

Blow-up of Solutions for Semilinear Parabolic Equations

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Section 1. Main results.

In this paper we consider the following initial boundary value problems

$$(IBVP1) \quad \begin{cases} u_t(x, t) = \Delta u + \lambda e^u(x, t) & \text{in } \Omega \times (0, T), \\ u(x, t) = 0 & \text{on } \partial\Omega \times (0, T), \\ u(x, 0) = \varphi(x) & \text{in } \Omega, \end{cases}$$

$$(IBVP2) \quad \begin{cases} u_t(x, t) = \Delta u + u^p(x, t) & \text{in } \Omega \times (0, T), \\ u(x, t) = 0 & \text{on } \partial\Omega \times (0, T), \\ u(x, 0) = \varphi(x) & \text{in } \Omega. \end{cases}$$

Here $\Omega \subset \mathbb{R}^n$ ($n \geq 2$) is a bounded domain with smooth boundary $\partial\Omega$, $\lambda > 0$ and $1 < p < (n+2)/(n-2)$ ($n \geq 3$), $p > 1$ ($n=2$).

Throughout the present paper we assume that an initial value φ satisfies the following conditions:

- (1) $\varphi \in C^1(\bar{\Omega}) \cap C^2(\Omega)$; (2) $\varphi(x) \geq 0$ ($x \in \Omega$), $\varphi(x) = 0$ ($x \in \partial\Omega$);
(3) $\Delta\varphi + \lambda e^\varphi \geq 0$ ($x \in \Omega$) (resp. $\Delta\varphi + \varphi^p \geq 0$ ($x \in \Omega$)).

Then, by the maximum principle, it follows that $u(x, t) > 0$,
 $u_t(x, t) > 0$ for $(x, t) \in \Omega \times (0, T)$.

Suppose that the solution $u(x, t)$ blows up as $t \rightarrow T$ ($< \infty$),
i. e., $\lim_{t \rightarrow T} \max_{x \in \Omega} u(x, t) = \infty$. We are interested in the behavior of

the limit function $\lim_{t \rightarrow T} u(x, t)$ near a blow-up point. (A point $x \in \bar{\Omega}$

is, by definition, a blow-up point if there exists a sequence

$(x_n, t_n) \in \Omega \times (0, T)$ such that $x_n \rightarrow x$, $t_n \rightarrow T$, and $u(x_n, t_n) \rightarrow \infty$ as $n \rightarrow \infty$.)

If Ω is a ball with center 0 and if φ satisfies the conditions (1), (2), (3) and if

(4) φ is radially symmetric with $\varphi_r \leq 0$,

then a blow-up point is a single point $x = 0$ ((5)) and the following estimates are known. Upper estimates for (IBVP1) ((4)):

There exists a constant C such that

$$(5) \quad u(x, t) \leq -2 \ln |x| + \ln |\ln |x|| + C \quad (|x| > 0).$$

The inequality (5) cannot be improved as follows :

$$u(x, t) \leq -2 \ln |x| + C.$$

In other words there does not exist such a constant C (see (3)).

Upper estimates for (IBVP2) ((5)): For any $0 < q < p$, there exists a constant C_q such that

$$u(x, t) \leq C_q (1/|x|)^{2/(q-1)} \quad (|x| > 0).$$

Lower estimates for (IBVP1) ((3)): Let $n \geq 3$. Then there exists $r_1 > 0$ such that

$$\lim_{t \rightarrow T} u(x, t) > \ln (2(n-2)/(\lambda |x|^2)) \quad (0 < |x| < r_1).$$

Lower estimates for (IBVP2) ((3)): Let $n \geq 3$ and let $p > n/(n-2)$. Then there exists $r_1 > 0$ such that ($\beta = 1/(p-1)$)

$$\lim_{t \rightarrow T} u(x, t) > (-4\beta(\beta + (2-n)/2)/|x|^2)^\beta \quad (0 < |x| < r_1).$$

We shall investigate lower estimates for solutions to (IBVP1,2)

in the case that a domain in $\mathbb{R}^n$ ($n \geq 2$) is not necessary a ball. Our results on lower estimates are as follows.

