

Blocks and strongly p -embedded Frobenius subgroups

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原田耕一郎氏は論文 [H] の中で次の予想を与えています。

予想 G を有限群, p を素数, B を G の p -ブロックとする. もし, $\text{Irr}(B)$ の空でない部分集合 J で, $\omega = \sum_{\chi_j \in J} \chi_j(1)\chi_j$ がすべての p -singular 元上でゼロの値を取るとすると, J は $\text{Irr}(B)$ と一致する.
(この予想の逆の命題は、よく知られた結果です.)

この予想は以下の場合には証明されています。

- (a) B が巡回不足群を持つとき, [H],
- (b) G が p -可解であるとき, [KO],
- (c) G のシロー p -部分群が $p = 2$ で dihedral, semidihedral, または quaternion であるか, $p = 3$ で位数 9 の群であるとき, [K],
- (d) $G = \text{PSL}_2(q)$ [I1],
- (e) $G = \text{PSp}(4, q)$ または $G_2(q)$ で, $(q, 2p) = 1$ となるとき, [I2]
- (f) もし, B のすべての既約ブラウワー指標が liftable であるとき, [I2].

これらの結果は分解行列に関する知識を使って行われています。この論文では、別の方法を紹介します。即ち、シロー p -部分群 P の正規化群 $N_G(P)$ が strongly p -embedded フロベニウス部分群という条件の下に、上の予想が正しいことを分解行列を使わずに証明するわけです。

定理 G を有限群とし, p を素数, P を G のシロー p -部分群とします. もし, P がアーベル群で, $N_G(P)$ が strongly p -embedded フロベニウス部分群とすると, G と p に対して予想が正しい。

定理の証明 Suppose false and let B be a block and $J \subsetneq \text{Irr}(B)$ is a counterexample to the conjecture, that is, $\emptyset \neq J \neq \text{Irr}(B)$ and $\omega = \sum_{\chi \in J} \chi(1)\chi$ vanishes on all p -singular elements. In order to simplify the arguments, we may assume that P is not cyclic by [H]. Set $N = N_G(P)$ and let K be a Frobenius kernel of N and H a complement of K in N , then $K = P \times C$ and $|H| < |P|$, where $C = O_{p'}(C_G(P))$.

Lemma 1. K is a T.I.-set.

[Proof] For $1 \neq c \in C$, $C_G(c)$ contains a strongly p -embedded subgroup $N_G(P) \cap C_G(c) = P \times C_C(c)$. Therefore, $C_G(c) = O_{p'}(C_G(c)) \rtimes P$. Since $\text{rank}(P) \geq 2$, we have

$C_G(c) \subseteq \langle C_G(r) \mid 1 \neq r \in P \rangle \subseteq N$ for any $1 \neq c \in C$. Therefore, for any $1 \neq E \subseteq C$, $P < C_G(E)$ and so $N_G(E) = N_{N_G(E)}(P)C_G(E) \leq N$, which implies that K is a T.I.-set. $\square$

In this paper, ρ_P denotes the regular representation of P . We note that $\mathbb{Z}\rho_P$ is an ideal of the character ring $\text{ch}(P)$ of P . The assumption of the theorem implies $\omega|_P \in \mathbb{Z}\rho_P$. We will divide the proof into three parts.

(1) Assume that $K = P$ and H acts on $P - \{1\}$ transitively.

Then P is an elementary abelian group and $\text{Irr}(P) = \{1_P, \xi_2^h \mid h \in H\}$ for some nontrivial linear character ξ_2 of P . Since $\xi_2^N - (1_P)^N$ vanishes on p -regular elements, we have

$$\langle 1_P^G - \xi_2^G, 1_P^G - \xi_2^G \rangle = \langle 1_P^N - \xi_2^N, 1_P^N - \xi_2^N \rangle = |H| + 1.$$

For any $\mu \in \text{Irr}(G)$, if $\langle \mu, 1_P^G - \xi_2^G \rangle = 0$, then $\langle \mu|_P, 1_P \rangle = \langle \mu|_P, \xi_2^h \rangle$ for all $h \in H$ and so $\mu|_P = a\rho_P$ for some $a \in \mathbb{Z}$. In this case, μ has a trivial defect group and $\{\mu\}$ is a block. Therefore, we may assume that for any $\mu \in B$, $\langle \mu, 1_P^G - \xi_2^G \rangle_G = a_\mu \neq 0$. Then

$$\mu|_P = (a_\mu + t)1_P + t\left(\sum_{h \in H} \xi_2^h\right) \equiv a_\mu 1_P \pmod{\mathbb{Z}\rho_P}$$

