

Necessary and sufficient condition for global stability of a Lotka-Volterra system with two delays

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1. Introduction

We consider the following symmetrical Lotka-Volterra type predator-prey system with two delays τ_1 and τ_2

$$\begin{cases} x'(t) = x(t)[r_1 + ax(t) + \alpha x(t - \tau_1) - \beta y(t - \tau_2)] \\ y'(t) = y(t)[r_2 + ay(t) + \beta x(t - \tau_1) + \alpha y(t - \tau_2)]. \end{cases} \quad (1)$$

The initial condition of (1) is given as

$$\begin{cases} x(s) = \phi(s) \geq 0, -\tau_1 \leq s \leq 0 ; \phi(0) > 0 \\ y(s) = \psi(s) \geq 0, -\tau_2 \leq s \leq 0 ; \psi(0) > 0. \end{cases} \quad (2)$$

Here $a, \alpha, \beta, r_1, r_2, \tau_1$ and τ_2 are constants with $a < 0, \tau_1 \geq 0$ and $\tau_2 \geq 0$, and ϕ, ψ are continuous functions. Obviously, we can take $\beta \geq 0$ without loss of generality. We assume that (1) has a positive equilibrium (x^*, y^*) , that is

$$x^* = \frac{-(a + \alpha)r_1 - \beta r_2}{(a + \alpha)^2 + \beta^2} > 0, \quad y^* = \frac{\beta r_1 - (a + \alpha)r_2}{(a + \alpha)^2 + \beta^2} > 0.$$

The positive equilibrium (x^*, y^*) is said to be globally asymptotically stable if (x^*, y^*) is stable and attracts any solution of (1) with (2). Our purpose is to seek a sharp condition for the global asymptotic stability of (x^*, y^*) for all τ_1 and τ_2 , making the best use of the symmetry of (1). In this paper we give the following necessary and sufficient condition for the global asymptotic stability of (x^*, y^*) for all $\tau_1 \geq 0$ and $\tau_2 \geq 0$,

Theorem. *The positive equilibrium (x^*, y^*) of (1) is globally asymptotically stable for all $\tau_1 \geq 0$ and $\tau_2 \geq 0$ if and only if*

$$\sqrt{\alpha^2 + \beta^2} \leq -a$$

holds.

Gopalsamy [2] showed that if $|\alpha| + |\beta| < -a$ holds, then the positive equilibrium (x^*, y^*) is globally asymptotically stable for all $\tau_1 \geq 0$ and $\tau_2 \geq 0$. It is clear that Theorem improves the Gopalsamy's condition for (1). Recently, Lu and Wang [7] also considered the global asymptotic stability of (x^*, y^*) for (1) with $\alpha = 0$.

When the system (1) has no delay, that is $\tau_1 = \tau_2 = 0$, it is easy to see that (x^*, y^*) is globally asymptotically stable if and only if $a + \alpha < 0$ [cf. Appendix]. So we can see that the condition $\sqrt{\alpha^2 + \beta^2} \leq -a$ in Theorem reflects the delay effects.

In the proof of the sufficiency of Theorem, we use an extended LaSalle's invariance principle (also see [8] and [9] for ODE), by which our proof is more complete than that in [7].

2. Proof of Theorem

In order to consider the global asymptotic stability of the positive equilibrium (x^*, y^*) of (1), we first introduce an extension of the LaSalle's invariance principle.

For some constant $\Delta > 0$, let $C^n = C([- \Delta, 0], R^n)$. Consider the delay differential equations

$$z'(t) = f(z_t) \quad (3)$$

where $z_t \in C^n$ is defined as $z_t(\theta) = z(t + \theta)$ for $-\Delta \leq \theta \leq 0$, $f : C^n \rightarrow R^n$ is completely continuous, and solutions of (3) are continuously dependent on the initial data in C^n . The following lemma is actually a corollary of LaSalle invariance principle and the proof is omitted. (see, for example, [4, 5]).

Lemma. Assume that for a subset G of C^n and $V : G \rightarrow R$,

- (i) V is continuous on G .
- (ii) For any $\phi \in \partial G$ (the boundary of G), the limit $l(\phi)$

$$l(\phi) = \lim_{\substack{\psi \rightarrow \phi \\ \psi \in G}} V(\psi)$$

exists or is $+\infty$.

