

固有値が 1 の二階非線形差分方程式に 帰着される、ある関数方程式について

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1 Introduction

We consider the following functional equation

$$\Psi(X(x, \Psi(x))) = Y(x, \Psi(x)), \quad (1.1)$$

where $X(x, y)$, $Y(x, y)$ are function of $(x, y) \in \mathbb{C}^2$, holomorphic in a neighborhood U of $(0, 0)$.

Here we suppose that $X(x, y)$ and $Y(x, y)$ are written in a neighborhood U of $(0, 0)$ as :

$$\begin{cases} X(x, y) = x + y + \sum_{i+j \geq 2} c_{ij} x^i y^j = x + X_1(x, y), \\ Y(x, y) = y + \sum_{i+j \geq 2} d_{ij} x^i y^j = y + Y_1(x, y). \end{cases} \quad (1.2)$$

For the equation (1.1), in which X and Y are written as follows

$$\begin{cases} X(x, y) = \lambda x + \lambda' y + \sum_{i+j \geq 2} c_{ij} x^i y^j = \lambda x + X_1(x, y), \\ Y(x, y) = \mu y + \sum_{i+j \geq 2} d_{ij} x^i y^j = \mu y + Y_1(x, y), \end{cases}$$

we considered the case $|\lambda| > 1, \lambda' = 0$ and $|\lambda| < 1, \lambda' = 0$ in [5], the case $\lambda = \mu$, $|\lambda| \neq 1, \lambda' = 0$ and $\lambda = \mu$, $|\lambda| \neq 1, \lambda' = 1$ in [8], $\lambda = \mu = 1, \lambda' = 0$ in [6], the case $\lambda = 1, |\mu| = 1, \lambda' = 0$ in [7]. In this present paper, we consider the equation (1.1) in the case $\lambda = \mu = \lambda' = 1$.

When we consider a nonlinear simultaneous system of difference equations:

$$\begin{cases} x(t+1) = X(x(t), y(t)), \\ y(t+1) = Y(x(t), y(t)), \end{cases} \quad (1.3)$$

we can reduce it to the following single equation (see [8])

$$x(t+1) = X(x(t), \Psi(x(t))),$$

making use of the equation (1.1). In [3], Kimura consider the first order nonlinear difference equation, in which eigenvalue is equal to 1. If we can have a solution of (1.1), then we have an analytic solution of (1.3) making use of the theorem in [3].

In this present paper we have the following theorem 1.

Theorem 1. Suppose $X(x, y)$ and $Y(x, y)$ are defined in (1.2). Suppose $d_{20} = 0$, $\frac{2c_{20} + d_{11} \pm \sqrt{(2c_{20} - d_{11})^2 + 8d_{30}}}{4} \in \mathbb{R}$, $\frac{2c_{20} + d_{11} + \sqrt{(2c_{20} - d_{11})^2 + 8d_{30}}}{4} < 0$, (1.4)

and we assume the following conditions,

$$(g_0^+(c_{20}, d_{11}, d_{30}) + c_{20})n \neq c_{20} - d_{11} - g_0^+(c_{20}, d_{11}, d_{30}) \quad (1.5)$$

$$(g_0^-(c_{20}, d_{11}, d_{30}) + c_{20})n \neq c_{20} - d_{11} - g_0^-(c_{20}, d_{11}, d_{30}) \quad (1.6)$$

for all $n \in \mathbb{N}$, ($n \geq 4$), where

$$g_0^\pm(c_{20}, d_{11}, d_{30}) = \frac{-(2c_{20} - d_{11}) \pm \sqrt{(2c_{20} - d_{11})^2 + 8d_{30}}}{4}, \quad (1.7)$$

respectively, then we have a formal solution $\Psi(x) = \sum_{n \geq 2}^\infty a_n x^n$ of (1.1). Further, for any κ , $0 < \kappa \leq \frac{\pi}{2}$, there are a $\delta > 0$ and a solution $\Psi(x)$ of (1.1), which is holomorphic and can be expanded asymptotically as