for some $t \in \mathbb{Z}$ and so

$$\mu(1) \equiv a_\mu \pmod{|P|}.$$

Hence we have

$$\begin{aligned} \mu(1)\mu|_P &\equiv a_\mu^2 1_P \pmod{(|P|, \rho_P)} \quad \text{and} \\ 0 &\equiv \sum_{\mu \in J} \mu(1)\mu|_P \equiv \sum_{\mu \in J} a_\mu^2 1_P \pmod{(|P|, \rho_P)}, \end{aligned}$$

where $(|P|, \rho_P)$ denotes an ideal of $\text{ch}(P)$ generated by $|P|$ and ρ_P . On the other hand, since $0 \leq \sum_{\mu \in \text{Irr}(B)} a_\mu^2 \leq |H| + 1 \leq |P|$ and $\text{Irr}(B) - J \neq \emptyset$ and $a_\mu \neq 0$ for $\mu \in \text{Irr}(B)$, we have $0 < \sum_{\mu \in J} a_\mu^2 < |P|$, a contradiction.

(2) Assume that $K = P$ and H does not act on $P - \{1\}$ transitively. Set $\text{Irr}(P) = \{1_P, \phi_2^h, \dots, \phi_r^h \mid h \in H\}$. By the theory of exceptional characters, there are $\chi_i \in \text{Irr}(G)$ and $\epsilon \in \{\pm 1\}$ such that

$$(\phi_i - \phi_j)^G = \epsilon(\chi_i - \chi_j)$$

for $i, j \geq 2$. There is also a virtual character A satisfying $\langle A, \chi_i \rangle = 0$ such that

$$(1_P - \phi_2)^G = \epsilon(A - \chi_2 + s \sum_{i=2}^r \chi_i)$$

for some $s \in \mathbb{Z}$ since $\langle (1_P - \phi_2)^G, (\phi_i - \phi_j)^G \rangle = -\delta_{i2} + \delta_{j2}$. For $\mu \in \text{Irr}(G)$, if

$$\langle \mu, (\phi_i - \phi_2)^G \rangle = 0 = \langle \mu, (1_P - \phi_2)^G \rangle$$

for all i , then $\langle \mu|_P, \phi_i \rangle = \langle \mu|_P, 1_P \rangle$ for all i and so

$$\mu|_P \in \mathbb{Z}\rho_P,$$

which implies that $\{\mu\}$ is a block with trivial defect. Therefore we may assume

$$\text{Irr}(B) \subseteq \{\chi_i \mid i = 2, \dots, r\} \cup \text{Irr}(A),$$

where $\text{Irr}(A) = \{\mu \in \text{Irr}(G) \mid \langle \mu, A \rangle \neq 0\}$. Set $\langle \mu, A \rangle = a_\mu$. Since $(\phi_i - \phi_j)^G$ vanishes on all p -regular elements, we have $\langle \omega, (\phi_i - \phi_j)^G \rangle = 0$. Therefore, if J contains some χ_i , then J contains all χ_j . Taking J or $B - J$ as J , we may assume

$$J \subseteq \text{Irr}(A).$$

For any $\mu \in \text{Irr}(A)$, since $\langle \mu, (\phi_i - \phi_j)^G \rangle = 0$ and $\langle \mu, (1_P - \phi_2)^G \rangle = \epsilon a_\mu$, we have

$$\mu|_P \equiv \epsilon a_\mu 1_P \pmod{\rho_P}$$

and so

$$\mu(1) \equiv \epsilon a_\mu \pmod{|P|}.$$

Hence

$$0 \equiv \omega|_P = \sum_{\mu \in J} \mu(1) \mu|_P \equiv \sum a_\mu^2 1_P \pmod{(|P|, \rho_P)},$$

which contradicts to

$$0 < \sum_{\mu \in \text{Irr}(A)} a_\mu^2 = \langle A, A \rangle = |H| < |P|.$$

(3) Assume $C \neq 1$. Since H acts on C fixed point freely, C is nilpotent. Set $\text{Irr}(P) = \{1_P = \phi_1, \phi_2, \dots, \phi_{|P|}\}$ and $\text{Irr}(C) = \{1_C = \xi_1, \xi_2^h, \dots, \xi_s^h \mid h \in H\}$, where $\deg(\xi_2) = 1$. Then $\text{Irr}(K) = \{\phi_i \otimes \xi_1, \phi_i \otimes \xi_2^h, \dots, \phi_i \otimes \xi_s^h \mid h \in H, i = 1, \dots, |P|\}$ and $(\phi_i \otimes \xi_j)^N$ are irreducible for $(i, j) \neq (1, 1)$ since N is a Frobenius group with the kernel K .