Theorem 1. Let Ω be a bounded domain in $\mathbb{R}^2$ with smooth boundary $\partial\Omega$. Assume that φ satisfies the conditions (1), (2), (3). Suppose that the solution $u(x, t)$ to (IBVP1) blows up at a point $x_0 \in \Omega$ as $t \rightarrow T$. Then there exists $r_1 > 0$ and the following inequality holds. For any r with $0 < r \leq r_1$

$$(6) \quad \lim_{t \rightarrow T} \{1/(2\pi) \int_0^{2\pi} e^{u(x_0 + (r \cos \theta, r \sin \theta), t)/2} d\theta\}^2 > 2\lambda^{-1} r^{-2}.$$

As a direct consequence of Theorem 1 we have

Corollary 2. Let Ω be a ball with center 0 in $\mathbb{R}^2$. Assume that $\varphi(x)$ is radially symmetric and satisfies the conditions (1), (2), (3). Suppose that the solution $u(x, t)$ to (IBVP1) blows up at $x = 0$ as $t \rightarrow T$. Then there exists $r_1 > 0$ and the following inequality holds. For any x with $0 < |x| < r_1$

$$(7) \quad \lim_{t \rightarrow T} u(x, t) > -2 \ln |x| + \ln(2\lambda^{-1}).$$

Next we state the results on (IBVP2).

Theorem 3. Let Ω be a bounded convex domain in $\mathbb{R}^n$ ($n \geq 2$) with smooth boundary $\partial\Omega$. If x_0 is a blow-up point with a blow-up time T , then for any $k > 0$, there exists $r_k > 0$ such that

$$\lim_{t \rightarrow T} u(x, t) \geq k(1/|x - x_0|)^{2/(p-1)}$$

for any x with $0 < |x - x_0| < r_k$.

Lastly we state results on the Hausdorff dimension of the set of blow-up points of the solution to (IBVP2). Let

$\Gamma \equiv \{ \gamma : \text{the solution } u(x, t) \text{ blows up in } L^\gamma \text{ as } t \rightarrow T \}$
and set $\gamma_0 \equiv \inf \Gamma$.

By the result of Giga and Kohn (9), it follows that $n(p-1)/2 \in \Gamma$ and so $\gamma_0 \leq n(p-1)/2$.

Theorem 4. Let Ω be a bounded convex domain with smooth boundary $\partial\Omega$. Then the Hausdorff dimension D_H of the set of blow-up points for (IBVP2) satisfies the following inequality

$$D_H \leq -2 \gamma_0 / (p-1) + n.$$

For the proof of Theorem 1 we use Bandle's mean value theorem, which is proved by using Bol's isoperimetric inequality. In Section 2 we recall her result and prove Theorem 1. For the proof of Theorem 3, Giga and Kohn's results ((10)) play an essential role. As a consequence of Theorem 3 we obtain Theorem 4.

Section 2. Proof of Theorem 1.

For the proof we use Bandle's mean value theorem. Before proceeding to the proof we recall her result.

Let D be a domain in $\mathbb{R}^2$, and let v satisfies a differential inequality $\Delta v + \lambda e^v \geq 0$ in D . Let $B(r, x_0) \subset D$ be the disk of radius r with center $x_0 \in D$. We define a Riemann metric $d\sigma$ in D by $d\sigma^2 = e^{v(x)} (dx_1^2 + dx_2^2)$. Then the length of $\partial B(r, x_0)$ and the area of $B(r, x_0)$ are respectively

$$l(r) \equiv \int_0^{2\pi} e^{v(r, \theta)/2} r d\theta \quad \text{and} \quad a(r) \equiv \int_0^{2\pi} \int_0^r e^{v(r, \theta)} r dr d\theta$$

where we write $v(r, \theta)$ for $v(x)$ ($x = x_0 + (r \cos \theta, r \sin \theta)$).