- (iii) $\dot{V}_{(3)} \leq 0$ on G , where $\dot{V}_{(3)}$ is the upper right-hand derivative of V along the solution of (3).

Let $E = \{\phi \in \bar{G} \mid l(\phi) < \infty \text{ and } \dot{V}(\phi) = 0\}$ and M denote the largest subset in E that is invariant with respect to (3). Then every bounded solution of (3) that remains in G approaches M as $t \rightarrow +\infty$.

Proof of Theorem.

(Sufficiency.) By using the transformation

$$\bar{x} = x - x^*, \quad \bar{y} = y - y^*,$$

the system (1) is reduced to

$$\begin{cases} x'(t) = (x^* + x(t))[ax(t) + \alpha x(t - \tau_1) - \beta y(t - \tau_2)] \\ y'(t) = (y^* + y(t))[ay(t) + \beta x(t - \tau_1) + \alpha y(t - \tau_2)] \end{cases} \quad (4)$$

where we used $x(t)$ and $y(t)$ again instead of $\bar{x}(t)$ and $\bar{y}(t)$ respectively. Define

$$G = \left\{ \phi = (\phi_1, \phi_2) \in C^2 \mid \phi_i(s) + x_i^* \geq 0, \phi_i(0) + x_i^* > 0, i = 1, 2 \right\}$$

where $C^2 = C([- \Delta, 0], R^2)$, $\Delta = \max\{\tau_1, \tau_2\}$ and $(x_1^*, x_2^*) = (x^*, y^*)$. We consider the functional V defined on G ,

$$V(\phi) = -2a \sum_{i=1}^2 \left\{ \phi_i(0) - x_i^* \log \frac{\phi_i(0) + x_i^*}{x_i^*} \right\} + (\alpha^2 + \beta^2) \sum_{i=1}^2 \int_{-\tau_i}^0 \phi_i^2(\theta) d\theta. \quad (5)$$

It is clear that V is continuous on G and that

$$\lim_{\substack{\psi \rightarrow \phi \in \partial G \\ \psi \in G}} V(\psi) = +\infty.$$

Furthermore,

$$\begin{aligned} \dot{V}_{(4)}(\phi) &= -2a [a\phi_1(0) + \alpha\phi_1(-\tau_1) - \beta\phi_2(-\tau_2)] \phi_1(0) \\ &\quad - 2a [a\phi_2(0) + \beta\phi_1(-\tau_1) + \alpha\phi_2(-\tau_2)] \phi_2(0) \\ &\quad + (\alpha^2 + \beta^2) \left\{ [\phi_1^2(0) - \phi_1^2(-\tau_1)] + [\phi_2^2(0) - \phi_2^2(-\tau_2)] \right\} \\ &= -[a\phi_1(0) + \alpha\phi_1(-\tau_1) - \beta\phi_2(-\tau_2)]^2 \\ &\quad - [a\phi_2(0) + \beta\phi_1(-\tau_1) + \alpha\phi_2(-\tau_2)]^2 \\ &\quad - [a^2 - (\alpha^2 + \beta^2)] [\phi_1^2(0) + \phi_2^2(0)] \leq 0 \end{aligned} \quad (6)$$

on G . From (5) and (6), we see that the trivial solution of (4) is stable and that every solution is bounded.

Let

$$E = \{ \phi \in \bar{G} \mid l(\phi) < \infty \text{ and } \dot{V}(\phi) = 0 \},$$

M : the largest subset in E that is invariant with respect to (4).

For $\phi \in M$, the solution $z_t(\phi) = (x(t + \theta), y(t + \theta))$ ($-\Delta \leq \theta \leq 0$) of (4) through $(0, \phi)$ remains in M for $t \geq 0$ and satisfies for $t \geq 0$,

$$\dot{V}_{(4)}(z_t(\phi)) = 0.$$

Hence, for $t \geq 0$,

$$\begin{cases} ax(t) + \alpha x(t - \tau_1) - \beta y(t - \tau_2) = 0 \\ ay(t) + \beta x(t - \tau_1) + \alpha y(t - \tau_2) = 0, \end{cases} \quad (7)$$

which implies that for $t \geq 0$,

$$x'(t) = y'(t) = 0.$$

Thus, for $t \geq 0$,

$$x(t) = c_1, \quad y(t) = c_2 \quad (8)$$

for some constants c_1 and c_2 . From (7) and (8), we have

$$\begin{bmatrix} a + \alpha & -\beta \\ \beta & a + \alpha \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