By the theory of exceptional characters [S], there are $\chi_{i,j} \in \text{Irr}(G)$ for $(i, j) \neq (1, 1)$ and $\epsilon_j \in \{\pm 1\}$ for j such that

$$(\phi_i \otimes \xi_j)^G - (\phi_h \otimes \xi_k)^G = \epsilon_j(\chi_{i,j} - \chi_{h,k})$$

for $(i, j), (h, k) \neq (1, 1)$ and $\deg(\xi_j) = \deg(\xi_k)$. We note that since $\deg(\xi_1) = \deg(\xi_2) = 1$, $\chi_{i,j}$ are all well-defined for $(i, j) \neq (1, 1)$. We also note that since $\langle (\phi_i \otimes \xi_j)^G - (\phi_h \otimes \xi_j)^G, (\phi_a \otimes \xi_h)^G - (\phi_b \otimes \xi_h)^G \rangle = 0$ for $j \neq h$, we have $\chi_{i,j} \neq \chi_{h,k}$ for $(i, j) \neq (h, k)$ except $j = 1 = k$ and ϕ_i is H -conjugate to ϕ_h . Since $\langle \phi_1 \otimes \xi_1 - \phi_2 \otimes \xi_1, \phi_i \otimes \xi_2 - \phi_h \otimes \xi_2 \rangle = 0$ and $\langle \phi_1 \otimes \xi_1 - \phi_2 \otimes \xi_1, \phi_2 \otimes \xi_1 - \phi_h \otimes \xi_2 \rangle = -1$ for $h \neq 2$, we also have a virtual character A of G satisfying $\langle A, \chi_{2,1} \rangle = 0 = \langle A, \chi_{i,2} \rangle$ for $i = 1, \dots, |P|$ and $r \in \mathbb{Z}$ such that

$$(\phi_1 \otimes \xi_1)^G - (\phi_2 \otimes \xi_1)^G = \epsilon(A + (r-1)\chi_{2,1} + r(\sum_{i=1}^{|P|} \chi_{i,2})).$$

However, since $\langle (\phi_1 \otimes \xi_1)^G - (\phi_h \otimes \xi_1)^G, (\phi_1 \otimes \xi_1)^G - (\phi_h \otimes \xi_1)^G \rangle = 1 + |H|$ and the number of $\chi_{i,2}$ is greater than $|H|$, we have $r = 0$ and $\langle A, A \rangle = |H|$. Moreover, since $\langle (\phi_1 \otimes \xi_1)^G - (\phi_1 \otimes \xi_2)^G, (\phi_i \otimes \xi_h)^G - (\phi_j \otimes \xi_h)^G \rangle = 0$ for $h \geq 2$, if $\chi_{i,h} \in \text{Irr}(A)$, then $\text{Irr}(A)$ contains all $\{\chi_{i,h} \mid i = 1, \dots, |P|\}$, which contradicts to $\langle A, A \rangle = |H| < |P|$. Therefore $\langle A, \chi_{i,h} \rangle = 0$ for $h \geq 2$.

If $\mu \in \text{Irr}(G)$ satisfies $\langle \mu, (\phi_i \otimes \xi_j)^G - (\phi_h \otimes \xi_j)^G \rangle = 0$ for all $j, (h, k)$, then there are $\lambda_j \in \mathbb{Z}$ such that $\mu|_K = \sum_j \lambda_j (\sum_{a \in H} \sum_i (\phi_i \otimes \xi_j)^a)$, which implies $\mu|_P \in \mathbb{Z}\rho_P$ and $\{\mu\}$ is a block as we did in the first part. Therefore, we may assume

$$\text{Irr}(B) \subseteq \{\chi_{i,j} \mid i, j\} \cup \text{Irr}(A).$$

We also have:

Lemma 2. For $(s, t) \neq (1, 1)$,

$$(\chi_{s,t})|_K \equiv \epsilon \sum_{h \in H} (\phi_s \otimes \xi_t)^h \pmod{\rho_P \otimes \text{ch}(K)}.$$

[Proof] For $t \neq j$, since

$$\langle \chi_{s,t}, \chi_{i,j} - \chi_{h,j} \rangle = \langle (\chi_{s,t})|_K, \epsilon_j (\phi_i \otimes \xi_j - \phi_h \otimes \xi_j) \rangle = 0,$$

