

Vertex Operators of the Elliptic Quantum Toroidal Algebra and the Elliptic Stable Envelopes

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1 Introduction

This is a review of the works given in [12]. The main result is a new formulation of the vertex operators of the elliptic quantum toroidal algebra (EQTA) $U_{t_1, t_2, p}(\mathfrak{gl}_{1, tor})$ by combining its representations and the notions of the elliptic stable envelopes (ESE) for the instanton moduli space $\mathcal{M}(n, r)$.

The EQTA $U_{t_1, t_2, p}(\mathfrak{gl}_{1, tor})$ is an elliptic quantum group associated with the toroidal algebra of type $\mathfrak{gl}_1$ [11]. The Hopf algebroid structure associated with the Drinfeld comultiplication allows us to construct two types of vertex operators, the type I and the type II dual, as intertwining operators of $U_{t_1, t_2, p}(\mathfrak{gl}_{1, tor})$ -modules [11]. It turns out that they give a realization of the affine quiver W -algebra associated with the Jordan quiver varieties [8]. In addition, the same vertex operators realize the refined topological vertices [7], which are relevant to the calculation of the instanton partition functions of the 5d and 6d lift of the 4d $\mathcal{N} = 2^*$ $U(M)$ gauge theory [13, 14]. However their relations to the elliptic stable envelopes [1] and to the vertex functions [15] of the corresponding quiver variety were missing. These relations have been observed in the case of the elliptic quantum group $U_{q, p}(\widehat{\mathfrak{sl}}_N)$ [9, 10] and expected to be possessed in the intertwining operators w.r.t. the standard comultiplication, which preserves the RLL -relation.

We here propose a new formulation of the vertex operators. We realize both the type I and the type II dual vertex operators as screened vertex operators, i.e. operator valued integrals with the ESE's for $E_T(\mathcal{M}(n, r))$ as their integration kernels. We then make several checks on their consistency such as

- a derivation of the shuffle product formula of ESE's [2] by considering a composition of the vertex operators
- a construction of the K-theoretic vertex functions for $\mathcal{M}(n, r)$ as the highest to

highest expectation values of the corresponding vertex operators

- exchange relations among the vertex operators, whose coefficients are given by the elliptic instanton R -matrices defined as transition matrices of the ESE's for $\text{Hilb}^n(\mathbb{C}^2)$
- a construction of the L -operator $L^+(u)$ satisfying the RLL -relation by combining the type I and the type II dual vertex operators
- exchange relations between the L -operator and the vertex operators.

The last relations indicates that our new vertex operators are the intertwining operators of the $U_{t_1, t_2, p}(\mathfrak{gl}_{1, \text{tor}})$ -modules w.r.t. the standard comultiplication.

2 Elliptic Quantum Toroidal Algebra $U_{t_1, t_2, p}(\mathfrak{gl}_{1, \text{tor}})$

The elliptic quantum toroidal algebra $\mathcal{U}_{t_1, t_2, p}(\mathfrak{gl}_{1, \text{tor}})$ was introduced in [11]. The parameters $t_1, t_2, \hbar = t_1 t_2$ in this paper correspond to $q^{-1}, t, t/q$ in [11], respectively.

2.1 Definition

Let us consider the Heisenberg algebras generated by $c, \Lambda_0, c^\perp, \Lambda_0^\perp, h, \alpha, P, Q$ satisfying the commutation relations

$$[c, \Lambda_0] = 1 = [c^\perp, \Lambda_0^\perp], \quad [h, \alpha] = 2 = [P, Q], \quad (2.1)$$

the others are zero. We set $\gamma = \hbar^{c^\perp/2}$, $C = \hbar^{c/2}$, $\mathfrak{z}^* = \hbar^P$ and $\mathfrak{z} = \hbar^{P+c}$. We call $\mathfrak{z}^*$ the dynamical parameter. Let $\mathbb{F}$ be the field of meromorphic functions of $\mathfrak{z}$ and $\mathfrak{z}^*$. We have

$$g(\mathfrak{z}, \mathfrak{z}^*) e^{\Lambda_0 - Q} = e^{\Lambda_0 - Q} g(\mathfrak{z}, \mathfrak{z}^* \hbar^{-2}), \quad g(\mathfrak{z}, \mathfrak{z}^*) e^{-\Lambda_0} = e^{-\Lambda_0} g(\mathfrak{z} \hbar^{-2}, \mathfrak{z}^*) \quad \forall g(\mathfrak{z}, \mathfrak{z}^*) \in \mathbb{F}.$$

Set

$$\begin{aligned} \kappa_m &= -(1 - t_1^m)(1 - t_2^m)(1 - \hbar^{-m}), \\ G^\pm(z) &= (1 - t_1^{\mp 1} z)(1 - t_2^{\mp 1} z)(1 - \hbar^{\pm 1} z). \end{aligned}$$

Definition 2.1. The elliptic quantum toroidal algebra $\mathcal{U} = U_{t_1, t_2, p}(\mathfrak{gl}_{1, \text{tor}})$ is a topological associative algebra over $\mathbb{F}[[p]]$ generated by $\alpha_m, x_n^\pm$, ($m \in \mathbb{Z} \setminus \{0\}, n \in \mathbb{Z}$) and $C, \gamma^{1/2}$. Let $x^\pm(z), \psi^\pm(z)$ be the following generating functions¹.

$$\begin{aligned} x^\pm(z) &:= \sum_{n \in \mathbb{Z}} x_n^\pm z^{-n}, \\ \psi^+(z) &:= C \exp \left(- \sum_{m>0} \frac{p^m}{1-p^m} \alpha_{-m} (\gamma^{-1/2} z)^m \right) \exp \left(\sum_{m>0} \frac{1}{1-p^m} \alpha_m (\gamma^{-1/2} z)^{-m} \right), \\ \psi^-(z) &:= C^{-1} \exp \left(- \sum_{m>0} \frac{1}{1-p^m} \alpha_{-m} (\gamma^{1/2} z)^m \right) \exp \left(\sum_{m>0} \frac{p^m}{1-p^m} \alpha_m (\gamma^{1/2} z)^{-m} \right). \end{aligned}$$

We call them the elliptic currents. The defining relations are given by

$$\begin{aligned} &\gamma^{1/2}, C : \text{central}, \\ &[\alpha_m, \alpha_n] = -\frac{\kappa_m}{m} (\gamma^m - \gamma^{-m}) \gamma^{-m} \frac{1-p^m}{1-p^{*m}} \delta_{m+n,0}, \\ &[\alpha_m, x^+(z)] = -\frac{\kappa_m}{m} \frac{1-p^m}{1-p^{*m}} \gamma^{-m} z^m x^+(z) \quad (m \neq 0), \\ &[\alpha_m, x^-(z)] = \frac{\kappa_m}{m} z^m x^-(z) \quad (m \neq 0), \\ &[x^+(z), x^-(w)] = -\frac{(1-t_1)(1-t_2)}{(1-\hbar)} (\delta(\gamma^{-1} z/w) \psi^+(\gamma^{1/2} w) - \delta(\gamma z/w) \psi^-(\gamma^{-1/2} w)), \\ &z^3 G^+(w/z) g(w/z; p^*) x^+(z) x^+(w) = -w^3 G^+(z/w) g(z/w; p^*) x^+(w) x^+(z), \\ &z^3 G^-(w/z) g(w/z; p)^{-1} x^-(z) x^-(w) = -w^3 G^-(z/w) g(z/w; p)^{-1} x^-(w) x^-(z), \\ &g(w/z; p^*) g(u/w; p^*) g(u/z; p^*) \left(\frac{w}{u} + \frac{w}{z} - \frac{z}{w} - \frac{u}{w} \right) x^+(z) x^+(w) x^+(u) \\ &+ \text{permutations in } z, w, u = 0, \\ &g(w/z; p)^{-1} g(u/w; p)^{-1} g(u/z; p)^{-1} \left(\frac{w}{u} + \frac{w}{z} - \frac{z}{w} - \frac{u}{w} \right) x^-(z) x^-(w) x^-(u) \\ &+ \text{permutations in } z, w, u = 0, \end{aligned}$$

where we set $p^* = p\gamma^{-2}$ and

$$g(z; s) = \exp \left(\sum_{m>0} \frac{\kappa_m}{m} \frac{s^m}{1-s^m} z^m \right) \in \mathbb{C}[[z]]$$

for $s = p, p^*$. The dynamical parameters $\mathfrak{z}, \mathfrak{z}^*$ commute with $\alpha_m, x_n^\pm$.