Bol's isoperimetric inequality says that

$$(8) \quad l(r)^2 \geq 4\pi a(r) - (\lambda/2) a(r)^2.$$

By the Schwarz inequality we have $2\pi r a_r(r) \geq l^2$. Hence this together with (8) yields a differential inequality

$$2\pi r a_r(r) \geq 4\pi a(r) - (\lambda/2) a(r)^2.$$

The equality holds if and only if

$$a(r) = a_0(r) \equiv \pi b r^2 / (1 + \lambda b r^2 / 8) \quad (b : \text{a positive parameter}).$$

Note that $a_0(r)$ is the area of $B(r, x_0)$ with respect to a Riemann metric $d\sigma_0^2 = e^{v_0(x)} (dx_1^2 + dx_2^2)$, where

$$(9) \quad v_0(x) = \ln (b / (1 + \lambda b r^2 / 8))^2$$

and it satisfies $\Delta v_0 + \lambda e^{v_0} = 0$.

We denote the length of $\partial B(r, x_0)$ with respect to $d\sigma_0$ by $l_0(r)$.

Now we state Bandle's mean value theorem ((1, 2)).

Lemma 1 (Bandle's mean value theorem).

Let v be as above. Let $B(r_1, x_0) \subset D$. Suppose that there exists $b > 0$ such that

$$(10) \quad \ell(r_1) = \ell_0(r_1).$$

Then if $\lambda a(r_1) \leq 4\pi$, then

$$(i) \quad a(r) \leq a_0(r) \quad \text{for } 0 < r \leq r_1,$$

$$(ii) \quad v(0) \leq \ln b.$$

For the proof see (1).

Lemma 2. Let $r_1 > 0$ be fixed. Suppose that $\lambda a(r_1) \leq 4\pi$. Then if

$$e^{\bar{v}(r_1)} \equiv \left((1/2\pi) \int_0^{2\pi} e^{v(r_1, \theta)/2} d\theta \right)^2 < 2/(\lambda r_1^2),$$

then there exists $b > 0$ such that (10) holds and

$$(11) \quad \lambda a_0(r_1) < 4\pi.$$

Proposition. Let $u(x, t)$ be a continuous function in $\Omega \times (0, T_0)$. Assume that for each $0 \leq t < T_0$, $u(x, t)$ satisfies a differential inequality $\Delta u + \lambda e^u \geq 0$ in Ω . Suppose that $\lambda a(r_1, 0) < 4\pi$ and that for each $0 \leq t < T_0$

$$e^{\bar{u}(r_1, t)} < 2/(\lambda r_1^2). \quad \text{Then the followings hold for each } 0 \leq t < T_0$$

$$(i) \quad \lambda a(r_1, t) < 4\pi, \text{ and}$$

$$(ii) \quad e^{u(x_0, t)} \leq 4 e^{\bar{u}(r_1, t)},$$

$$\text{where } a(r_1, t) = \int_{B(r_1, x_0)} e^{u(x, t)} dx,$$

$$(12) \quad e^{\bar{u}(r_1, t)} \equiv \left((1/2\pi) \int_0^{2\pi} e^{u(r_1, \theta, t)/2} d\theta \right)^2,$$

$$u(r_1, t, \theta) = u(x, t) \quad (x = x_0 + (r \cos \theta, r \sin \theta)).$$

Proof of Theorem 1. Since $u_t > 0$, we have differential inequalities $\Delta u + \lambda e^u > 0$. Let $r_1 > 0$ be so small that

$$(i) \quad \lambda a(r_1, 0) < 4\pi; \quad (ii) \quad e^{\bar{u}(r_1, 0)} < 2/(\lambda r_1^2).$$

We prove that there exists $T_1 < T$ such that $e^{\bar{u}(r_1, T_1)} = 2/(\lambda r_1^2)$.

Suppose that there does not exist such a T_1 . Then we have for $0 \leq t < T$, $e^{\bar{u}(r_1, t)} < 2/(\lambda r_1^2)$. Hence by (ii) of Proposition we have

$$u(x_0, t) \leq \ln \bar{u}(r_1, t) + 2 \ln 2 \quad (0 \leq t < T),$$

and so we have

$$u(x_0, t) \leq -2 \ln r_1 + \ln(8/\lambda) \quad (0 \leq t < T),$$

contradicting the fact that x_0 is a blow-up point and that T is a blow-up time.