$$\Psi(x) \sim \sum_{n=2}^\infty a_n x^n, \quad (1.8)$$

in the following domain $D(\kappa, \delta)$,

$$D(\kappa, \delta) = \{x; |\arg x| < \kappa, 0 < |x| < \delta\}. \quad (1.9)$$

2 Proof of the theorem

2.1 Determination of a formal solution

At first, we put a formal solution of (1.1) as $\Psi(x) = \sum_{n=1}^\infty a_n x^n$. To determine coefficients a_m , we substitute $\Psi(x) = \sum_{n=1}^\infty a_n x^n$ into (1.1) with (1.2), and have

$$\begin{aligned} & \sum_{n=1}^\infty a_n \left\{ (1 + a_1)x + \sum_{m=2}^\infty a_m x^m + \sum_{i+j \geq 2} c_{ij} \left(\sum_{k_1, \dots, k_j \geq 1} a_{k_1} \cdots a_{k_j} x^{k_1 + \dots + k_j + i} \right) \right\}^n \\ &= \sum_{n=1}^\infty a_n x^n + \sum_{i'+j' \geq 2} d_{i'j'} \left(\sum_{k_1, \dots, k_j \geq 1} a_{k_1} a_{k_2} \cdots a_{k_j} \cdot x^{k_1 + \dots + k_j + i} \right). \end{aligned} \quad (2.1)$$

We compare the coefficients of x^n , ($n = 1, 2, \dots$) in (2.1), then we have

$$\begin{cases} x^1 : a_1 = 0, \\ x^2 : d_{20} = 0, \\ x^3 : a_2\{2a_2 + (2c_{20} - d_{11})\} = d_{30}, \\ x^4 : a_3\{5a_2 + (3c_{20} - d_{11})\} = -2a_2(c_{30} + c_{11}a_2) - a_2(a_2 + c_{20})^2 + d_{21}a_2 + d_{02}a_2^2 + d_{40}, \\ \dots, \\ x^n : a_{n-1}\{(n+1)a_2 + (n-1)c_{20} - d_{11}\} = f_{n-1}(a_2, a_3, \dots, a_{n-2}, c_{ij}, d_{i'j'}), \quad (n \geq 4). \end{cases}$$

Where $f_n(a_2, a_3, \dots, a_{n-2}, c_{ij}, d_{i'j'})$ are polynomials for $a_2, a_3, \dots, a_{n-2}, c_{ij}, d_{i'j'}$, $i + j \leq n - 1$, $i' + j' \leq n - 1$.

From the coefficients of x and x^2 , we have $a_1 = 0$ and $d_{20} = 0$. From the coefficients of x^3 we have

$$a_2 = g_0^+(c_{20}, d_{11}, d_{30}), \quad g_0^-(c_{20}, d_{11}, d_{30}).$$

From the coefficients of x^n ($n \geq 4$), we have

$$a_{n-1}^+\{(g_0^+(c_{20}, d_{11}, d_{30}) + c_{20})n - c_{20} + d_{11} + g_0^+(c_{20}, d_{11}, d_{30})\} = f_{n-1}(a_2, a_3, \dots, a_{n-2}, c_{ij}, d_{i'j'}),$$

or

$$a_{n-1}^-\{(g_0^-(c_{20}, d_{11}, d_{30}) + c_{20})n - c_{20} + d_{11} + g_0^-(c_{20}, d_{11}, d_{30})\} = f_{n-1}(a_2, a_3, \dots, a_{n-2}, c_{ij}, d_{i'j'}).$$

From the following assumption (1.5) and (1.6), for all $n \in \mathbb{N}$, ($n \geq 4$), we have

$$a_{n-1}^\pm = \frac{f_{n-1}(a_2, a_3, \dots, a_{n-2}, c_{ij}, d_{i'j'})}{(n+1)g_0^\pm(c_{20}, d_{11}, d_{30}) + (n-1)c_{20} - d_{11}}, \quad (n \geq 4), \quad (2.2)$$

respectively. Therefore we can decide a formal solution

$$\Psi(x) = \sum_{n=2}^{\infty} a_n x^n. \quad (2.3)$$

2.2 Existence of a solution $\Psi(x)$

In this subsection we prove the existence a solution $\Psi(x)$ of (1.1) under the condition (1.4), (1.5) and (1.6).