¹Our $x^\pm(z)$ is $x^\pm(\gamma^{1/2} z)$ in [11].

It is convenient to set

$$\alpha'_m = \frac{1 - p^{*m}}{1 - p^m} \gamma^m \alpha_m \quad (m \in \mathbb{Z}_{\neq 0}).$$

Through this paper, we treat $t_1, t_2, p, p^* = p\gamma^{-2}$ as generic complex numbers with $|t_1|, |t_2|, |p|, |p^*| < 1$.

1. In particular, we have

$$g(z; p) = \frac{(pt_1^{-1}z; p)_\infty}{(pt_1z; p)_\infty} \frac{(pt_2^{-1}z; p)_\infty}{(pt_2z; p)_\infty} \frac{(p\hbar z; p)_\infty}{(p\hbar^{-1}z; p)_\infty},$$

where

$$(z; p)_\infty = \prod_{n=0}^{\infty} (1 - zp^n) \quad |z| < 1.$$

2.2 Representations of $U_{t_1, t_2, p}(\mathfrak{gl}_{1, \text{tor}})$

Let $\mathcal{V}$ be a $\mathcal{U}$ -module. For $(k, l) \in \mathbb{C}^2$, we say that $\mathcal{V}$ has level (k, l) , if the central elements γ and C act as²

$$\gamma \cdot \xi = \hbar^{k/2} \xi, \quad C \cdot \xi = \hbar^l \xi \quad \forall \xi \in \mathcal{V}.$$

2.2.1 The level- $(1, N)$ representation

Let h, α satisfy $[h, \alpha] = 1$ and commuting with the other generators. Define for $v \in \mathbb{C}^\times$

$$|0\rangle_v^{(1, N)} := v^\alpha e^{\Lambda_0^\perp} e^{N\Lambda_0} 1. \quad (2.2)$$

We assume $\gamma^{1/2} \cdot 1 = C \cdot 1 = e^{\pm h} \cdot 1 = 1$ and $e^Q \cdot 1 = e^Q 1$. One has

$$e^{\pm h} u^{\pm c} \cdot |0\rangle_v^{(1, N)} = v^{\pm 1} u^{\pm N} |0\rangle_{vu^{\pm 1}}^{(1, N)}, \quad \gamma \cdot |0\rangle_v^{(1, N)} = \hbar^{1/2} |0\rangle_v^{(1, N)}, \quad C \cdot |0\rangle_v^{(1, N)} = \hbar^{N/2} |0\rangle_v^{(1, N)}.$$

Let $\mathcal{F}_v^{(1, N)} = \mathbb{C}[\alpha_{-m} \ (m > 0)] |0\rangle_v^{(1, N)}$ be a Fock module of the Heisenberg subalgebra $\{\alpha_m \ (m \in \mathbb{Z}_{\neq 0})\}$.

²We changed the definition of the level of representation from the one given in [11] so that our level (k, l) is the level $(k, -l)$ there. Note also our $C = \hbar^{c/2}$ is ψ_0^+ in [11].

Theorem 2.2. *The following gives a level $(1, N)$ representation of $\mathcal{U}$ on $\mathcal{F}_v^{(1, N)}$.*

$$x^+(z) = e^h(z^{-1}\hbar^{1/2})^c \exp \left\{ - \sum_{n>0} \frac{\hbar^{n/2}}{1 - \hbar^n} \alpha_{-n} z^n \right\} \exp \left\{ \sum_{n>0} \frac{\hbar^{n/2}}{1 - \hbar^n} \alpha_n z^{-n} \right\}, \quad (2.3)$$

$$x^-(z) = e^{-h}(z^{-1}\hbar^{1/2})^{-c} \exp \left\{ \sum_{n>0} \frac{\hbar^{n/2}}{1 - \hbar^n} \alpha'_{-n} z^n \right\} \exp \left\{ - \sum_{n>0} \frac{\hbar^{n/2}}{1 - \hbar^n} \alpha'_n z^{-n} \right\}, \quad (2.4)$$

$$\psi^+(\hbar^{1/4}z) = \hbar^{-c/2} \exp \left\{ - \sum_{n>0} \frac{p^n}{1 - p^n} \alpha_{-n} z^n \right\} \exp \left\{ \sum_{n>0} \frac{1}{1 - p^n} \alpha_n z^{-n} \right\}, \quad (2.5)$$

$$\psi^-(\hbar^{-1/4}z) = \hbar^{c/2} \exp \left\{ - \sum_{n>0} \frac{1}{1 - p^n} \alpha_{-n} z^n \right\} \exp \left\{ \sum_{n>0} \frac{p^n}{1 - p^n} \alpha_n z^{-n} \right\}. \quad (2.6)$$

2.2.2 The level-(0,-1) representation

For $u \in \mathbb{C}^\times$, let $\mathcal{F}_u^{(0, -1)}$ be a vector space spanned by $|\lambda\rangle_u$ ($\lambda \in \mathcal{P}$), where

$$\mathcal{P} = \{ \lambda = (\lambda_1, \lambda_2, \dots) \mid \lambda_i \geq \lambda_{i+1}, \lambda_i \in \mathbb{Z}_{\geq 0}, \lambda_l = 0 \text{ for sufficiently large } l \}.$$

We denote by $\ell(\lambda)$ the length of $\lambda \in \mathcal{P}$ i.e. $\lambda_{\ell(\lambda)} > 0$ and $\lambda_{\ell(\lambda)+1} = 0$. We also set $|\lambda| = \sum_{i \geq 1} \lambda_i$.

Theorem 2.3. *The following action gives a level-(0,-1) representation of $\mathcal{U}$ on $\mathcal{F}_u^{(0, -1)}$.*

$$x^+(z)|\lambda\rangle_u = a^+(p) \sum_{i=1}^{\ell(\lambda)+1} A_{\lambda, i}^+(p) \delta(u_i/z) |\lambda + \mathbf{1}_i\rangle_u, \quad (2.7)$$

$$x^-(z)|\lambda\rangle_u = a^-(p) \sum_{i=1}^{\ell(\lambda)} A_{\lambda, i}^-(p) \delta(t_1 u_i/z) |\lambda - \mathbf{1}_i\rangle_u, \quad (2.8)$$

$$\psi^+(z)|\lambda\rangle_u = \prod_{i=1}^{\ell(\lambda)} \frac{\theta(t_2^{-1} u_i/z)}{\theta(t_1 u_i/z)} \prod_{i=1}^{\ell(\lambda)+1} \frac{\theta(\hbar u_i/z)}{\theta(u_i/z)} |\lambda\rangle_u, \quad (2.9)$$

$$\psi^-(z)|\lambda\rangle_u = \prod_{i=1}^{\ell(\lambda)} \frac{\theta(t_2 z/u_i)}{\theta(t_1^{-1} z/u_i)} \prod_{i=1}^{\ell(\lambda)+1} \frac{\theta(\hbar^{-1} z/u_i)}{\theta(z/u_i)} |\lambda\rangle_u, \quad (2.10)$$

where

$$\begin{aligned} a^+(p) &= (1-t) \frac{(p\hbar; p)_\infty (p/t_2; p)_\infty}{(p; p)_\infty (p/q; p)_\infty}, & a^-(p) &= (1-t^{-1}) \frac{(p/\hbar; p)_\infty (pt_2; p)_\infty}{(p; p)_\infty (pq; p)_\infty}, \\ A_{\lambda, i}^+(p) &= \prod_{j=1}^{i-1} \frac{\theta(t_2 u_i/u_j) \theta(\hbar^{-1} u_i/u_j)}{\theta(t_1^{-1} u_i/u_j) \theta(u_i/u_j)}, & A_{\lambda, i}^-(p) &= \prod_{j=i+1}^{\ell(\lambda)} \frac{\theta(\hbar^{-1} u_j/u_i)}{\theta(u_j/u_i)} \prod_{j=i+1}^{\ell(\lambda)+1} \frac{\theta(t_2 u_j/u_i)}{\theta(t_1^{-1} u_j/u_i)}. \end{aligned}$$

This is an elliptic analogue of a representation given in [5, 6]. In [11] it is conjectured that this gives a geometric action of $\mathcal{U}$ on the equivariant elliptic cohomology of the Hilbert schemes $\bigoplus_{n \geq 0} E_T(\text{Hilb}^n(\mathbb{C}^2))$ under the identification of $|\lambda\rangle_u$ with the fixed point class $[\lambda]$ in $\bigoplus_{n \geq 0} E_T(\text{Hilb}^n(\mathbb{C}^2))$.

