |\xi, \chi\rangle &= 0 \quad \text{for } n \geq 1. \end{aligned}$$

By Theorem 2.1, any simple $\mathcal{W}_k$ -module is of the form $L(\xi, \lambda)$ with some ξ and χ . (It is important that the lisse condition is defined independent of the choice of a conformal vector.)

For a $\mathcal{W}^k$ -module M set

$$M_{a,d} = \{m \in M; J_0 m = am, L_0 m = dm\}.$$

It is clear that $L(\xi, \chi) = \bigoplus_{\substack{(a,d) \in \mathbb{C}^2 \\ d \in \chi + \mathbb{Z}_{\geq 0}}} L(\xi, \chi)_{a,d}$, $\dim L(\xi, \chi)_{\xi, \chi} = 1$. Let

$$L(\xi, \chi)_{\text{top}} = \{v \in L(\xi, \chi); L_0 v = \chi v\} = \bigoplus_a L(\xi, \chi)_{a, \chi}.$$

By definition $L(\xi, \chi)_{\text{top}}$ is spanned by the vectors $(G_0^+)^i |\xi, \chi\rangle$ with $i \geq 0$.

Following [14] set

$$g(\xi, \chi) = -(3\xi^2 - (2k+3)\xi - (k+3)\chi),$$

so that $G_0^- G_0^+ |\xi, \chi\rangle = g(\xi, \chi) |\xi, \chi\rangle$. We have

$$G_0^- (G_0^+)^i |\xi, \chi\rangle = i h_i(\xi, \chi) (G_0^+)^{i-1} |\xi, \chi\rangle,$$

where

$$\begin{aligned} h_i(\xi, \chi) &= \frac{1}{i} (g(\xi, \chi) + g(\xi+1, \chi) + \cdots + g(\xi+i-1, \chi)) \\ &= -i^2 + ki - 3\xi i + 3i - 3\xi^2 - k + 2k\xi + 6\xi + k\chi + 3\chi - 2. \end{aligned}$$

Hence we have the following assertion.

Proposition 2.2. *If the space $L(\xi, \chi)_{\text{top}}$ is n -dimensional, then $h_n(\xi, \chi) = 0$.*

Define

$$\Delta(-J, z) = z^{-J_0} \exp \left(\sum_{k=1}^{\infty} (-1)^{k+1} \frac{-J_k}{k z^k} \right),$$

and set

$$\sum_{n \in \mathbb{Z}} \psi(a_{(n)}) z^{-n-1} = Y(\Delta(-J, z)a, z)$$

for $a \in \mathcal{W}^k$. For any $\mathcal{W}^k$ -module M , we can define on M a new $\mathcal{W}^k$ -module structure by twisting the action of $\mathcal{W}^k$ as $a_{(n)} \mapsto \psi(a_{(n)})$ ([11]). We denote by $\psi(M)$ thus obtained $\mathcal{W}^k$ -module from M .

Proposition 2.3. *Suppose that $\dim L(\xi, \chi)_{\text{top}} = i$. Then*

$$\psi(L(\xi, \chi)) \cong L\left(\xi + i - 1 - \frac{2k+3}{3}, \chi - (\xi - i + 1) + \frac{2k+3}{3}\right).$$

Proof. The assertion follows from the fact that

$$\begin{aligned}\psi(J_n) &= J_n - \frac{2k+3}{3}\delta_{n,0}, & \psi(L_n) &= L_n - J_n + \frac{2k+3}{3}, \\ \psi(G_n^+) &= G_{n-1}^+, & \psi(G_n^-) &= G_{n+1}^-.\end{aligned}$$

□

By solving the equation

$$h_i(\xi, \chi) = h_j(\xi + i - 1 - \frac{2k+3}{3}, \chi - (\xi - i + 1) + \frac{2k+3}{3})$$

we obtain the following assertion.