2.2.1 Map T

Put

$$u - Y(x, y) = 0, \quad (2.4)$$

$$f(u, x, y) = u - \{y + \sum_{i'+j' \geq 2} d_{i'j'} x^{i'} y^{j'}\}. \quad (2.5)$$

Since $f(0,0,0) = 0$, $\left. \frac{\partial f}{\partial y} \right|_{x=y=u=0} = -1 \neq 0$, thus, we obtain an inverse function $H(x, u)$, such that

$$y = H(x, u) = u + H_1(x, u), \quad H_1(x, u) = \sum_{i+j \geq 2} r_{ij} x^i y^j,$$

defined in $|x| < \epsilon_1$, $|u| < \epsilon_2$, where ϵ_1 and ϵ_2 are small positive constants. The range of $H(x, u)$ contains a disc $|y| < \epsilon_3$. Let $\epsilon = \min(\epsilon_1, \epsilon_2, \epsilon_3)$. Then the equation (1.1) is equivalent to the following equation (2.6)

$$\Psi(x) = H\left(x, \Psi(X(x, \Psi(x)))\right), \quad \text{for } |x| < \epsilon. \quad (2.6)$$

Let κ be a number such that $0 < \kappa < \pi/2$. Take a positive integer $N > 3$. Let $g_N(x) = \sum_{n=2}^N a_n x^n$ be the truncation of the formal solutions of (2.3). Put

$\mathfrak{F} = \mathfrak{F}(N, K, \delta) = \{\phi(x); \phi(x) \text{ is holomorphic and satisfies}$

$$|\phi(x)| \leq K|x|^N \text{ and } |g_N(x)| + K|x|^N < \delta, \text{ in } D(\kappa, \delta)\}$$

where N , K and δ are positive constants to be determined later. Note that K and δ may depend on N , and will be expressed, sometimes, as $K(N)$, $\delta(N)$, respectively.

Put $v = X(x, g_N(x) + \phi(x))$, we have

$$|v| = |x| \cdot |1 + (a_2 + c_{20})\mathbb{R}[x] + \text{higher terms}|, \quad (2.7)$$

$$\arg[v] = \arg[x] + \arg[1 + (a_2 + c_{20})x + x^2 F_0(x, \phi(x))]. \quad (2.8)$$

From the condition (1.4), we have $a_2 + c_{20} \leq \frac{2c_{20} + d_{11} + \sqrt{(2c_{20} - d_{11})^2 + 8d_{30}}}{4} < 0$. Since, $-\pi/2 < \arg[x] < \pi/2$, further if δ is sufficiently small, then we have $|x|/2 < |v| < |x|$ and $|\arg[v]| < |\arg[x]|$, (see Figure 1).

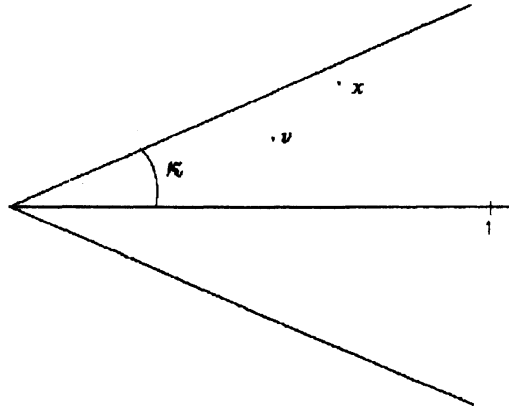


Figure 1

Thus, if $x \in D(\kappa, \delta)$, then $v \in D(\kappa, \delta)$ and $\phi(X(x, g_N(x) + \phi(x)))$ is defined for $\phi(x) \in \mathfrak{F}$. Hence we can define the following map T , for a $\phi(x) \in \mathfrak{F}$,

$$T[\phi](x) = H\left(x, g_N(X(x, g_N(x) + \phi(x))) + \phi(X(x, g_N(x) + \phi(x)))\right) - g_N(x). \quad (2.9)$$

If there is a unique fixed point $\phi_0(x)$ in $\mathfrak{F}$ and further it is independent of N , then we have a solution $\Psi(x)$ of (1.1) which is holomorphic and can be expanded asymptotically as in (1.8) in the domain $D(\kappa, \delta)$, such that $\Psi(x) = g_N(x) + \phi_0(x)$.