3 Elliptic Stable Envelopes for $E_T(\mathcal{M}(n, r))$

The elliptic stable envelopes for the equivariant elliptic cohomology of the instanton moduli space $E_T(\mathcal{M}(n, r))$ were constructed in [4, 16].

3.1 The instanton moduli space $\mathcal{M}(n, r)$

Let $\mathcal{M}(n, r)$ be the moduli space of framed rank r torsion free sheaves $\mathcal{S}$ on $\mathbb{P}^2$ with $c_2(\mathcal{S}) = n$. One has a natural action of $G = GL(r) \times GL(2)$ on $\mathcal{M}(n, r)$. Let T be the maximal torus of G and set $A = T \cap GL(r)$. The parameters t_1, t_2 are identified with the generators of the character group of T/A . The rank 1 case is isomorphic to the Hilbert scheme of n -points on $\mathbb{C}^2$.

$$\mathcal{M}(n, 1) \cong \text{Hilb}^n(\mathbb{C}^2).$$

Let us consider the case $r = 1$, the Hilbert scheme $\mathcal{H}_n = \text{Hilb}^n(\mathbb{C}^2)$, $A = \mathbb{C}^\times$. We denote the coordinate on A by u such that

$$t_1 = u\hbar^{1/2}, \quad t_2 = u^{-1}\hbar^{1/2}.$$

There are a finite number of the A -fixed points of $\mathcal{H}_n$ labeled by partitions of n . Let

$$\mathcal{P}_n = \{ \lambda \in \mathcal{P} \mid |\lambda| = n \}.$$

We regard $\lambda \in \mathcal{P}_n$ as a Young diagram with n boxes. For a box $\square = (i, j) \in \lambda$, we define

$$c_\square := i - j, \quad h_\square := i + j - 2, \quad \rho_\square := c_\square - \epsilon h_\square$$

with $0 < \epsilon \ll 1$. We introduce a canonical ordering on the n boxes of λ by

$$a < b \Leftrightarrow \rho_a < \rho_b \quad a, b \in \lambda$$

and define a bijection $\iota : \lambda \rightarrow [1, n]$ if $a \in \lambda$ is the $\iota(a)$ -th box in this order. In the following we often denote the box a by $\square_{\iota(a)}$ or simply $\iota(a)$.

Let us consider the following presentation of the equivariant K-theory of $\mathcal{H}_n$.

$$K_T(\mathcal{H}_n) = \mathbb{Z}[x_1^\pm, \dots, x_n^\pm, t_1^\pm, t_2^\pm]^{\mathfrak{S}_n} / R,$$

where $\mathfrak{S}_n$ denotes the symmetrization in x_a 's, and R the ideal of Laurent polynomials vanishing at all fixed points in $\mathcal{H}_n^T$. We often loosely use x_a as $x_{\iota(a)}$ for $a \in \lambda$ and vice versa. For $\square = (i, j) \in \lambda$, we set

$$\varphi_\square^\lambda = t_1^{-(j-1)} t_2^{-(i-1)} \in K_T(pt).$$

The restriction of a K-theory class $f(x_1, \dots, x_n, t_1, t_2)$ to a fixed point labeled by $\lambda \in \mathcal{P}_n$ is given by

$$i_\lambda^* f(x_1, \dots, x_n, t_1, t_2) = f(\varphi_{\square_1}^\lambda, \dots, \varphi_{\square_n}^\lambda, t_1, t_2).$$

Here $i_\lambda : \lambda \rightarrow \mathcal{H}_n$ denotes the canonical inclusion of a fixed point.

Let $\mathcal{V}$ be the rank n tautological bundle on $\mathcal{H}_n$. We present $\mathcal{V}$ as

$$\mathcal{V} = x_1 + \dots + x_n$$

regarding $x_1, \dots, x_n$ as the Chern roots of $\mathcal{V}$.

For $r \in \mathbb{Z}_{>0}$, fixed points in $\mathcal{M}(n, r)$ are labeled by a r -tuple partition $\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(r)})$ with $|\boldsymbol{\lambda}| = \sum_{i=1}^r |\lambda^{(i)}| = n$. We denote the Chern roots of the rank n tautological bundle $\mathcal{V}$ on $\mathcal{M}(n, r)$ by $\mathbf{x} = (x_1, \dots, x_n)$ and a coordinate of $A = (\mathbb{C}^\times)^r$ by $\mathbf{u} = (u_1, \dots, u_r)$. We define a canonical ordering on the n boxes of $\boldsymbol{\lambda}$ by extending the one for each partition $\lambda^{(i)}$ with adding the following condition, for $i < j$

$$a < b \Leftrightarrow a \in \lambda^{(i)}, b \in \lambda^{(j)}.$$

3.2 Elliptic stable envelopes

The elliptic stable envelopes for $E_T(\mathcal{H}_n)$ was constructed in [16]. Let $T^{1/2} \in K_T(\mathcal{H}_n)$ be a polarization of $\mathcal{H}_n$ satisfying

$$T\mathcal{H}_n = T^{1/2} + \hbar(T^{1/2})^\vee \in K_T(\mathcal{H}_n).$$

For $\lambda \in \mathcal{P}_n$, let us set

$$S_\lambda^{Ell}(x_1, \dots, x_n, u) := \frac{\prod_{\substack{a, b \in \lambda \\ \rho_a + 1 < \rho_b}} \theta(t_1 x_a / x_b) \prod_{\substack{a, b \in \lambda \\ \rho_a + 1 > \rho_b}} \theta(t_2 x_b / x_a) \prod_{\substack{a \in \lambda \\ \rho_a \leq 0}} \theta(x_a / u) \prod_{\substack{a \in \lambda \\ \rho_a > 0}} \theta(\hbar u / x_a)}{\prod_{\substack{a, b \in \lambda \\ \rho_a < \rho_b}} \theta(x_a / x_b) \theta(\hbar x_a / x_b)}. \quad (3.1)$$

Let $\mathbf{t}$ be a λ -tree, see Definition 1 in Section 4.2 of [16]. Namely, $\mathbf{t}$ is a rooted tree in a Young diagram λ with vertices corresponding to boxes of λ , edges connecting only adjacent boxes and the root at the box $(1, 1) \in \lambda$. Let us set (formula (54) in [16]):

$$W_{T^{1/2}}^{Ell}(\mathbf{t}; x_1, \dots, x_n, u, z) = (-1)^{\kappa_{\mathbf{t}}} \phi\left(\frac{x_r}{u}, z^n(t_1 t_2)^{\mathbf{v}_r}\right) \prod_{e \in \mathbf{t}} \phi\left(\frac{x_{h(e)} \varphi_{t(e)}^\lambda}{x_{t(e)} \varphi_{h(e)}^\lambda}, z^{\mathbf{w}_e}(t_1 t_2)^{\mathbf{v}_e}\right)$$

with

$$\phi(x, y) = \frac{\theta(xy)}{\theta(x)\theta(y)},$$

where the product runs over the edges of the tree $\mathbf{t}$ and $h(e) \in \lambda$, $t(e) \in \lambda$ denote the head and tail box of the edge e . And, $\kappa_{\mathbf{t}}, \mathbf{w}_e, \mathbf{v}_e \in \mathbb{Z}$ are certain integers computed from the tree in a combinatorial way, we refer to Sections 4.2-4.5 in [16] for definitions of these integers. The symbol z denotes the Kähler parameter in $E_T(\mathcal{H}_n)$ [1]. In the below we identify the dynamical parameter $\mathfrak{z}$ with the Kähler parameter z . Finally, let Υ_λ be the set of λ -trees without $\mathbf{J}$ -shaped subgraphs, see section 4.6 in [16]. Then, we have:

Theorem 3.1. [16] *The elliptic stable envelope of a fixed point $\lambda \in \mathcal{H}_n^A$ is given by*

$$\text{Stab}_{\mathfrak{E}, T^{1/2}}(\lambda; z) = \text{Sym} \left(S_\lambda^{Ell}(x_1, \dots, x_n, u) \sum_{\delta \in \Upsilon_\lambda} W_{T^{1/2}}^{Ell}(\mathbf{t}_\delta; x_1, \dots, x_n, u; z) \right), \quad (3.2)$$

where the symbol Sym stands for symmetrization over $x_1, \dots, x_n$. The chamber $\mathfrak{E}$ is taken as a stability condition $t_1/t_2 > 0$.