Proof of Proposition. First we prove that (i) holds for $0 \leq t < T_0$.

Suppose that it does not hold, then there exists $0 < T_1 < T_0$ such that $\lambda a(r_1, t) < 4\pi$ ($0 \leq t < T_1$), and

$$(13) \quad \lambda a(r_1, T_1) = 4\pi.$$

Since $e^{\bar{u}(r_1, T_1)} < 2/(\lambda r_1^2)$, we get by Lemmas 1, 2 that $\lambda a(r_1, T_1) < 4\pi$. This contradicts (13).

Next we prove that (ii) holds for $0 \leq t < T_0$. Since

$$(14) \quad (\lambda r_1^2/2) e^{\bar{u}(r_1, t)} < 1,$$

there exists $b = b(r_1, t)$ such that $\ell(r_1, t) = \ell_0(r_1, t)$, where $\ell(r_1, t)$ is the length of $\partial B(x_0, r_1)$ with respect to a Riemann metric $d\sigma^2 = e^{u(x, t)} (dx_1^2 + dx_2^2)$ (it equals to $2\pi r_1 e^{\bar{u}(r_1, t)/2}$) and $\ell_0(r_1, t) = 2\pi r_1 b^{1/2} / (1 + \lambda b r_1^2/8)$.

Indeed b is a solution of a quadratic equation

$$A^2 C b^2 + (2AC - 1)b + C = 0,$$

where $A = \lambda r_1^2/8$; $C = e^{\bar{u}(r_1, t)}$.

We choose $b = (1 - 2AC - \sqrt{1 - 4AC})/(2A^2C)$,

Hence by (i) and by Lemma 1 we get

$$(15) \quad u(x_0, t) \leq \ln b(r_1, t).$$

On the other hand, we have by (14), $b(r_1, t) < 4C$. Hence by (15) we obtain

$$u(x_0, t) < 2 \ln 2 + \bar{u}(r_1, t).$$

Thus we have proved Proposition.

Proof of Lemma 2. Set $A = \lambda r_1^2/8$ and $C = e^{\bar{u}(r_1, t)}$.

Since $4AC < 1$, there exists b such that $\ell(r_1) = \ell_0(r_1)$. We choose $b = (1 - 2AC - \sqrt{1 - 4AC})/(2A^2C)$. Then an inequality

$$\lambda a_0(r_1) (= \pi \lambda b r_1^2 / (1 + \lambda b r_1^2 / 8)) < 4\pi$$

holds if and only if $Ab < 1$. On the other hand this inequality

holds if and only if $(1 - X - \sqrt{1 - 2X})/X < 1$ ($X = 2AC$)

i.e., $4AC < 1$. Thus we have proved Lemma 2.

Section 3. Proofs of Theorems 3,4.

Proof of Theorem 3. We prove that for any $k > 0$ there exists $r_k > 0$ such that for $0 < |x| < r_k$ ($x_0 = 0$)

$$\lim_{t \rightarrow T} u^{p-1}(x, t) > k/|x|^2.$$

Let $k > 0$ be fixed. Suppose that there does not exist such r_k . Then there exists a sequence $\{x_\ell\}$ with $x_\ell \rightarrow 0$ such that

$$\lim_{t \rightarrow T} u^{p-1}(x_\ell, t) \leq k/|x_\ell|^2.$$

Since $u_t > 0$, it follows that

$$(16) \quad |x_\ell|^2 u^{p-1}(x_\ell, t) \leq k \quad \text{for } t > 0.$$

Choose $\varepsilon > 0$ with

$$(17) \quad \varepsilon < \beta/k \quad (\beta = 1/(p-1)) \text{ and set } t_\ell \equiv T - \varepsilon|x_\ell|^2. \quad \text{Then}$$

by (16) we get

$$(18) \quad \varepsilon^{-1}(T-t_\ell)u^{p-1}(x_\ell, t_\ell) \leq k.$$

On the other hand by the result of Giga and Kohn it follows that

$$(T-t_\ell)u^{p-1}(x_\ell, t_\ell) \rightarrow \beta.$$

Hence by (18) we get $\beta \leq \varepsilon k$. This contradicts (17).