$\langle (\chi_{s,t})|_K, \phi_i \otimes \xi_j \rangle$ does not depend on the choice of i , say a_j . Since $\delta_{s,i} - \delta_{s,h} = \langle \chi_{s,t}, \chi_{i,t} - \chi_{h,t} \rangle = \langle (\chi_{s,t})|_K, \epsilon_t (\phi_i \otimes \xi_t - \phi_h \otimes \xi_t) \rangle$, $\langle (\chi_{s,t})|_K, \phi_i \otimes \xi_t \rangle$ does not depend on the choice of $i \neq s$, say a_t . Therefore $(\chi_{s,t})|_K = \sum_j a_j \sum_{h \in H} (\rho_P \otimes \xi_j)^h + \epsilon \sum_{h \in H} (\phi_s \otimes \xi_t)^h$. $\square$

Lemma 3. $\chi_{i,j}$ and $\chi_{h,k}$ belong to the same block if and only if $j = k$. In particular, $\{\chi_{i,k} \mid i = 1, \dots, |P|\}$ is a p -block of G for $k \neq 1$.

[Proof] Since $\phi_i \otimes \xi_s - \phi_j \otimes \xi_s$ vanishes on all p -regular elements, so does $\chi_{i,s} - \chi_{j,s}$. Hence $\chi_{i,s}$ and $\chi_{j,s}$ belong to the same block. Let G^0 and N^0 denote the set of all p -regular elements of G and N , respectively. Since $G - G^0$ is a disjoint union of $\{(N - N^0)^g \mid g \in G/N\}$ and we have $N - N^0 = \{(g, c) \mid 1 \neq g \in P, c \in C\}$, if $j \neq k$, then we have:

$$\begin{aligned} \langle \chi_{i,j}, \chi_{h,k} \rangle_{G^0} &= -\langle \chi_{i,j}, \chi_{h,k} \rangle_{G-G^0} \\ &= -\langle (\chi_{i,j})|_N, (\chi_{h,k})|_N \rangle_{N-N^0} \\ &= -\frac{1}{|N|} \sum_{1 \neq g \in P, c \in C} \chi_{i,j}(gc) \overline{\chi_{h,k}(gc)} \\ &= -\frac{1}{|N|} \sum_{1 \neq g \in P} \sum_{c \in C} \chi_{i,j}(gc) \overline{\chi_{h,k}(gc)} \\ &= -\frac{1}{|N|} \sum_{1 \neq g \in P} \sum_{c \in C} \sum_{a \in H} \phi_i^a(g) \xi_j^a(c) \sum_{b \in H} \overline{\phi_h^b(g) \xi_k^b(c)} \\ &= -\frac{1}{|N|} \sum_{1 \neq g \in P} \sum_{a, b \in H} \phi_i^a(g) \overline{\phi_h^b(g)} \sum_{c \in C} \xi_j^a(c) \overline{\xi_k^b(c)} = 0, \end{aligned}$$

since $\sum_{c \in C} \xi_j^a(c) \overline{\xi_k^b(c)} = 0$ for any $a, b \in H$. Therefore, $\chi_{i,j}$ and $\chi_{h,k}$ don't belong to the same block. $\square$

If B is a block $\{\chi_{i,j} \mid i = 1, \dots, |P|\}$ for some $j \neq 1$, then since $\langle \omega, \chi_{i,j} - \chi_{k,j} \rangle = 0$, J contains all $\chi_{k,j}$ and so $J = B$. Therefore, we may assume

$$\text{Irr}(B) \subseteq \text{Irr}(A) \cup \{\chi_{i,1} \mid i = 1, \dots, |P|\}.$$

Since $\langle \omega, \chi_{i,1} - \chi_{j,1} \rangle = 0$, taking J or $\text{Irr}(B) - J$ as J , we may assume that $J \subseteq \text{Irr}(A)$ and $J \cap \{\chi_{i,1} \mid i = 1, \dots, |P|\} = \emptyset$.

Set $a_\mu = \langle \mu, A \rangle$. For $\mu \in J$, since $a_\mu = \epsilon \langle \mu, (\phi_1 \otimes \xi_1)^G - (\phi_h \otimes \xi_1)^G \rangle$ and $\langle \mu, (\phi_i \otimes \xi_j)^G - (\phi_h \otimes \xi_j)^G \rangle = 0$ for $(i, j), (h, j) \neq (1, 1)$, we have

$$\begin{aligned} \mu|_P &\equiv a_\mu 1_P \pmod{\rho_P} & \text{and} \\ \mu(1) &\equiv a_\mu \pmod{|P|} \end{aligned}$$

and so

$$0 \equiv \omega|_P = \sum_{\mu \in J} \mu(1) \mu|_P \equiv \sum_{\mu \in J} a_\mu^2 1_P \pmod{(|P|, \rho_P)}.$$

However, since

$$0 < \sum_{\mu \in J} a_\mu^2 \leq \sum_{\text{all } \mu} a_\mu^2 = |H| < |P|,$$

we have a contradiction.

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