The elliptic stable envelope for $E_T(\mathcal{M}(n, r))$ is constructed by taking the shuffle product [2] of those for the Hilbert schemes. Let λ', λ'' be two partitions with $|\lambda'| = n', |\lambda''| = n''$ and consider the elliptic stable envelopes $\text{Stab}_{T^{1/2}, \mathfrak{C}'}(\lambda'; z')$ and $\text{Stab}_{T^{1/2}, \mathfrak{C}''}(\lambda''; z'')$ for $E_T(\mathcal{H}_{n'})$ and $E_T(\mathcal{H}_{n''})$ with the equivariant parameters u_1, u_2 , respectively. Here one takes $\mathfrak{C}' = \mathfrak{C}''$ as $t_1/t_2 > 0$. We take the canonical ordering on the $n' + n''$ boxes in the double partition

(λ', λ'') as defined in Sec.3.1. Then the following $\text{Stab}_{\mathfrak{C}, T^{1/2}}((\lambda', \lambda''); z)$ gives the elliptic stable envelope for $E_T(\mathcal{M}(n' + n'', 2))$ with the chamber $\mathfrak{C}$ given by $|u_1| \ll |u_2|$.

$$\begin{aligned} & \text{Stab}_{\mathfrak{C}, T^{1/2}}((\lambda', \lambda''); z) \\ &= \text{Sym}_{\{x_a\}_{a \in (\lambda', \lambda'')}} \left(\prod_{a \in \lambda', b \in \lambda''} \frac{\theta(t_1 x'_a / x''_b) \theta(t_2 x'_a / x''_b)}{\theta(x'_a / x''_b) \theta(\hbar x'_a / x''_b)} \prod_{b \in \lambda''} \theta(\hbar u_1 / x''_b) \prod_{a \in \lambda'} \theta(x'_a / u_2) \right. \\ & \quad \left. \times \text{Stab}_{\mathfrak{C}', T^{1/2}}(\lambda'; z' \hbar^{-1}) \text{Stab}_{\mathfrak{C}'', T^{1/2}}(\lambda''; z'') \right). \end{aligned} \quad (3.3)$$

Here x'_a ($a \in \lambda'$) and x''_b ($b \in \lambda''$) are the Chern roots for the tautological bundle on $\mathcal{H}_{n'}$ and $\mathcal{H}_{n''}$, respectively, and we set $\{x_a\}_{a \in (\lambda', \lambda'')} = \{x'_a\}_{a \in \lambda'} \cup \{x''_b\}_{b \in \lambda''}$. The formula (3.3) is called the shuffle product [2]. By taking the shuffle product r times, one obtains the elliptic stable envelopes $\text{Stab}_{\mathfrak{C}, T^{1/2}}(\lambda; \mathfrak{z})$ for $E_T(\mathcal{M}(n, r))$. Here $\lambda = (\lambda^{(1)}, \dots, \lambda^{(r)})$ denotes a r -tuple partition with $|\lambda| = \sum_{i=1}^r |\lambda^{(i)}| = n$. An explicit formula for $\text{Stab}_{\mathfrak{C}, T^{1/2}}(\lambda; \mathfrak{z})$ is given in Proposition 3.2 in [12].

In the next sections, we use $\widehat{\text{Stab}}_{\mathfrak{C}, T^{1/2}}(\lambda; \mathfrak{z})$ defined by

$$\text{Stab}_{\mathfrak{C}, T^{1/2}}(\lambda; \mathfrak{z}) = (-)^{\varepsilon(\lambda, r)} \Theta(T^{1/2}) \widehat{\text{Stab}}_{\mathfrak{C}, T^{1/2}}(\lambda; \mathfrak{z}) \quad (3.4)$$

where $\varepsilon(\lambda, r) = \sum_{i=1}^r \sum_{\substack{a \in \lambda \\ \rho a > \rho r_i}} 1$ and

$$\Theta(T^{1/2}) = \prod_{\substack{a, b \in \lambda \\ \rho a \neq \rho b}} \frac{\theta(t_1 x_a / x_b)}{\theta(x_a / x_b)} \prod_{i=1}^r \prod_{a \in \lambda} \theta(x_a / u_i).$$

Proposition 3.2. *The shuffle product formula for $\widehat{\text{Stab}}_{\mathfrak{C}, T^{1/2}}(\lambda; \mathfrak{z})$'s is given by*

$$\begin{aligned} & \widehat{\text{Stab}}_{\mathfrak{C}, T^{1/2}}((\lambda', \lambda''); \mathfrak{z}) \\ &= \text{Sym}_{\{x_a\}_{a \in (\lambda', \lambda'')}} \left(\prod_{a \in \lambda', b \in \lambda''} \frac{\theta(x''_b / x'_a) \theta(t_2 x'_a / x''_b)}{\theta(t_1 x''_b / x'_a) \theta(\hbar x'_a / x''_b)} \prod_{i=1}^{r'} \prod_{b \in \lambda''} \left(-\frac{\theta(\hbar u'_i / x''_b)}{\theta(x''_b / u'_i)} \right) \right. \\ & \quad \left. \times \widehat{\text{Stab}}_{\mathfrak{C}', T^{1/2}}(\lambda'; \mathfrak{z} \hbar^{-r'' - 2n''}) \widehat{\text{Stab}}_{\mathfrak{C}'', T^{1/2}}(\lambda''; \mathfrak{z}) \right). \end{aligned} \quad (3.5)$$

We also use the elliptic stable envelopes for $E_T(\mathcal{M}(n, r))$ with the opposite polarization $T_{opp}^{1/2} = \hbar(T^{1/2})^\vee$, the elliptic nome p^* and the Kähler parameter $\mathfrak{z}^{*-1}$:

$$\text{Stab}_{\mathfrak{E}, T_{opp}^{1/2}}^*(\boldsymbol{\lambda}; \mathfrak{z}^{*-1}) := \text{Stab}_{\mathfrak{E}, T^{1/2}}(\boldsymbol{\lambda}; \mathfrak{z})|_{T^{1/2} \mapsto T_{opp}^{1/2}, p \mapsto p^*, \mathfrak{z} \mapsto \mathfrak{z}^{*-1}} \quad (3.6)$$

and its hatted version defined by

$$\text{Stab}_{\mathfrak{E}, T_{opp}^{1/2}}^*(\boldsymbol{\lambda}; \mathfrak{z}^{*-1}) = (-)^{\varepsilon^*(\boldsymbol{\lambda}, r)} \Theta(T_{opp}^{1/2}) \widehat{\text{Stab}}_{\mathfrak{E}, T_{opp}^{1/2}}^*(\boldsymbol{\lambda}; \mathfrak{z}^{*-1}), \quad (3.7)$$

where $\varepsilon^*(\boldsymbol{\lambda}, r) = \sum_{i=1}^r \sum_{\substack{a \in \boldsymbol{\lambda} \\ \rho_a \leq \rho_{r_i}}} 1$ and

$$\Theta(T_{opp}^{1/2}) = \prod_{\substack{a, b \in \boldsymbol{\lambda} \\ \rho_a \neq \rho_b}} \frac{\theta^*(t_2 x_a / x_b)}{\theta^*(\hbar x_a / x_b)} \prod_{i=1}^r \prod_{a \in \boldsymbol{\lambda}} \theta^*(\hbar u_i / x_a).$$

4 Vertex Operators of $U_{t_1, t_2, p}(\mathfrak{gl}_{1, tor})$

We first define the basic vertex operators which correspond to $\text{Hilb}^n(\mathbb{C}^2)$ and then construct the ones for general $\mathcal{M}(n, r)$ by composing the basic ones.