which implies that $c_1 = c_2 = 0$ by our assumptions and thus we have

$$x(t) = y(t) = 0 \quad \text{for } t \geq 0.$$

Therefore, for any $\phi \in M$, we have

$$\phi(0) = (x(0), y(0)) = 0.$$

By Lemma, any solution $z_t = (x(t + \theta), y(t + \theta))$ tends to M . Thus

$$\lim_{t \rightarrow +\infty} x(t) = \lim_{t \rightarrow +\infty} y(t) = 0.$$

Hence, (x^*, y^*) is globally asymptotically stable for all $\tau_1 \geq 0$ and $\tau_2 \geq 0$.

(*Necessity.*) The proof is by contradiction. Assume the assertion were false. That is, let (x^*, y^*) be globally asymptotically stable for all $\tau_1 \geq 0$ and $\tau_2 \geq 0$ and $\sqrt{\alpha^2 + \beta^2} > -a$.

Linearizing (4), we have

$$\begin{cases} x'(t) = x^*[ax(t) + \alpha x(t - \tau_1) - \beta y(t - \tau_2)] \\ y'(t) = y^*[ay(t) + \beta x(t - \tau_1) + \alpha y(t - \tau_2)]. \end{cases} \quad (9)$$

Now, we will show that there exists a characteristic root λ_0 of (9) such that

$$\operatorname{Re}(\lambda_0) > 0 \quad (10)$$

for some τ_1 and τ_2 , which implies that the trivial solution of (4) is not stable (see [1, p.160, 161]).

When $\alpha \geq -a$, (x^*, y^*) is not globally asymptotically stable in case $\tau_1 = \tau_2 = 0$ [cf. Appendix]. Therefore, we have only to consider the case $\alpha < -a$.

(I) The case $0 < |\alpha| < -a$.

Let $\tau_1 = \tau_2 = \tau$, then the characteristic equation of (9) takes the form

$$\lambda^2 + p\lambda + q + (r + s\lambda)e^{-\lambda\tau} + ve^{-2\lambda\tau} = 0 \quad (11)$$

where $p = -a(x^* + y^*)$, $q = a^2x^*y^*$, $r = 2a\alpha x^*y^*$, $s = -\alpha(x^* + y^*)$ and $v = (\alpha^2 + \beta^2)x^*y^*$.

When $x^* = y^*$, (11) can be factorized as

$$[\lambda - x^*\{a + (\alpha + i\beta)e^{-\lambda\tau}\}][\lambda - x^*\{a + (\alpha - i\beta)e^{-\lambda\tau}\}] = 0. \quad (12)$$

Let us consider the equation

$$\lambda - x^*\{a + (\alpha + i\beta)e^{-\lambda\tau}\} = 0. \quad (13)$$

Set $\alpha = b \cos \theta$ and $\beta = b \sin \theta$, where b and θ are constants with $b \geq 0$. Then, we note that $b > 0$ because of $a < 0$ and $\sqrt{\alpha^2 + \beta^2} > -a$. Substituting $\lambda = iy$ into (13), we have

$$iy - x^*[a + b\{\cos(y\tau - \theta) - i\sin(y\tau - \theta)\}] = 0. \quad (14)$$

By separating the real and imaginary parts of (14), we obtain

$$\begin{cases} bx^* \cos(y\tau - \theta) = -ax^* \\ bx^* \sin(y\tau - \theta) = -y. \end{cases} \quad (15)$$

From (15), we have

$$(bx^*)^2 = (ax^*)^2 + y^2.$$

In order to solve y in (15), define the following function

$$f_1(Y) = Y + (ax^*)^2 - (bx^*)^2 \quad (16)$$

where $Y = y^2$. Then f_1 is an increasing linear function and

$$f_1(0) = x^{*2}\{a^2 - (\alpha^2 + \beta^2)\} < 0.$$

Thus, it follows that there exists a positive root Y_0 of $f_1(Y) = 0$. Substituting y_0 , which satisfies $Y_0 = y_0^2$, into (15), we can get τ_0 such that (13) has a characteristic root iy_0 when $\tau = \tau_0$.