Proposition 2.4. *Suppose that $\dim L(\xi, \chi)_{\text{top}} = i$ and $\dim \psi(L(\xi, \chi))_{\text{top}} = j$. Then*

$$\begin{aligned}\xi &= \xi_{i,j} \stackrel{\text{def}}{=} \frac{1}{3}(-2i - j + 2k + 6), \\ \chi &= \chi_{i,j} \stackrel{\text{def}}{=} \frac{i^2 + ji - ki - 3i + j^2 - 6j - 2jk + 3k + 6}{3(k+3)}.\end{aligned}$$

Proposition 2.5. *Let k, p be as in Main Theorem. Then $(G_{-1}^+)^{p-2}\mathbf{1}$ belongs to the maximal ideal of $\mathcal{W}^k$.*

Proof. Since $\xi_{1,p-2} = \chi_{1,p-2} = 0$, the correspondence $\mathbf{1} \mapsto |\xi_{1,p-2}, \chi_{1,p-2}\rangle$ gives an isomorphism $\mathcal{W}_k \cong L(\xi_{1,p-2}, \chi_{1,p-2})$. Because

$$h_{p-2}(\xi_{1,p-2} - (2k+3)/2, \chi_{1,p-2} + (2k+3)/3) = 0,$$

from Proposition 2.3 it follows that $\psi(\mathcal{W}_k)_{\text{top}}$ is at most $p-2$ -dimensional. Hence $(G_{-1}^+)^{p-2}\mathbf{1} = 0$. □

Remark 2.6. One can show that in fact $(G_{-1}^+)^{p-2}$ generates the maximal ideal of $\mathcal{W}^k$. However we do not need this fact.

Proposition 2.7. *Let k, p be as in Main Theorem. Then any simple $\mathcal{W}_k$ -module is isomorphic to $L(\xi_{i,j}, \chi_{i,j})$ for some (i, j) such that $1 \leq i \leq p-2$, $1 \leq j \leq p-i-1$.*

Proof. Let $L(\xi, \chi)$ be a simple $\mathcal{W}_k$ -module. As $G^+(z)^{p-2} := 0$ on $L(\xi, \chi)$ by Proposition 2.5, $L(\xi, \chi)_{\text{top}}$ is at most $(p-2)$ -dimensional. Since $\psi(L(\xi, \chi))$ is also a $\mathcal{W}_k$ -module we have $(\xi, \chi) = (\xi_{i,j}, \chi_{i,j})$ for some $1 \leq i, j \leq p-2$. Because $\psi(\psi(L(\xi_{i,j}, \chi_{i,j})))$ is also a $\mathcal{W}_k$ -module it follows that $\xi_{i,j} + i - 1 - \frac{2k+3}{3} = \frac{i-j}{3} \leq \frac{-2j-1+2k+6}{3} = \frac{p-2j-1}{3}$. Hence $j \leq p-i-1$. □

The simple $\mathcal{W}^k$ -modules $L(\xi_{i,j}, \chi_{i,j})$ with $1 \leq i \leq p-2$, $1 \leq j \leq p-i-1$, are mutually non-isomorphic since their highest weights are distinct.

3. PROOF OF MAIN THEOREM

Let k, p be as in Main Theorem.

Let $\mathfrak{g} = \mathfrak{sl}_3$ as in introduction, $\mathfrak{h} \subset \mathfrak{g}$ be the Cartan subalgebra of $\mathfrak{g}$ consisting of diagonal matrixes. Set $h_i = E_{i,i} - E_{i+1,i+1}$, $h_\theta = h_1 + h_2$, $e_i = e_{\alpha_i} = E_{i,i+1}$, $f_i = f_{\alpha_i} = E_{i+1,i}$ for $i = 1, 2$, $e_\theta = E_{1,3}$, $f_\theta = E_{3,1}$, where $E_{i,j}$ is the matrix element. We equip $\mathfrak{g}$ the invariant form $(x|y) = \text{tr}(xy)$. Set $\bar{\Lambda}_1 = (2h_1 + h_2)/3$, $\bar{\Lambda}_1 = (h_1 + 2h_2)/3$, so that $(\bar{\Lambda}_i|h_j) = \delta_{i,j}$.

Let $\widehat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}K \oplus \mathbb{C}D$ be the (non-twisted) affine Kac-Moody algebra associated with $\mathfrak{g}$, where K is the central element and D is the degree operator. Let $\widehat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}K \oplus \mathbb{C}D \subset \widehat{\mathfrak{g}}$ the standard Cartan subalgebra, $\widehat{\mathfrak{h}}^* = \mathfrak{h}^* \oplus \mathbb{C}\Lambda_0 \oplus \mathbb{C}\delta$ the dual of $\widehat{\mathfrak{h}}$, where Λ_0 and δ are elements dual to K and D , respectively.