4.1 OPE of the elliptic currents

Let us consider the level $(1, N)$ representation given in Sec.2.2.1, on which $p^* = p\hbar^{-1}$. For the elliptic currents $x^+(u), x^-(v)$, one gets the following operator product expansion (OPE).

$$x^+(u)x^+(v) = \langle x^+(u)x^+(v) \rangle^{sym} \frac{\theta^*(t_1 v / u) \theta^*(\hbar u / v)}{\theta^*(v / u) \theta^*(t_2 u / v)} : x^+(u)x^+(v) :, \quad (4.1)$$

$$x^-(u)x^-(v) = \langle x^-(u)x^-(v) \rangle^{sym} \frac{\theta(v / u) \theta(t_2 u / v)}{\theta(t_1 v / u) \theta(\hbar u / v)} : x^-(u)x^-(v) :, \quad (4.2)$$

with

$$\langle x^+(u)x^+(v) \rangle^{sym} = t_1^{-1} \frac{(p^* t_2 v / u, p^* t_2 u / v, v / u, u / v; p^*)_\infty}{(p^* \hbar v / u, p^* \hbar u / v, v / t_1 u, u / t_1 v; p^*)_\infty}, \quad (4.3)$$

$$\langle x^-(u)x^-(v) \rangle^{sym} = t_1^{-1} \frac{(p t_1^{-1} v / u, p t_1^{-1} u / v, \hbar v / u, \hbar u / v; p)_\infty}{(p v / u, p u / v, t_2 v / u, t_2 u / v; p)_\infty}. \quad (4.4)$$

Here we set

$$(a_1, \dots, a_M; p)_\infty = \prod_{i=1}^M (a_i; p)_\infty.$$

4.2 Vertex operators for $\text{Hilb}^n(\mathbb{C}^2)$

Let us define

$$\Phi_\emptyset(u) := (-u)^{-\alpha} e^{-\Lambda_0} \exp \left\{ - \sum_{m>0} \frac{1}{\kappa_m} \alpha'_{-m} (\hbar^{1/2} u)^m \right\} \exp \left\{ \sum_{m>0} \frac{1}{\kappa_m} \alpha'_m (\hbar^{1/2} u)^{-m} \right\}, \quad (4.5)$$

$$\Psi_\emptyset^*(u) := (-u)^\alpha e^{\Lambda_0 - Q/2} \exp \left\{ \sum_{m>0} \frac{1}{\kappa_m} \alpha_{-m} (\hbar^{1/2} u)^m \right\} \exp \left\{ - \sum_{m>0} \frac{1}{\kappa_m} \alpha_m (\hbar^{1/2} u)^{-m} \right\}. \quad (4.6)$$

One can show the following commutation relations.

Proposition 4.1.

$$\Psi_\emptyset^*(u) x^+(v) = - \frac{\theta^*(v/u)}{\theta^*(\hbar u/v)} x^+(v) \Psi_\emptyset^*(u), \quad (4.7)$$

$$\Phi_\emptyset(u) x^-(v) = - \frac{\theta(\hbar u/v)}{\theta(v/u)} x^-(v) \Phi_\emptyset(u), \quad (4.8)$$

$$x^+(v) \Phi_\emptyset(u) = \Phi_\emptyset(u) x^+(v), \quad x^-(v) \Psi_\emptyset^*(u) = \Psi_\emptyset^*(u) x^-(v), \quad (4.9)$$

$$\mathfrak{z} \Phi_\emptyset(u) = \hbar^{-1} \Phi_\emptyset(u) \mathfrak{z}, \quad \Psi_\emptyset^*(u) \mathfrak{z}^* = \hbar \mathfrak{z}^* \Psi_\emptyset^*(u), \quad (4.10)$$

$$[x^\pm(x), \mathfrak{z}] = [\Psi_\emptyset^*, \mathfrak{z}] = 0, \quad [x^\pm(x), \mathfrak{z}^*] = [\Phi_\emptyset(u), \mathfrak{z}^*] = 0. \quad (4.11)$$

Definition 4.2. We define the type I $\Phi(u)$ and the type II dual $\Psi^*(u)$ vertex operators to be the following linear maps.

$$\begin{aligned} \Phi(u) &: \mathcal{F}_v^{(1,N)} \rightarrow \mathcal{F}_u^{(0,-1)} \widetilde{\otimes} \mathcal{F}_{-v/u}^{(1,N+1)}, \\ \Psi^*(u) &: \mathcal{F}_v^{(1,N)} \widetilde{\otimes} \mathcal{F}_u^{(0,-1)} \rightarrow \mathcal{F}_{-vu}^{(1,N-1)}, \end{aligned}$$

with

$$\begin{aligned} \Phi(u) &= \sum_{\lambda \in \mathcal{P}} |\lambda\rangle_u \widetilde{\otimes} \Phi_\lambda(u), \\ \Phi_\lambda(u) &= \int_{\mathcal{C}} \prod_{a \in \lambda} dx_a : \prod_{a \in \lambda} x^-(x_a) : \Phi_\emptyset(u) \prod_{\rho_a < \rho_b} \langle x^-(x_a) x^-(x_b) \rangle^{sym} \widehat{\text{Stab}}_{\mathfrak{C}, T^{1/2}}(\lambda; \mathfrak{z}), \end{aligned}$$

and

$$\begin{aligned} \Psi_\lambda^*(u) \xi &= \Psi^*(u) (\xi \widetilde{\otimes} |\lambda\rangle_u), \quad \forall \xi \in \mathcal{F}_v^{(1,N)}, \\ \Psi_\lambda^*(u) &= \int_{\mathcal{C}^*} \prod_{a \in \lambda} dx_a \widehat{\text{Stab}}_{\mathfrak{C}, T_{opp}^{1/2}}^*(\lambda; \mathfrak{z}^{*-1}) : \prod_{a \in \lambda} x^+(x_a) : \Psi_\emptyset^*(u) \prod_{\rho_a < \rho_b} \langle x^+(x_a) x^+(x_b) \rangle^{sym}. \end{aligned}$$

The integration cycles $\mathcal{C}, \mathcal{C}^*$ are chosen appropriately depending on the situation of the application. We call $\Phi_\lambda(u)$ (resp. $\Psi_\lambda^*(u)$) with $\lambda \in \mathcal{P}_n$ the type I (resp. type II dual) vertex operator for $\text{Hilb}^n(\mathbb{C}^2)$.

4.3 Vertex operators for $\mathcal{M}(n, r)$

Let λ', λ'' be two partitions with $|\lambda'| = n', |\lambda''| = n''$. Let us consider the following composition of the two type I vertex operators for the Hilbert schemes.

$$\begin{aligned} & \Phi_{\lambda'}(u_1) \Phi_{\lambda''}(u_2) \\ &= \int_{\mathcal{C}} \prod_{a \in \lambda'} dx'_a \int_{\mathcal{C}} \prod_{b \in \lambda''} dx''_b : \prod_{a \in \lambda'} x^-(x'_a) : \Phi_\emptyset(u_1) \prod_{\rho_a < \rho_b} < x^-(x'_a) x^-(x'_b) >^{sym} \widehat{\text{Stab}}_{\mathfrak{C}', T^{1/2}}(\lambda'; \mathfrak{z}) \\ & \quad \times : \prod_{b \in \lambda''} x^-(x''_b) : \Phi_\emptyset(u_2) \prod_{\rho_c < \rho_d} < x^-(x''_c) x^-(x''_d) >^{sym} \widehat{\text{Stab}}_{\mathfrak{C}'', T^{1/2}}(\lambda''; \mathfrak{z}). \end{aligned}$$

Here the chambers $\mathfrak{C}', \mathfrak{C}''$ are the same and taken as the stability condition $t_1/t_2 > 0$. We also assume $|u_1| \ll |u_2|$. In a similar way to the type A linear quiver case studied in [9], let us arrange the order of the elements in the integrand as follows.

1. Move $\widehat{\text{Stab}}_{\mathfrak{C}', T^{1/2}}(\lambda'; \mathfrak{z})$ to the right of all operators.
2. Move $: \prod_{b \in \lambda''} x^-(x''_b) :$ to the left of $\Phi_\emptyset(u_1)$ by using the formula (4.8).
3. Make $: \prod_{a \in \lambda'} x^-(x'_a) :: \prod_{b \in \lambda''} x^-(x''_b) :$ totally normal ordered product by the formula

$$: \prod_{a \in \lambda'} x^-(x'_a) :: \prod_{b \in \lambda''} x^-(x''_b) := \prod_{a \in \lambda', b \in \lambda''} < x^-(x'_a) x^-(x''_b) > : \prod_{a \in (\lambda', \lambda'')} x^-(x_a) : .$$

Here we set $\{x_a\}_{a \in (\lambda', \lambda'')} = \{x'_a\}_{a \in \lambda'} \cup \{x''_b\}_{b \in \lambda''}$. We define the order of boxes in the different partitions by $\rho_a < \rho_b$ for $a \in \lambda', b \in \lambda''$.

4. Divide $< x^-(x'_a) x^-(x''_b) >$ into the symmetric and the non-symmetric parts as (4.2).
5. Symmetrize the integrand over $\{x_a\}_{a \in (\lambda', \lambda'')}$.