Furthermore, taking the derivative of λ with τ on (13), we have

$$\frac{d\lambda}{d\tau} = \frac{-x^*be^{i\theta}\lambda e^{-\lambda\tau}}{1 + x^*b\tau e^{i\theta}e^{-\lambda\tau}}.$$

Using (13), we obtain

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{1}{-\lambda(\lambda - x^*a)} - \frac{\tau}{\lambda}.$$

Hence,

$$\begin{aligned} \operatorname{sign} \left[\operatorname{Re} \left(\frac{d\lambda}{d\tau} \Big|_{\lambda=iy_0, \tau=\tau_0} \right) \right] &= \operatorname{sign} \left[\operatorname{Re} \left(\left(\frac{d\lambda}{d\tau} \right)^{-1} \Big|_{\lambda=iy_0, \tau=\tau_0} \right) \right] \\ &= \operatorname{sign} \left[\operatorname{Re} \left(\frac{1}{-iy_0(iy_0 - x^*a)} - \frac{\tau_0}{iy_0} \right) \right] = \operatorname{sign} \left[\operatorname{Re} \left(\frac{1}{y_0^2 + iy_0x^*a} \right) \right] > 0, \end{aligned}$$

which implies that (10) holds. Therefore, the trivial solution of (4) is not stable, that is, (x^*, y^*) is not stable near τ_0 , which is a contradiction.

When $x^* \neq y^*$, (11) cannot be factorized as (12). Substituting $\lambda = iy$ into (11), we have

$$(-y^2 + piy + q)e^{iy\tau} + r + siy + ve^{-iy\tau} = 0. \quad (17)$$

By separating the real and imaginary parts of (17), we have

$$\begin{cases} [(-y^2 + q)^2 - v^2 + p^2y^2] \cos(y\tau) = (r - sp)y^2 - r(q - v) \\ [(-y^2 + q)^2 - v^2 + p^2y^2] \sin(y\tau) = sy^3 + [rp - s(q + v)]y \end{cases} \quad (18)$$

and thus

$$[(-y^2 + q)^2 - v^2 + p^2y^2]^2 = [(r - sp)y^2 - r(q - v)]^2 + [sy^3 + [rp - s(q + v)]y]^2.$$

Define the following function

$$\begin{aligned} f_2(Y) &= [(-Y + q)^2 - v^2 + p^2Y]^2 - [(r - sp)Y - r(q - v)]^2 \\ &\quad - Y[sY + rp - s(q + v)]^2 \end{aligned} \quad (19)$$

where $Y = y^2$, then f_2 is a quartic fuction such that $f_2 \rightarrow +\infty$ as $|Y| \rightarrow +\infty$. Since

$$f_2(0) = [a^2 - (\alpha^2 + \beta^2)]^2[(a + \alpha)^2 + \beta^2][(a - \alpha)^2 + \beta^2](x^*y^*)^4 > 0,$$

we cannot immediately find positive zeros of (19) and so we have to investigate f_2 in more detail. Define

$$\begin{aligned} F(Y) &= [(-Y + q)^2 - v^2 + p^2Y]^2 \\ G(y) &= -[(r - sp)Y - r(q - v)]^2 \\ H(y) &= -Y[sY + rp - s(q + v)]^2, \end{aligned}$$

then $f_2 = F + G + H$. It is easy to see that positive zeroes of F , G and H are mutually different as long as $x^* \neq y^*$. Hence, the value of f_2 at the positive zero of F is negative, which, together with $f_2(0) > 0$, implies that there exists a positive root of $f_2(Y) = 0$. It is also clear that there exists another positive root of $f_2(Y) = 0$ because $f_2 \rightarrow +\infty$ as $Y \rightarrow +\infty$. Thus, one of the two positive roots is a simple root at least.

Let Y_0 be such a simple root. Substituting y_0 , which satisfies $Y_0 = y_0^2$, into (18), we can get some τ such that (11) has a characteristic root iy_0 at τ . We note that iy_0 is a simple root of (11) because Y_0 is a simple root of $f_2(Y) = 0$.