2.2.2 Existence of a fixed point of T

From (2.9) we have

$$\begin{aligned} T[\phi](x) &= \{H(x, (g_N + \phi)(X(x, g_N(x) + \phi(x)))) - H(x, g_N(X(x, g_N(x) + \phi(x))))\} \\ &\quad + \{H(x, g_N(X(x, g_N(x) + \phi(x)))) - H(x, g_N(X(x, g_N(x))))\} \\ &\quad + \{H(x, g_N(X(x, g_N(x)))) - g_N(x)\} \\ &= U[\phi](x) + V[\phi](x) + W[\phi](x). \end{aligned} \quad (2.10)$$

Since $g_N(x)$ is the truncated formal solution, we have

$$|W(x)| \leq K_1(N)|x|^{N+1}, \quad (2.11)$$

for a constant $K_1(N)$ which is dependent on N . Put $u_1 = g_N(X(x, g_N(x) + \phi(x)))$, $u_2 = g_N(X(x, g_N(x)))$. Then we have

$$\begin{aligned} |u_1 - u_2| &\leq 2|a_2|(1 + |a_2|)|x|\{|\phi(x)|(1 + K_2(N)|x|)\}, \\ |1 + r_{11}x + r_{02}(u_1 + u_2) + \text{higher terms}| &\leq 2 \end{aligned}$$

Therefore, we have

$$|V[\phi](x)| = |H(x, u_1) - H(x, u_2)| \leq 4(1 + K_2(N)|x|)|a_2|(1 + |a_2|)K|x|^{N+1} \quad (2.12)$$

where $K_2(N)$ is a constant which is dependent on N . Furthermore,

$$|U[\phi](x)| \leq \left| \phi(X(x, g_N(x) + \phi(x))) \right| \int_0^1 \left\{ 1 + |x|(|r_{11}| + K_3(N)|x|) \right\} dt$$

where $K_3(N)$ is a constant which is dependent on N . Here we take δ sufficiently small such that $K_3(N)|x| < 1$, for $x \in D(\kappa, \delta)$, we have the (2.7) in before. Put $\theta = \arg[x]$, then $|\theta| < \kappa < \pi/2$, and $|x| \cos \theta > |x| \cos \kappa$. Since $a_2 + c_{20} < 0$, if δ is sufficiently small, then

$$|v| \leq |x| \cdot \left(1 - \frac{1}{2}|a_2 + c_{20}| \cdot |x| \cos \kappa\right) \leq |x|. \quad (2.13)$$

Hence

$$\left| \phi(X(x, g_N(x) + \phi(x))) \right| \leq K|x|^N \left(1 - \frac{N}{3}|a_2 + c_{20}| \cdot |x| \cos \kappa\right),$$

for sufficiently small δ . Thus,

$$|U[\phi](x)| \leq K|x|^N \left(1 - \frac{N}{3}|a_2 + c_{20}| \cdot |x| \cos \kappa\right) (1 + (|r_{11}| + 1)|x|). \quad (2.14)$$

From (2.11), (2.12) and (2.14), we have

$$\begin{aligned} |T[\phi](x)| &\leq K|x|^N \left\{ \left(\frac{K_1(N)}{K} + 4(1 + K_2(N)\delta)|a_2|(1 + |a_2|) + (|r_{11}| + 1) \right. \right. \\ &\quad \left. \left. - \frac{N}{3}|a_2 + c_{20}| \cos \kappa (1 + (|r_{11}| + 1)|x|) \right) |x| + 1 \right\}. \end{aligned}$$

If we take N to be large enough, then $\frac{N}{3}|a_2 + c_{20}| \cos \kappa \left(1 + (|r_{11}| + 1)|x|\right) > A > 0$, for a positive constant A . Thus

$$|T[\phi](x)| \leq K|x|^N \left\{ \left(\frac{K_1(N)}{K} + 4 \left(1 + K_2(N)\delta\right) |a_2|(1 + |a_2|) + (|r_{11}| + 1) - A \right) |x| + 1 \right\}$$

Let A be sufficiently large, i.e., N be large, then we take δ small enough such that

$$K_2(N)\delta < \frac{A + (|r_{11}| + 1)}{4|a_2|(1 + |a_2|)} - 1, \quad (2.15)$$

i.e., $A - 4|a_2|(1 + |a_2|)(1 + K_2(N)\delta) + (|r_{11}| + 1) > 0$, for the constant $K_2(N)$.