The vector f_θ is a the minimal nilpotent element of $\mathfrak{g}$. Let $\mathfrak{g} = \bigoplus_{j \in \frac{1}{2}\mathbb{Z}} \mathfrak{g}_j$ be the corresponding Dynkin grading: $\mathfrak{g}_j = \{u \in \mathfrak{g}; [h_\theta, u] = 2ju\}$. Denote by $H_{f_\theta}^{\frac{\infty}{2}+0}(?)$ the BRST cohomology of the generalized quantized Drinfeld-Sokolov reduction associated with $(\mathfrak{g}, f_\theta)$ and the Dynkin grading. We have [7, 9] the vertex algebra isomorphism

$$\mathcal{W}^k \xrightarrow{\sim} H_{f_\theta}^{\frac{\infty}{2}+0}(V^k(\mathfrak{g})),$$

which is given by the following assignment:

$$\begin{aligned} J(z) &\mapsto J^{-\bar{\Lambda}_1 + \bar{\Lambda}_2}(z) - : \Phi_1(z) \Phi_2(z) :, \\ G^+(z) &\mapsto J^{f_1}(z) - : J^{h_1}(z) \Phi_2(z) : + : \Phi_1(z) \Phi_2(z)^2 : - (k+1) \partial \Phi_2(z), \\ G^+(z) &\mapsto -J^{f_2}(z) - : J^{h_2}(z) \Phi_1(z) : - : \Phi_1(z)^2 \Phi_2(z) : - (k+1) \partial \Phi_1(z), \end{aligned}$$

Here

$$J^u(z) = u(z) - \sum_{\beta, \gamma \in \{\alpha_1, \alpha_2, \theta\}} c_{u, f_\beta}^{f_\gamma} : \psi_\beta^*(z) \psi_\gamma(z) :$$

for $u \in \mathfrak{g}$, $c_{u_1, u_2}^{u_3}$ is the structure constant, $\psi_\alpha(z)$, $\psi_\alpha^*(z)$ with $\alpha \in \{\alpha_1, \alpha_2, \theta\}$ are fermionic ghosts satisfying

$$(1) \quad \psi_\alpha(z) \psi_\beta^*(w) \sim \frac{\delta_{\alpha, \beta}}{z-w}, \quad \psi_\alpha(z) \psi_\beta(w) \sim \psi_\alpha^*(z) \psi_\beta^*(w) \sim 0,$$

$\Phi_1(z)$, $\Phi_2(z)$ are bosonic ghosts satisfying

$$\Phi_1(z) \Phi_2(w) \sim \frac{1}{z-w}, \quad \Phi_i(z) \Phi_i(w) \sim 0,$$

and the BRST differential is the zero mode of the field

$$\begin{aligned} Q(z) &= \sum_{\alpha \in \{\alpha_1, \alpha_2, \theta\}} e_\alpha(z) \psi_\alpha^*(w) - : \psi_{\alpha_1}^*(z) \psi_{\alpha_2}^*(z) \psi_\theta(z) : \\ &\quad + \Phi_1(z) \psi_{\alpha_1}^*(z) + \Phi_2(z) \psi_{\alpha_2}^*(z) + \psi_\theta(z). \end{aligned}$$

Let $\mathcal{O}_k$ be the category $\mathcal{O}$ of $\widehat{\mathfrak{g}}$ at level k , $\mathbf{L}_\lambda$ the irreducible representation of $\widehat{\mathfrak{g}}$ with highest weight λ . Denote by $\mathcal{W}^k\text{-Mod}$ the category of $\mathcal{W}^k$ -modules.

Theorem 3.1 ([1]).

- (i) The functor $H_{f_\theta}^{\frac{\infty}{2}+0}(?) : \mathcal{O}_k \rightarrow \mathcal{W}^k\text{-Mod}$, $M \mapsto H_{f_\theta}^{\frac{\infty}{2}+0}(M)$, is exact.
- (ii) For $\lambda \in \widehat{\mathfrak{h}}^*$ we have $H_{f_\theta}^{\frac{\infty}{2}+0}(\mathbf{L}_\lambda) = 0$ if and only if $\lambda(\alpha_0^\vee) \in \{0, 1, 2, 3, \dots\}$. Otherwise $H_{f_\theta}^{\frac{\infty}{2}+0}(\mathbf{L}_\lambda)$ is irreducible.