Furthermore, taking the derivative of λ with τ on (11), we have

$$\frac{d\lambda}{d\tau} = \frac{-2\lambda(\lambda^2 + p\lambda + q) - \lambda(r + s\lambda)e^{-\lambda\tau}}{2\lambda + p + 2\tau(\lambda^2 + p\lambda + q) + e^{-\lambda\tau}[s + \tau(r + s\lambda)]},$$

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{2\lambda + p + se^{-\lambda\tau}}{-2\lambda(\lambda^2 + p\lambda + q) - \lambda(r + s\lambda)e^{-\lambda\tau}} - \frac{\tau}{\lambda}.$$

Hence, we have

$$\begin{aligned} \text{sign} \left[\text{Re} \left(\frac{d\lambda}{d\tau} \Big|_{\lambda=iy_0} \right) \right] &= \text{sign} \left[\text{Re} \left(\left(\frac{d\lambda}{d\tau} \right)^{-1} \Big|_{\lambda=iy_0} \right) \right] \\ &= \text{sign} \left[\text{Re} \left(\frac{2iy_0 + p + se^{-iy_0\tau}}{-2iy_0(-y_0^2 + piy_0 + q) - iy_0(r + siy_0)e^{-iy_0\tau}} - \frac{\tau}{iy_0} \right) \right] \\ &= \text{sign} \left[\text{Re} \left\{ \left(\frac{2iy_0 + p + se^{-iy_0\tau}}{-2iy_0(-y_0^2 + piy_0 + q) - iy_0(r + siy_0)e^{-iy_0\tau}} \right)^{-1} \right\} \right] \\ &= \text{sign} \left[1 + \frac{(a^2 + a\alpha \cos(y_0\tau))(x^* - y^*)^2}{(p + s \cos(y_0\tau))^2 + (2y_0 - s \sin(y_0\tau))^2} \right]. \end{aligned} \quad (20)$$

Since

$$(a^2 + a\alpha \cos(y_0\tau))(x^* - y^*)^2 \geq a(a + |\alpha|)(x^* - y^*)^2 > 0,$$

the last expression in (20) is positive. This implies that (10) holds, which is a contradiction.

(II) The case $\alpha = 0$.

Let $\tau_1 = \tau_2 = \tau$, then the characteristic equation of (9) takes the form

$$\lambda^2 + p\lambda + q + ve^{-2\lambda\tau} = 0. \quad (21)$$

Substituting $\lambda = iy$ into (21), we have

$$-y^2 + piy + q + ve^{-2iy\tau} = 0. \quad (22)$$

By separating the real and imaginary parts of (22), we have

$$\begin{cases} v \cos(2y\tau) = y^2 - q \\ v \sin(2y\tau) = py \end{cases} \quad (23)$$

and

$$v^2 = (y^2 - q)^2 + (py)^2.$$

Define the following function

$$f_3(Y) = (Y - q)^2 + p^2Y - v^2 \quad (24)$$

where $Y = y^2$, then f_3 is a downwards convex quadratic function and

$$f_3(0) = (a^4 - \beta^4)x^{*2}y^{*2} < 0.$$

Thus, it follows that there exists a positive simple root Y_0 of $f_3(Y) = 0$. Substituting y_0 , which satisfies $Y_0 = y_0^2$, into (23), we can get some τ such that (21) has a characteristic root iy_0 at τ . Here iy_0 is a simple root of (21) by the same reason as above.

Taking the derivative of λ with τ on (21), we have

$$\begin{aligned} \frac{d\lambda}{d\tau} &= \frac{2v\lambda e^{-2\lambda\tau}}{2\lambda + p - 2v\tau e^{-2\lambda\tau}}, \\ \left(\frac{d\lambda}{d\tau}\right)^{-1} &= \frac{2\lambda + p}{2\lambda(-\lambda^2 - p\lambda - q)} - \frac{\tau}{\lambda}. \end{aligned}$$

Hence,

$$\begin{aligned} \text{sign} \left[\text{Re} \left(\frac{d\lambda}{d\tau} \Big|_{\lambda=iy_0} \right) \right] &= \text{sign} \left[\text{Re} \left(\left(\frac{d\lambda}{d\tau} \right)^{-1} \Big|_{\lambda=iy_0} \right) \right] \\ &= \text{sign} \left[\text{Re} \left(\frac{2iy_0 + p}{2iy_0(y_0^2 - piy_0 - q)} - \frac{\tau}{iy_0} \right) \right] \\ &= \text{sign} \left[\text{Re} \left(\frac{2iy_0 + p}{2y_0[py_0 + i(y_0^2 - q)]} \right) \right] \\ &= \text{sign} [2y_0^2 + a^2(x^{*2} + y^{*2})] > 0. \end{aligned}$$

This implies that (10) holds, which is a contradiction.