For the N and δ which satisfy the condition (2.15), we take K sufficiently large such that

$$K > \frac{K_1(N)}{A - 4|a_2|(1 + |a_2|)(1 + K_2(N)\delta) + (|r_{11}| + 1)},$$

then we have $|T[\phi](x)| \leq K|x|^N$, i.e., T in (2.9) maps $\mathfrak{F}$ into $\mathfrak{F}$.

$\mathfrak{F}$ is clearly convex, and a normal family by the theorem of Montel. Since T is obviously continuous, we obtain a fixed point $\phi_N(x)$ by Schauder's fixed point theorem [4], we conclude the existence of some fixed point $\phi(x) \in \mathfrak{F}$.

2.2.3 Uniqueness of the fixed point

Next, we show the uniqueness of the fixed point ϕ . Suppose there were two fixed points $\phi_j(x) \in \mathfrak{F}$, $j = 1, 2$. then we have

$$g_N \left(X(x, g_N(x) + \phi_j(x)) \right) + \phi_j \left(X(x, g_N(x) + \phi_j(x)) \right) = Y(x, g_N(x) + \phi_j(x)), \quad (j = 1, 2).$$

Put $v_j = v_j(x) = X(x, g_N(x) + \phi_j(x))$, $j = 1, 2$. Then

$$\begin{cases} g_N(v_1) + \phi_1(v_1) = Y(x, g_N(x) + \phi_1(x)), \\ g_N(v_1) + \phi_2(v_1) = Y(x, g_N(x) + \phi_2(x)). \end{cases} \quad (2.16)$$

$$\begin{aligned} v_1 - v_2 &= (1 + \text{higher order terms of } x)(\phi_1(x) - \phi_2(x)), \\ g_N(v_1) - g_N(v_2) &= (2a_2x + \text{higher order terms of } x)(\phi_1(x) - \phi_2(x)), \end{aligned} \quad (2.17)$$

and

$$\phi_2(v_1) - \phi_2(v_2) = (\phi_1(x) - \phi_2(x))(1 + \text{higher order terms of } x) \int_0^1 \phi_2'(v_2 + t(v_1 - v_2)) dt.$$

Put $D_1 = \overline{D(\kappa/2, (1/2)\delta)}$ and $C = \{\xi \mid |\xi - x| = r = |x| \sin \frac{\kappa}{2}, \text{ for } x \in D_1\}$. Then $C \subset D$ and by the Cauchy's integral formula, we see that, for $x \in D_1 \setminus \{0\}$,

$$|\phi_2'(x)| \leq \frac{1}{2\pi} \int_C \frac{|\phi_2(\xi)|}{|\xi - x|^2} |d\xi| \leq \frac{1}{2\pi} \int_C \frac{K|\xi|^N}{(|x| \sin \frac{\kappa}{2})^2} |d\xi|.$$

Since $|\xi| \leq |x| + |\xi - x| \leq |x|(1 + \sin \frac{\kappa}{2})$, $|\phi_2'(x)| \leq K \frac{(1 + \sin \frac{\kappa}{2})^N}{\sin \frac{\kappa}{2}} |x|^{N-1}$. Thus,

$$|\phi_2(v_1) - \phi_2(v_2)| \leq K \frac{(1 + \sin \frac{\kappa}{2})^N}{\sin \frac{\kappa}{2}} |1 + \text{higher order terms of } x| \cdot |\phi_1(x) - \phi_2(x)| \cdot |x|^{N-1}.$$

Hence, for a fixed $N > 3$,

$$|\phi_2(v_1) - \phi_2(v_2)| \leq K_4(N) |x|^2 |\phi_1(x) - \phi_2(x)|, \quad (2.18)$$

where $K_4(N)$ is a constant which is dependent on N . On the other hand,

$$\begin{aligned} Y(x, g_N(x) + \phi_1(x)) - Y(x, g_N(x) + \phi_2(x)) \\ = (1 + d_{11}x + \text{higher order terms of } x)(\phi_1(x) - \phi_2(x)). \end{aligned} \quad (2.19)$$