One thus obtains

$$\begin{aligned} \Phi_{\lambda'}(u_1) \Phi_{\lambda''}(u_2) &= \int_{\mathcal{C} \times \mathcal{C}} \prod_{a \in (\lambda', \lambda'')} dx_a : \prod_{a \in (\lambda', \lambda'')} x^-(x_a) : \Phi_\emptyset(u_1) \Phi_\emptyset(u_2) \\ & \quad \times \prod_{\substack{a, b \in (\lambda', \lambda'') \\ \rho_a < \rho_b}} < x^-(x_a) x^-(x_b) >^{sym} \widehat{\text{Stab}}_{\mathfrak{C}, T^{1/2}}((\lambda', \lambda''); \mathfrak{z}), \end{aligned}$$

where we set

$$\widehat{\text{Stab}}_{\mathfrak{e}, T^{1/2}}((\lambda', \lambda''); \mathfrak{z}) = \text{Sym}_{\{x_a\}_{a \in (\lambda', \lambda'')}} \left(\prod_{a \in \lambda', b \in \lambda''} \frac{\theta(x_b''/x_a') \theta(t_2 x_a'/x_b'')}{\theta(t_1 x_b''/x_a') \theta(\hbar x_a'/x_b'')} \prod_{b \in \lambda''} \left(-\frac{\theta(\hbar u_1/x_b'')}{\theta(x_b''/u_1)} \right) \right. \\ \left. \times \widehat{\text{Stab}}_{\mathfrak{e}', T^{1/2}}(\lambda'; \mathfrak{z} \hbar^{-1}) \widehat{\text{Stab}}_{\mathfrak{e}'', T^{1/2}}(\lambda''; \mathfrak{z}) \right).$$

Then it is remarkable that $\widehat{\text{Stab}}_{\mathfrak{e}, T^{1/2}}((\lambda', \lambda''); \mathfrak{z})$ coincides with the hatted version of the elliptic stable envelope for $E_T(\mathcal{M}(n' + n'', 2))$ given in (3.3). We hence regard the composition $\Phi_{\lambda'}(u_1) \Phi_{\lambda''}(u_2)$ as a vertex operator for $\mathcal{M}(n' + n'', 2)$.

In general, one obtains the type I vertex operator for $\mathcal{M}(n, r)$ by composing the basic vertex operators repeatedly.

$$(\text{id} \tilde{\otimes} \cdots \tilde{\otimes} \text{id} \tilde{\otimes} \Phi(u_1)) \circ \cdots (\text{id} \tilde{\otimes} \Phi(u_{r-1})) \circ \Phi(u_r) \\ : \mathcal{F}_v^{(1, N)} \rightarrow \mathcal{F}_{u_r}^{(0, -1)} \tilde{\otimes} \cdots \tilde{\otimes} \mathcal{F}_{u_1}^{(0, -1)} \tilde{\otimes} \mathcal{F}_{(-)^r v / u_1 \cdots u_r}^{(1, N+r)}.$$

Defining the components $\Phi_{\lambda}(u_1, \dots, u_r)$ by

$$(\text{id} \tilde{\otimes} \cdots \tilde{\otimes} \text{id} \tilde{\otimes} \Phi(u_1)) \circ \cdots (\text{id} \tilde{\otimes} \Phi(u_{r-1})) \circ \Phi(u_r) \\ = \sum_{n \in \mathbb{Z}_{\geq 0}} \sum_{\substack{\lambda = (\lambda^{(1)}, \dots, \lambda^{(r)}) \\ |\lambda| = n}} |\lambda^{(r)}\rangle_{u_r} \tilde{\otimes} \cdots \tilde{\otimes} |\lambda^{(1)}\rangle_{u_1} \tilde{\otimes} \Phi_{\lambda}(u_1, \dots, u_r), \quad (4.12)$$

one finds

$$\Phi_{\lambda}(u_1, \dots, u_r) = \Phi_{\lambda^{(1)}}(u_1) \cdots \Phi_{\lambda^{(r)}}(u_r) \\ = \int_{C^r} \prod_{a \in \lambda} dx_a : \prod_{a \in \lambda} x^-(x_a) : \Phi_{\emptyset}(u_1) \cdots \Phi_{\emptyset}(u_r) \prod_{\substack{a, b \in \lambda \\ \rho_a < \rho_b}} \langle x^-(x_a) x^-(x_b) \rangle^{sym} \widehat{\text{Stab}}_{\mathfrak{e}, T^{1/2}}(\lambda; \mathfrak{z}), \quad (4.13)$$

where $\widehat{\text{Stab}}_{\mathfrak{e}, T^{1/2}}(\lambda; \mathfrak{z})$ is the hatted version of the elliptic stable envelope for $E_T(\mathcal{M}(n, r))$ satisfying the shuffle product formula (3.5). We hence regard $\Phi_{\lambda}(u_1, \dots, u_r)$ as the type I vertex operator for $\mathcal{M}(n, r)$.

We also have a similar construction for the type II dual vertex operators.

5 The K-Theoretic Vertex functions for $\mathcal{M}(n, r)$

A vertex function for a quiver variety X is a generating function of counting quasi maps from $\mathbb{P}^1$ to X [15]. We show that the vacuum expectation value of the vertex operator constructed in the last section gives the K-theoretic vertex function for $X = \mathcal{M}(n, r)$.

Let λ, μ be two partitions with $|\lambda| = |\mu| = n$. There is a bijection ς from boxes in λ to those in μ defined by $\varsigma(a) = b \in \mu$ for $a \in \lambda$ if $\iota(a) = \iota(b)$. Here ι is defined in Sec.3.1. For a box $a = (i, j) \in \mu$, we set $\varphi_a^\mu = t_1^{-(j-1)} t_2^{-(i-1)}$ as before. For the Chern root x_a ($a \in \lambda$) we take the Jackson integral

$$\int_0^{\varphi_{\varsigma(a)}^\mu} d_p x_a f(x_a) = (1-p) \varphi_{\varsigma(a)}^\mu \sum_{d \in \mathbb{N}} f(\varphi_{\varsigma(a)}^\mu p^d) p^d$$

in the vertex operators for $\text{Hilb}^n(\mathbb{C}^2)$. Let $|0\rangle_v^{(1,N)}$ be the vacuum state (2.2) in $\mathcal{F}_v^{(1,N)}$ and ${}^{(1,N)}\langle 0|_v$ be the dual state satisfying

$${}^{(1,N)}\langle 0|_v |0\rangle_v^{(1,N)} = 1.$$

One finds that the following normalized vacuum expectation value gives the K-theoretic vertex function for $\text{Hilb}^n(\mathbb{C}^2)$.

$$\begin{aligned} V_\lambda^\mu(\hbar^{-1}u, \mathfrak{z}) &= \frac{1}{\mathcal{N}_\mu} {}^{(1,N-1)}\langle 0|_{-v/u} \langle 0| \Phi_\lambda(u) |0\rangle_v^{(1,N)} \\ &= \sum_{\mathbf{d} \in \mathbb{N}^n} (\hbar p^{-N-1} \mathfrak{z})^{-\sum_a d_a} \prod_{a \in \lambda} \frac{(\hbar \varphi_{\varsigma(a)}^\mu / u; p)_{d_a}}{(p \varphi_{\varsigma(a)}^\mu / u; p)_{d_a}} \prod_{\substack{a, b \in \lambda \\ \rho a \neq \rho b}} \frac{(p \varphi_{\varsigma(a)}^\mu / \varphi_{\varsigma(b)}^\mu; p)_{d_a - d_b} (t_2 \varphi_{\varsigma(a)}^\mu / \varphi_{\varsigma(b)}^\mu; p)_{d_a - d_b}}{(p t_1^{-1} \varphi_{\varsigma(a)}^\mu / \varphi_{\varsigma(b)}^\mu; p)_{d_a - d_b} (\hbar \varphi_{\varsigma(a)}^\mu / \varphi_{\varsigma(b)}^\mu; p)_{d_a - d_b}}. \end{aligned}$$

Here we take $\mathcal{N}_\mu$ as the the specialization of the integrand of ${}_{-v/u} \langle 0| \Phi_\lambda(\hbar^{-1}u) |0\rangle_v^{(1,N)}$ to $x_a = \varphi_{\varsigma(a)}^\mu$ ($a \in \lambda$). We also use the following quasi-periodicity of the ESE.

$$\widehat{\text{Stab}}_{\mathfrak{C}, T^{1/2}}(\lambda; \mathfrak{z}) \Big|_{\substack{x_a = \varphi_{\varsigma(a)}^\mu p^{d_a} \\ (a \in \lambda)}} = \mathfrak{z}^{-\sum_{a \in \lambda} d_a} \times \widehat{\text{Stab}}_{\mathfrak{C}, T^{1/2}}(\lambda; \mathfrak{z}) \Big|_\mu,$$

where

$$\widehat{\text{Stab}}_{\mathfrak{C}, T^{1/2}}(\lambda; \mathfrak{z}) \Big|_\mu := \widehat{\text{Stab}}_{\mathfrak{C}, T^{1/2}}(\lambda; \mathfrak{z}) \Big|_{\substack{x_a = \varphi_{\varsigma(a)}^\mu \\ (a \in \lambda)}}.$$

The special case $\mu = \lambda$, hence $\varsigma = \text{id}$, of this expression coincides with the formula obtained geometrically in [17].