(III) The case $\alpha \leq a$.

Let $\tau_1 = \tau$ and $\tau_2 = 0$, then the characteristic equation of (9) takes the form

$$\lambda^2 + \tilde{p}\lambda + \tilde{q} + (\tilde{r} + \tilde{s}\lambda)e^{-\lambda\tau} = 0 \quad (25)$$

where $\tilde{p} = -ax^* - (a + \alpha)y^*$, $\tilde{q} = a(a + \alpha)x^*y^*$, $\tilde{r} = [\alpha(a + \alpha) + \beta^2]x^*y^*$, $\tilde{s} = -\alpha x^*$. Let us use p , q , r and s again instead of $\tilde{p}$, $\tilde{q}$, $\tilde{r}$ and $\tilde{s}$ respectively. Substituting $\lambda = iy$ into (25), we have

$$-y^2 + piy + q + (r + siy)e^{-iy\tau} = 0. \quad (26)$$

By separating the real and imaginary parts of (26), we have

$$\begin{cases} (r^2 + s^2y^2) \cos(y\tau) = r(y^2 - q) - spy^2 \\ (r^2 + s^2y^2) \sin(y\tau) = sy(y^2 - q) + pry \end{cases} \quad (27)$$

and

$$[r^2 + s^2 y^2]^2 = [r(y^2 - q) - spy^2]^2 + [sy(y^2 - q) + pry]^2.$$

Define the following function

$$f_4(Y) = Y[s(Y - q) + pr]^2 + [r(Y - q) - spY]^2 - [r^2 + s^2 Y]^2 \quad (28)$$

where $Y = y^2$, then f_4 is an upwards cubic function to the right and

$$f_4(0) = [\alpha(a + \alpha) + \beta^2]^2[(a + \alpha)^2 + \beta^2][a^2 - (\alpha^2 + \beta^2)](x^* y^*)^4 < 0.$$

Thus, there can exist some positive roots of $f_4(Y) = 0$. Now, let us show that there exists a simple root in such positive roots. We see that

$$f'_4(Y) = 3s^2 Y^2 + 2[s^2(p^2 - 2q - s^2) + r^2]Y + s^2(q^2 - 2r^2) + r^2(p^2 - 2q)$$

and

$$f''_4(Y) = 6s^2 Y + 2[s^2(p^2 - 2q - s^2) + r^2].$$

Let $f''_4(Y) = 0$, then

$$3s^2 Y + [s^2(p^2 - 2q - s^2) + r^2] = 0,$$

and thus we have

$$\begin{aligned} -3s^2 f'_4(Y) &= [s^2(p^2 - 2q - s^2) + r^2]^2 - 3s^2 [s^2(q^2 - 2r^2) + r^2(p^2 - 2q)] \\ &= x^{*4} y^{*2} [\alpha^2(4\alpha^2 - a^2)x^{*2} + \{\alpha(a + \alpha) + \beta^2\}^2 y^{*2}] \\ &\quad \times [\{\alpha(a + \alpha) + \beta^2\}^2 - \alpha^2(a + \alpha)^2] \\ &\quad + \alpha^4 x^{*4} [(a^2 - \alpha^2)x^{*2} - (a + \alpha)^2 y^{*2}]^2. \end{aligned} \quad (29)$$

Since $\alpha \leq a < 0$, (29) is positive. This prove that there exists no triple root of $f_4(Y) = 0$, which implies that there exists at least a positive simple root Y_0 of $f_4(Y) = 0$.

Substituting y_0 , which satisfies $Y_0 = y_0^2$, into (27), we can get some τ such that (25) has a characteristic root iy_0 at τ . Here again iy_0 is a simple root of (25).