For $x \in D_1$, by substituting (2.17)-(2.19) into (2.16), we have

$$\phi_1(v_1) - \phi_2(v_1) = (1 + (d_{11} - 2a_2)x - K_4(N)x^2 + O(x^2))(\phi_1(x) - \phi_2(x)).$$

Write $h(x) = 1 + (d_{11} - 2a_2)x - K_4(N)x^2 + O(x^2)$, then

$$\phi_1(v_1) - \phi_2(v_1) = h(x)(\phi_1(x) - \phi_2(x)). \quad (2.20)$$

Next, for sufficiently small δ , we have $\frac{|x|}{2} < |x|(1 - |a_2 + c_{20}||x|(1 + \frac{\cos \kappa}{2}))$. Since $\cos \kappa < 1 + \frac{\cos \kappa}{2}$, further from (2.13), if we let $p_1 = |a_2 + c_{20}|(1 + \frac{1}{2} \cos \kappa) > 0$ and $p_2 = \frac{1}{2}|a_2 + p_{20}| \cos \kappa$, we have

$$|x|(1 - p_1|x|) \leq |v_1(x)| \leq |x|(1 - p_2|x|), \quad (2.21)$$

for sufficiently small x . In the case where $x \in D(\kappa, \delta)$, then $v_1 \in D(\kappa, \delta)$, and hence, the following estimations hold:

$$|v_1^{n-1}(x)|(1 - p_1|v_1^{n-1}(x)|) \leq |v_1^n(x)| \leq |v_1^{n-1}(x)|(1 - p_2|v_1^{n-1}(x)|), \quad (n \geq 1) \quad (2.22)$$

where $v_1^{k+1}(x) = v(v^k(x))$, $v_1^0(x) = x$. From these inequalities, we have

$$|x| \prod_{k=0}^{n-1} (1 - p_1|v_1^k(x)|) \leq |v_1^n(x)| \leq |x|(1 - p_2|x|) \prod_{k=1}^{n-1} (1 - p_2|v_1^k(x)|). \quad (2.23)$$

On the other hand, from the condition (1.4), we have $d_{11} - 2a_2$, hence, if we take δ sufficiently small, then we have $|h(x)| \geq 1 - 2|d_{11} - 2a_2| \cdot |x|$. Put $b = 2|d_{11} - 2a_2| > 0$, from (2.20), we have the following inequalities:

$$|\phi_1(v_1^n(x)) - \phi_2(v_1^n(x))| \geq (1 - b|v_1^{n-1}(x)|) \cdot |\phi_1(v_1^{n-1}(x)) - \phi_2(v_1^{n-1}(x))|, \quad (n \geq 1).$$

From these, we have

$$|\phi_1(x) - \phi_2(x)| \leq \frac{|\phi_1(v_1^n(x)) - \phi_2(v_1^n(x))|}{\prod_{k=0}^{n-1} (1 - b|v_1^k(x)|)}. \quad (2.24)$$

From the definition of ϕ_1 and ϕ_2 , we have

$$|\phi_1(v_1^n(x)) - \phi_2(v_1^n(x))| \leq 2K|v_1^n(x)|^N, \quad (n = 0, 1, 2, \dots, n-1).$$

Similarly, from (2.23) and (2.24), we have

$$|\phi_1(x) - \phi_2(x)| \leq 2K|x|^N \prod_{k=0}^{n-1} \frac{(1 - p_2|v_1^k(x)|)^N}{1 - b|v_1^k(x)|}.$$

Furthermore, we can take N sufficiently large, for a given δ , such that $p_2N - p_1 - b \geq 0$. Then we have

$$(1 - p_1|v_1^k(x)|) - \frac{(1 - p_2|v_1^k(x)|)^N}{1 - b|v_1^k(x)|} \geq 0.$$

Here, we put $q(t) = t(1 - p_1t)$, $r_0 = r = |x|$, $r_k = q^k(t) = q(q^{k-1}(t)) = q(r_{k-1})$, $k \geq 2$ and $r_1 = q(t)$. From (2.21) and (2.22), by induction, we have $|v_1^k(t)| \leq r_{k-1}$, ($r_0 = r = |x|$). Note that $q'(t) = 1 - 2p_1t$, $q''(t) = -2p_1$, thus for $0 \leq t < \frac{p_1}{2}$, we have $0 < q'(t) < 1$ and $q''(t) < 0$. Then, making use of [1], for $r < \frac{1}{2p_1}$, $r_n = q^n(r) \rightarrow 0$, (as $n \rightarrow \infty$). Hence, from (2.23), we have

$$|x| \prod_{k=0}^{n-1} (1 - p_1|v_1^k(x)|) \leq |v_1^n(x)| \leq r_{n-1} \rightarrow 0, \quad (\text{as } n \rightarrow \infty).$$