Let Adm^k be the set of admissible weights [8] of $\widehat{\mathfrak{g}}$ of level k , and put

$$\text{Adm}_+^k = \{\lambda \in \text{Adm}^k; \bar{\lambda} \text{ is an integral dominant weight of } \mathfrak{g}\},$$

where $\widehat{\mathfrak{h}}^* \ni \lambda \mapsto \bar{\lambda} \in \mathfrak{h}^*$ is the restriction. Then

$$\text{Adm}_+^k = \{\bar{\mu} + k\Lambda_0; \mu \in \widehat{P}_{++}^{p-3}\},$$

where $\widehat{P}_{++}^{p-3}$ is the set of integral dominant weights of $\widehat{\mathfrak{g}}$ of level $p-3$. Explicitly, we have

$$\text{Adm}_+^k = \{\lambda_{i,j}; 1 \leq i \leq p-2, 1 \leq j \leq p-i-1\},$$

where

$$\lambda_{i,j} = (i-1)\bar{\Lambda}_1 + (p-i-j-1)\bar{\Lambda}_2 + k\Lambda_0.$$

Note that

$$(2) \quad \xi_{i,j} = (\lambda_{i,j} | -\bar{\Lambda}_1 + \bar{\Lambda}_2), \quad \chi_{i,j} = \frac{(\lambda_{i,j} | \lambda_{i,j} + 2\bar{\rho})}{2(k+3)} - (\lambda_{i,j} | \bar{\Lambda}_2),$$

where $\bar{\rho} = \bar{\Lambda}_1 + \bar{\Lambda}_2$.

Recall the following result of Malikov and Frenkel [12].

Theorem 3.2 ([12, Corollary 5.2.2]). *For $\lambda \in \text{Adm}_+^k$, $\mathbf{L}_\lambda$ is a module over $\mathbf{L}_{k\Lambda_0}$.*

Proposition 3.3. *For $\lambda_{i,j} \in \text{Adm}_+^k$, $H_{f_\theta}^{\frac{\infty}{2}+0}(\mathbf{L}_{\lambda_{i,j}})$ is a simple $\mathcal{W}_k$ -module isomorphic to $L(\xi_{i,j}, \chi_{i,j})$.*

Proof. By Theorem 3.1 we have $\mathcal{W}_k \cong H_{f_\theta}^{\frac{\infty}{2}+0}(\mathbf{L}_{k\Lambda_0})$. Hence by the functoriality of $H_{f_\theta}^{\frac{\infty}{2}+0}(\cdot)$, Theorem 3.2 immediately gives that $H_{f_\theta}^{\frac{\infty}{2}+0}(\mathbf{L}_{\lambda_{i,j}})$ is a module over $\mathcal{W}_k$. By Theorem 3.1, $H_{f_\theta}^{\frac{\infty}{2}+0}(\mathbf{L}_{\lambda_{i,j}})$ is (nonzero and) irreducible. Let v be the image of the highest weight vector of $\mathbf{L}_{\lambda_{i,j}}$ in $H_{f_\theta}^{\frac{\infty}{2}+0}(\mathbf{L}_{\lambda_{i,j}})$. By (2) and the fact [9] that the image of $L(z)$ in $\mathcal{W}^k$ is cohomologous to

$$L_{\mathfrak{g}}(z) + L_{\text{ch}}(z) + L_{\Phi}(z) + \partial J^{\bar{\Lambda}_2}(z),$$

where $L_{\mathfrak{g}}(z)$ is the Sugawara operator of $\mathfrak{g}$, $L_{\text{ch}}(z) = -\sum_{\alpha=\alpha_1, \alpha_2, \theta} \phi_\alpha(z) \partial \phi_\alpha^*(z)$, $L_{\Phi}(z) = \frac{1}{2}(\Phi_2(z) \partial \Phi_1(z) - \partial \Phi_1(z) \Phi_2(z))$, it is straightforward to check that the assignment $|\xi_{i,j}, \chi_{i,j}\rangle \mapsto v$ gives a $\mathcal{W}^k$ -module homomorphism. By the irreducibility, this must be an isomorphism. $\square$

By Propositions 2.7 and 3.3, the set $\{H_{f_\theta}^{\frac{\infty}{2}+0}(\mathbf{L}_\lambda); \lambda \in \text{Adm}_+^k\}$ gives the complete set of isomorphism classes of simple $\mathcal{W}_k$ -modules. Therefore Main Theorem now follows immediately from the following important result of Gorelik and Kac [5].

Theorem 3.4 ([5, Corollary 8.8.9]). *For any $\lambda, \mu \in \text{Adm}^k$, we have*

$$\text{Ext}_{\mathcal{W}^k\text{-Mod}}^1(H_{f_\theta}^{\frac{\infty}{2}+0}(\mathbf{L}_\lambda), H_{f_\theta}^{\frac{\infty}{2}+0}(\mathbf{L}_\mu)) = 0.$$

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