Taking the derivative of λ with τ on (25), we have

$$\begin{aligned} \frac{d\lambda}{d\tau} &= \frac{\lambda(r + s\lambda)e^{-\lambda\tau}}{2\lambda + p + e^{-\lambda\tau}[s - \tau(r + s\lambda)]}, \\ \left(\frac{d\lambda}{d\tau}\right)^{-1} &= \frac{2\lambda + p + se^{-\lambda\tau}}{\lambda(r + s\lambda)e^{-\lambda\tau}} - \frac{\tau}{\lambda} \\ &= \frac{2\lambda + p}{-\lambda(\lambda^2 + p\lambda + q)} + \frac{s}{\lambda(r + s\lambda)} - \frac{\tau}{\lambda}. \end{aligned}$$

Hence, we have

$$\begin{aligned}
 \operatorname{sign} \left[\operatorname{Re} \left(\frac{d\lambda}{d\tau} \Big|_{\lambda=iy_0} \right) \right] &= \operatorname{sign} \left[\operatorname{Re} \left(\left(\frac{d\lambda}{d\tau} \right)^{-1} \Big|_{\lambda=iy_0} \right) \right] \\
 &= \operatorname{sign} \left[\operatorname{Re} \left(\frac{2iy_0 + p}{-iy_0(-y_0^2 + piy_0 + q)} + \frac{s}{iy_0(r + siy_0)} - \frac{\tau}{iy_0} \right) \right] \\
 &= \operatorname{sign} \left[\frac{s^2y_0^4 + 2r^2y_0^2 - s^2q^2 - 2r^2q + p^2r^2}{[(py_0)^2 + (y_0^2 - q)^2][r^2 + (sy_0)^2]} \right].
 \end{aligned} \tag{30}$$

Since

$$\begin{aligned}
 &-s^2q^2 - 2r^2q + p^2r^2 \\
 &= [a^2x^{*2} + (a + \alpha)^2y^{*2}][\alpha(a + \alpha) + \beta^2]^2x^{*2}y^{*2} - a^2\alpha^2(a + \alpha)^2x^{*4}y^{*2} \\
 &\geq [a^2x^{*2} + (a + \alpha)^2y^{*2}]\alpha^2(a + \alpha)^2x^{*2}y^{*2} - a^2\alpha^2(a + \alpha)^2x^{*4}y^{*2} \\
 &= \alpha^2(a + \alpha)^4x^{*2}y^{*4} > 0,
 \end{aligned}$$

the last expression in (30) is positive. This implies that (10) holds, which is a contradiction. This completes the proof.

Here, we give the following three portraits of the trajectory of (1) with (2), drawn by a computer using the Runge-Kutta method, to illustrate Theorem ($r_1 = 10$, $r_2 = -10$).

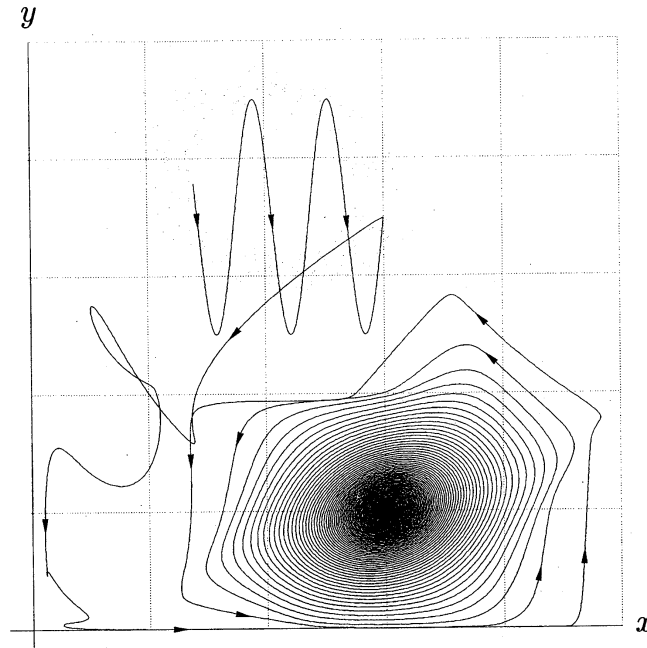


Fig.1 $a = -5$, $\alpha = 3$, $\beta = 3.99$ ($\sqrt{\alpha^2 + \beta^2} < -a$)
 $\tau_1 = 1$, $\tau_2 = 2$, $(\phi, \psi) = (3 + 0.8t, 3.5 + \sin(8t))$