In the same way, the vertex function for $\mathcal{M}(n, r)$ is obtained from the vertex operator (4.13). One obtains the following result.

$$\begin{aligned} V_{\lambda}^{\mu}(u_1/\hbar, \dots, u_r/\hbar, \mathfrak{z}) &:= \frac{1}{\mathcal{N}_{\mu}} (-\hbar)^{(1, N-r)}_{v/u_1 \dots u_r} \langle 0 | \Phi_{\lambda}(u_1/\hbar, \dots, u_r/\hbar) | 0 \rangle_v^{(1, N)} \\ &= \sum_{\mathbf{d} \in \mathbb{N}^n} \prod_{i=1}^r (\hbar^{-(r-2)} p^{r-N-1} \mathfrak{z})^{-\sum_{a \in \lambda(i)} d_a} \prod_{i=1}^r \prod_{a \in \lambda} \frac{(\hbar \varphi_{\zeta(a)}^{\mu}/u_i; p)_{d_a}}{(p \varphi_{\zeta(a)}^{\mu}/u_i; p)_{d_a}} \\ &\quad \times \prod_{\substack{a, b \in \lambda \\ \rho_a \neq \rho_b}} \frac{(p \varphi_{\zeta(a)}^{\mu}/\varphi_{\zeta(b)}^{\mu}; p)_{d_a-d_b} (t_2 \varphi_{\zeta(a)}^{\mu}/\varphi_{\zeta(b)}^{\mu}; p)_{d_a-d_b}}{(p t_1^{-1} \varphi_{\zeta(a)}^{\mu}/\varphi_{\zeta(b)}^{\mu}; p)_{d_a-d_b} (\hbar \varphi_{\zeta(a)}^{\mu}/\varphi_{\zeta(b)}^{\mu}; p)_{d_a-d_b}}. \end{aligned}$$

Here $\mathbf{d} = (d_a)$, $a \in \lambda = (\lambda^{(1)}, \dots, \lambda^{(r)})$. This agrees with the formula in [4].

6 Exchange Relations of the Vertex Operators

Let $(\mathcal{P}^2)_n = \{\alpha = (\alpha', \alpha'') \in \mathcal{P} \times \mathcal{P} \mid |\alpha| = |\alpha'| + |\alpha''| = n\}$. For $\alpha = (\alpha', \alpha'')$, let $\bar{\alpha} = (\alpha'', \alpha')$. We define the elliptic dynamical instanton R -matrix $R_{T^{1/2}}(u_1, u_2; \mathfrak{z}) \in \text{End}_{\mathbb{F}}(\mathcal{F}_{u_1}^{(0, -1)} \widetilde{\otimes} \mathcal{F}_{u_2}^{(0, -1)})$ as the following transition matrix of the elliptic stable envelopes. For $\alpha, \beta, \gamma \in (\mathcal{P}^2)_n$,

$$R_{T^{1/2}}(u_1, u_2; \mathfrak{z}) |\beta'\rangle_{u_1} \widetilde{\otimes} |\beta''\rangle_{u_2} = \sum_{\alpha \in (\mathcal{P}^2)_n} R_{T^{1/2}}(u_1, u_2; \mathfrak{z})_{\alpha}^{\beta} |\alpha'\rangle_{u_1} \widetilde{\otimes} |\alpha''\rangle_{u_2}, \quad (6.1)$$

$$R_{T^{1/2}}(u_1, u_2; \mathfrak{z})_{\alpha}^{\beta} = \mu(u_1/u_2) \bar{R}_{T^{1/2}}(u_1, u_2; \mathfrak{z})_{\alpha}^{\beta}, \quad (6.2)$$

$$\widehat{\text{Stab}}_{\bar{\mathfrak{C}}, T^{1/2}}(\bar{\alpha}; \mathfrak{z})|_{\bar{\gamma}} = \sum_{\beta \in (\mathcal{P}^2)_n} \widehat{\text{Stab}}_{\mathfrak{C}, T^{1/2}}(\beta; \mathfrak{z})|_{\gamma} \bar{R}_{T^{1/2}}(u_1, u_2; \mathfrak{z})_{\alpha}^{\beta}. \quad (6.3)$$

Here $\mu(u)$ is a scalar function defined by

$$\mu(u_1/u_2) \Phi_{\emptyset}(u_1) \Phi_{\emptyset}(u_2) = \Phi_{\emptyset}(u_2) \Phi_{\emptyset}(u_1). \quad (6.4)$$

It is explicitly calculated by using (4.5) as

$$\mu(u) = \frac{\Gamma(\hbar u; t_1, t_2, p)}{\Gamma(pu; t_1, t_2, p)},$$

where $\Gamma(z; t_1, t_2, p)$ denotes the triple Gamma function defined by

$$\begin{aligned} \Gamma(z; t_1, t_2, p) &= (z; t_1, t_2, p)_{\infty} (t_1 t_2 p / z; t_1, t_2, p)_{\infty}, \\ (z; t_1, t_2, p)_{\infty} &= \prod_{m_1, m_2, m_3=0}^{\infty} (1 - z t_1^{m_1} t_2^{m_2} p^{m_3}). \end{aligned}$$

By definition, $R_{T^{1/2}}(u_1, u_2; \mathfrak{z})$ preserves the representation level w.r.t. $C = \hbar^{c/2}$:

$$[R_{T^{1/2}}(u_1, u_2; \mathfrak{z}), c^{(1)} + c^{(2)}] = 0. \quad (6.5)$$

In addition, we assume the property

$$R_{T^{1/2}}(u_1, u_2; \mathfrak{z}\hbar^{c^{(1)}+c^{(2)}}) = R_{T^{1/2}}(u_1, u_2; \mathfrak{z}). \quad (6.6)$$

We use this in Proposition 6.1, 6.3 and as a consistency condition in a derivation of the dynamical Yang-Baxter equation (6.12). See [12] for detail. Note also that $\widehat{\text{Stab}}_{\mathfrak{C}, T^{1/2}}(\alpha; \mathfrak{z})$ is depend on u_1, u_2 only through the chamber $\mathfrak{C}$: $u_1 \ll u_2$ essentially. Hence for any $a \in \mathbb{C}^\times$

$$R_{T^{1/2}}(au_1, au_2; \mathfrak{z}) = R_{T^{1/2}}(u_1, u_2; \mathfrak{z}). \quad (6.7)$$

Proposition 6.1. *The type I vertex operators satisfy the following exchange relation.*

$$\Phi_{\omega''}(u_2)\Phi_{\omega'}(u_1) = \sum_{\lambda=(\lambda', \lambda'') \in (\mathcal{P}^2)_n} R_{T^{1/2}}(u_1, u_2; \mathfrak{z})_{\omega}^{\lambda} \Phi_{\lambda'}(u_1)\Phi_{\lambda''}(u_2),$$

where $\omega = (\omega', \omega'') \in (\mathcal{P}^2)_n$.