Thus,

$$|\phi_1(x) - \phi_2(x)| \leq 2K|x|^{N-1}|x| \prod_{k=0}^{\infty} (1 - p_1|v_1^k(x)|) = 0.$$

Therefore,

$$\phi_1(x) \equiv \phi_2(x) \text{ for } x \in D(\kappa, \delta).$$

From the above discussion, if N is fixed, then there can only be a unique solution $\phi_N(x)$ which is dependent on N such that

$$\Psi_N(x) - g_N(x) = \phi_N(x), \quad |\phi_N(x)| \leq K_N|x|^N,$$

where Ψ_N is a solution of (1.1).

2.2.4 Independence of N

Let $\Psi_{N'}$ and Ψ_N , ($N' > N$) be solutions of (1.1). Put $\delta = \min(\delta_N, \delta_{N'})$ and

$$\Psi_{N'}(x) = g_{N'}(x) + \phi_{N'}(x) = g_N(x) + (g_{N'}(x) - g_N(x) + \phi_{N'}(x)), \text{ for } x \in D(\kappa, \delta).$$

From the uniqueness of $\phi_{N'}$, we see that $g_{N'}(x) - g_N(x) + \phi_{N'}(x) = \phi_N(x)$, for $x \in D(\kappa, \delta)$. Then we can define $\Psi_{N,N'}$ as

$$\Psi_{N,N'} = \begin{cases} \Psi_N & \left(x \in D(\kappa, \delta_N) \right), \\ \Psi_{N'} & \left(x \in D(\kappa, \delta_{N'}) \right), \end{cases}$$

and if $\delta = \min(\delta_N, \delta_{N'})$, we see that

$$\Psi_{N'} = \Psi_N \text{ for } x \in D(\kappa, \delta).$$

In that way, we can obtain a solution Ψ of (1.1), which is independent of N .

2.2.5 Solutions of the equation (1.1)

Take $N' = N + 1$ and $\delta = \min(\delta_N, \delta_{N'})$ in the subsection 2.2.4. Then, for $x \in D(\kappa, \delta)$,

$$|\phi_{N'}(x)| = |\Psi_{N+1}(x) - g_{N+1}(x)| = |\Psi_N(x) - g_{N+1}(x)| \leq (K_N + |a_{N+1}|)|x|^{N+1}.$$

We put $C_N = K_N + |a_{N+1}|$. Then we have

$$|\Psi(x) - g_N(x)| \leq C_N|x|^{N+1}, \text{ for } x \in D(\kappa, \delta),$$

where C_N is a constant and δ is sufficiently small.

This also completes the proof of Theorem 1.

References

- [1] R. L. Devaney, , *An Introduction to Chaotic Dynamical Systems, 2nd ed.*, Addison-Wesley, 1989.
- [2] M. Kuczma, B. Choczewski, R. Ger, *Iterative functional equations*, Encyclopedia of mathematics and its applications, vol. 32, Cambridge University Press, 1990.
- [3] T. Kimura, *On the Iteration of Analytic Functions*, Funkcialaj Ekvacioj, **14**, (1971), 197-238.
- [4] D.R. Smart, *Fixed point theorems*, Cambridge Univ. Press, 1974.
- [5] M. Suzuki, *Holomorphic solutions of some functional equations*, Nihonkai Math. J., **5** , (1994), 109-114.
- [6] M. Suzuki, *Holomorphic solutions of some functional equations II*, Southeast Asian Bulletin of Mathematics, **24** , (2000), 85-94.
- [7] M. Suzuki, *Holomorphic solutions of some functional equations III*, J. of Difference Equations and Applications, **6** , (2000), 369-386.
- [8] M. Suzuki, *Holomorphic solutions of a functional equation and their application to nonlinear second order difference equations* , Aequationes Mathematicae, to appear.

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