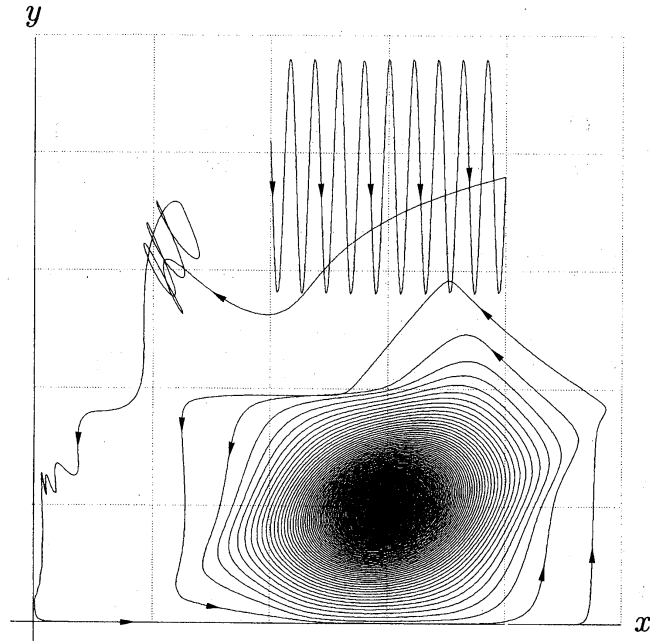


Fig.2 $a = -5, \alpha = 3, \beta = 4$ ($\sqrt{\alpha^2 + \beta^2} = -a$)
 $\tau_1 = 1, \tau_2 = 2, (\phi, \psi) = (4 + t, 3.8 + \sin(30t))$

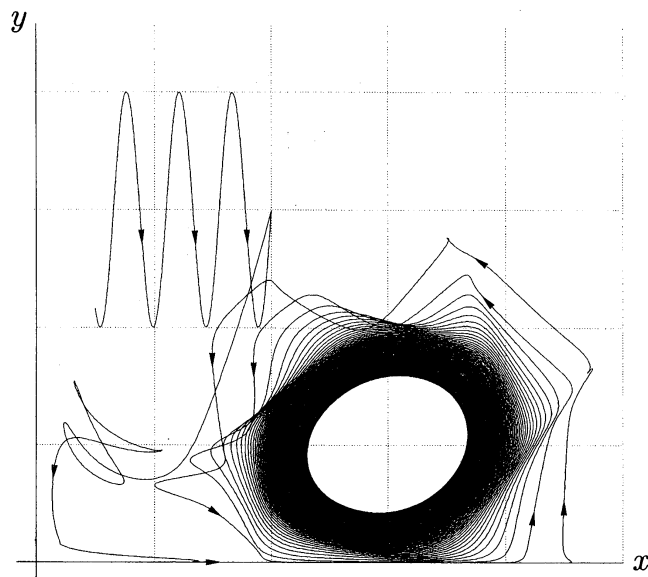


Fig.3 $a = -5, \alpha = 3, \beta = 4.01$ ($\sqrt{\alpha^2 + \beta^2} > -a$)
 $\tau_1 = 2, \tau_2 = 3, (\phi, \psi) = (2 + 0.5t, 3 + \sin(7t))$

3. Appendix

When $\tau_1 = \tau_2 = 0$, the system (1) become

$$\begin{cases} x'(t) = x(t)[r_1 + (a + \alpha)x(t) - \beta y(t)] \\ y'(t) = y(t)[r_2 + \beta x(t) + (a + \alpha)y(t)]. \end{cases} \quad (31)$$

By using the transformation

$$\bar{x} = x - x^*, \quad \bar{y} = y - y^*,$$

(31) is reduced to

$$\begin{cases} x'(t) = (x^* + x(t))[(a + \alpha)x(t) - \beta y(t)] \\ y'(t) = (y^* + y(t))[\beta x(t) + (a + \alpha)y(t)], \end{cases} \quad (32)$$

where we used $x(t)$ and $y(t)$ again instead of $\bar{x}(t)$ and $\bar{y}(t)$, respectively. Consider the following Liapunov function

$$V(x, y) = \left(x - x^* \log \frac{x + x^*}{x^*} \right) + \left(y - y^* \log \frac{y + y^*}{y^*} \right) \quad (33)$$

for $x > -x^*$ and $y > -y^*$, then V is positive definite. Calculating the derivative of V along the solution of (32), we have

$$\dot{V}_{(32)}(x, y) = (a + \alpha)(x^2 + y^2).$$

Clearly, $\dot{V}_{(32)}$ is negative definite if and only if $a + \alpha < 0$ holds. The well-known Liapunov theorem shows that the origin $(0, 0)$ is globally asymptotically stable if and only if $a + \alpha < 0$ holds.

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