Proof of Theorem 4. Before proceeding to the proof we recall the definition of the Hausdorff dimension. Let X be a metric space and let Y be a subset of X . For $\varepsilon > 0$ and $D > 0$ we set

$$\mu_{D, \varepsilon}(Y) \equiv \inf_{\text{diam } B_j < \varepsilon} \sum_{Y \subset \cup B_j} (\text{diam } B_j)^D,$$

$$\mu_D(Y) \equiv \lim_{\varepsilon \downarrow 0} \mu_{D, \varepsilon}(Y) = \sup_{\varepsilon > 0} \mu_{D, \varepsilon}(Y).$$

Then $\mu_D(Y)$ is called the D -dimensional Hausdorff measure of Y .

The Hausdorff dimension of Y is defined as follows:

$$\inf \{ D : \mu_D(Y) < \infty \} \quad (= \inf \{ D : \mu_D(Y) = 0 \}).$$

We prove the following proposition. Theorem 4 is an immediate consequence of it.

Proposition. Suppose that u does not blow-up in L^γ as $t \rightarrow T$, i.e., $\lim_{t \rightarrow T} \|u\|_{L^\gamma} < \infty$. Let $D \equiv -2\gamma/(p-1) + n$. Then the D -dimensional Hausdorff measure μ_D of the blow-up points satisfies

$$(19) \quad \mu_D \leq c \lim_{t \rightarrow T} \|u\|_{L^\gamma}^\gamma \quad (c: \text{some constant}).$$

Proof of Proposition. Let $\varepsilon > 0$ be fixed. Let x_0 be a blow-up point. Since Ω is convex, it follows that $x_0 \in \Omega$ ((5)). Hence by Theorem 3, there exists $0 < r_{x_0} \leq 2/3 \varepsilon$ such that

$$B(x_0, r_{x_0}) \equiv \{x : |x - x_0| < r_{x_0}\} \subset \Omega,$$

$$\lim_{t \rightarrow T} u(x, t) \geq |x - x_0|^{-2/(p-1)}, \quad x \in B(x_0, r_{x_0}).$$

Since the set of blow-up points is compact ((5)), there exists a finite subset $\{x_j\}$ of the blow-up points and r_j with $0 < r_j \leq 2/3 \varepsilon$ such that

$$(20) \quad \begin{aligned} & B(x_i, 3/2 r_i) \subset \Omega; \\ & B(x_i, r_i) \cap B(x_j, r_j) = \emptyset \quad (i \neq j); \\ & \{\text{blow-up points}\} \subset \bigcup_i B(x_i, 3/2 r_i); \end{aligned}$$

$$(21) \quad \lim_{t \rightarrow T} u^\gamma(x, t) dx \geq |x - x_0|^{-2\gamma/(p-1)}, \quad x \in B(x_i, r_i).$$

By (21) we get

$$(22) \quad \int_{|x-x_i| \leq r_i} \lim_{t \rightarrow T} u^\gamma(x, t) dx \geq \int_0^{r_i} \int_{S^{n-1}} r^{-2\gamma/(p-1)} r^{n-1} dr d\omega \equiv C r_i^D.$$

Hence by (20), (22) we get

$$\begin{aligned}
 \sum_i (3/2 r_i)^D &\leq (3/2)^D C^{-1} \sum_i \int_{|x-x_i| \leq r_i} \lim_{t \rightarrow T} u^\gamma(x, t) dx \\
 &\leq (3/2)^D C^{-1} \int_{\Omega} \lim_{t \rightarrow T} u^\gamma(x, t) dx \\
 &= (3/2)^D C^{-1} \lim_{t \rightarrow T} \int_{\Omega} u^\gamma(x, t) dx < \infty.
 \end{aligned}$$

Since C is independent of ε , we obtain (19).

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