Similarly, let us define $R_{T_{opp}^{1/2}}^*(u_1, u_2; \mathfrak{z}^{*-1}) \in \text{End}_{\mathbb{F}}(\mathcal{F}_{u_1}^{(0,-1)} \widetilde{\otimes} \mathcal{F}_{u_2}^{(0,-1)})$ as the following transition matrix of the elliptic stable envelopes $\widehat{\text{Stab}}_{\mathfrak{C}, T_{opp}^{1/2}}^*(\bullet; \mathfrak{z}^{*-1})$. For $\alpha, \beta, \gamma \in (\mathcal{P}^2)_n$,

$$R_{T_{opp}^{1/2}}^*(u_1, u_2; \mathfrak{z}^{*-1})_{\alpha}^{\beta} = \mu^*(u_1/u_2) \bar{R}_{T_{opp}^{1/2}}^*(u_1, u_2; \mathfrak{z}^{*-1})_{\alpha}^{\beta}, \quad (6.8)$$

$$\widehat{\text{Stab}}_{\mathfrak{C}, T_{opp}^{1/2}}^*(\bar{\alpha}; \mathfrak{z}^{*-1})|_{\bar{\gamma}} = \sum_{\beta \in (\mathcal{P}^2)_n} \widehat{\text{Stab}}_{\mathfrak{C}, T_{opp}^{1/2}}^*(\beta; \mathfrak{z}^{*-1})|_{\gamma} \bar{R}_{T_{opp}^{1/2}}^*(u_1, u_2; \mathfrak{z}^{*-1})_{\alpha}^{\beta}, \quad (6.9)$$

where $\mu^*(u)$ is a scalar function satisfying

$$\Psi_{\emptyset}^*(u_1)\Psi_{\emptyset}^*(u_2) = \mu^*(u_1/u_2)\Psi_{\emptyset}^*(u_2)\Psi_{\emptyset}^*(u_1). \quad (6.10)$$

Explicitly it is given by

$$\mu^*(u) = \frac{\Gamma(p^*\hbar u; t_1, t_2, p^*)}{\Gamma(u; t_1, t_2, p^*)}.$$

Proposition 6.2.

$$R_{T^{1/2}}(u_1, u_2; \mathfrak{z}) = {}^t R_{T_{opp}^{1/2}}(u_1, u_2; \mathfrak{z}^{-1}). \quad (6.11)$$

Proposition 6.3.

$$\Psi_{\omega'}^*(u_1)\Psi_{\omega''}^*(u_2) = \sum_{\lambda=(\lambda',\lambda'')\in(\mathcal{P}^2)_n} \Psi_{\lambda'}^*(u_2)\Psi_{\lambda''}^*(u_1)R_{T^{1/2}}^*(u_1,u_2;\mathfrak{z}^*)_{\lambda}^{\omega},$$

where $\omega = (\omega', \omega'') \in (\mathcal{P}^2)_n$.

Proposition 6.1 and the associativity for the composition of three type I vertex operators yield the following dynamical Yang-Baxter equation under the assumption (6.6).

$$\begin{aligned} R_{T^{1/2}}^{(12)}(u_1, u_2; \mathfrak{z}\hbar^{c(3)})R_{T^{1/2}}^{(13)}(u_1, u_3; \mathfrak{z})R_{T^{1/2}}^{(23)}(u_2, u_3; \mathfrak{z}\hbar^{c(1)}) \\ = R_{T^{1/2}}^{(23)}(u_2, u_3; \mathfrak{z})R_{T^{1/2}}^{(13)}(u_1, u_3; \mathfrak{z}\hbar^{c(2)})R_{T^{1/2}}^{(12)}(u_1, u_2; \mathfrak{z}). \end{aligned} \quad (6.12)$$

Similarly, Proposition 6.3 and the associativity for the composition of the type II dual vertex operators yield the same dynamical Yang-Baxter equation for $R_{T^{1/2}}^*(u_1, u_2; \mathfrak{z}^*)$.

Finally, the type I and the type II dual vertex operators exchange by a scalar function.

Proposition 6.4. *In the level $(1, N)$ representation, one has*

$$\begin{aligned} \Phi_{\lambda}(u)\Psi_{\mu}^*(v) &= \chi(u/v)\Psi_{\mu}^*(v)\Phi_{\lambda}(u) \quad \forall \lambda, \mu \in \mathcal{P}_n, \\ \chi(u) &= \frac{1}{\Gamma(\hbar^{1/2}u; t_1, t_2)} = \Gamma(\hbar^{1/2}/u; t_1, t_2). \end{aligned}$$

Here $\Gamma(z; t_1, t_2)$ denotes the elliptic Gamma function given by

$$\Gamma(z; t_1, t_2) = \frac{(t_1 t_2 / z; t_1, t_2)_{\infty}}{(z; t_1, t_2)_{\infty}}, \quad (z; t_1, t_2)_{\infty} = \prod_{m_1, m_2=0}^{\infty} (1 - z t_1^{m_1} t_2^{m_2}).$$

7 L -operator of $U_{t_1, t_2, p}(\mathfrak{gl}_{1, \text{tor}})$

Combining the type I and the type II dual vertex operators, we construct the L -operator L^+ satisfying the RLL -relation. We then derive the exchange relations between L^+ and the vertex operators, which can be regarded as the intertwining relations w.r.t. the standard comultiplication Δ .

7.1 L -operator on $\mathcal{F}_\bullet^{(1,N)}$

Let $\sigma^{op} : \xi \tilde{\otimes} \eta \rightarrow \eta \tilde{\otimes} \xi$ and consider the following composition of the type I and type II vertex operators.

$$\mathcal{F}_u^{(0,-1)} \tilde{\otimes} \mathcal{F}_{\hbar^{1/2}v}^{(1,N)} \xrightarrow{\sigma^{op}} \mathcal{F}_{\hbar^{1/2}v}^{(1,N)} \tilde{\otimes} \mathcal{F}_u^{(0,-1)} \xrightarrow{\Phi(\hbar^{1/2}u) \tilde{\otimes} \text{id}} \mathcal{F}_{\hbar^{1/2}u}^{(0,-1)} \tilde{\otimes} \mathcal{F}_{-v/u}^{(1,N+1)} \tilde{\otimes} \mathcal{F}_u^{(0,-1)} \xrightarrow{\text{id} \tilde{\otimes} \Psi^*(u)} \mathcal{F}_{\hbar^{1/2}u}^{(0,-1)} \tilde{\otimes} \mathcal{F}_v^{(1,N)}.$$

Hence we have the operator

$$L^+(u) := g(\text{id} \tilde{\otimes} \Psi^*(u)) \circ (\Phi(\hbar^{1/2}u) \tilde{\otimes} \text{id}) \sigma^{op} : \mathcal{F}_u^{(0,-1)} \tilde{\otimes} \mathcal{F}_{\hbar^{1/2}v}^{(1,N)} \rightarrow \mathcal{F}_{\hbar^{1/2}u}^{(0,-1)} \tilde{\otimes} \mathcal{F}_v^{(1,N)}$$

for $N \in \mathbb{Z}$, $v \in \mathbb{C}^\times$. Here we set $g = (\hbar; t_1, t_2)_\infty$. Define the components of $L^+(u)$ by

$$L^+(u) \cdot |\nu\rangle_u \tilde{\otimes} \xi = \sum_{\mu} |\mu\rangle_{\hbar^{1/2}u} \tilde{\otimes} L_{\mu\nu}^+(u) \xi,$$

for $|\mu\rangle_u \tilde{\otimes} \xi \in \mathcal{F}_u^{(0,-1)} \tilde{\otimes} \mathcal{F}_{\hbar^{1/2}v}^{(1,N)}$. One finds

$$L_{\mu\nu}^+(u) = g\Psi_\nu^*(u)\Phi_\mu(\hbar^{1/2}u). \quad (7.1)$$

Now let us consider the following elliptic dynamical R -matrices.

$$R_{T^{1/2}}^+(u, v; \mathfrak{z})_\alpha^\beta := \rho^+(u/v) \bar{R}_{T^{1/2}}(u, v; \mathfrak{z})_\alpha^\beta, \quad (7.2)$$

where $\bar{R}_{T^{1/2}}$ is given in (6.3) and $R_{T^{1/2}}^{+*} = R_{T^{1/2}}^+|_{p \rightarrow p^*}$.

Proposition 7.1. *The L^+ operator satisfies the following relation.*

$$\sum_{\mu', \nu'} R_{T^{1/2}}^+(u, v; \mathfrak{z})_{\mu\nu}^{\mu'\nu'} L_{\mu'\mu''}^+(u) L_{\nu'\nu''}^+(v) = \sum_{\mu', \nu'} L_{\nu\nu'}^+(v) L_{\mu\mu'}^+(u) R_{T^{1/2}}^{+*}(u, v; \mathfrak{z}^*)_{\mu'\nu'}^{\mu''\nu''}.$$

7.2 Intertwining relations

Proposition 7.2. *The type I and the type II vertex operators satisfy the following relations.*

$$\Phi_\nu(\hbar^{1/2}v) L_{\mu\mu''}^+(u) = \sum_{\mu'\nu'} R_{T^{1/2}}^+(u, v; \mathfrak{z})_{\mu\nu}^{\mu'\nu'} L_{\mu'\mu''}^+(u) \Phi_{\nu'}(\hbar^{1/2}v), \quad (7.3)$$

$$L_{\mu\mu''}^+(u) \Psi_{\nu'}^*(v) = \sum_{\mu'\nu'} \Psi_{\nu'}^*(v) L_{\mu\mu'}^+(u) R_{T^{1/2}}^{+*}(u, v; \mathfrak{z}^*)_{\mu'\nu'}^{\mu''\nu''}. \quad (7.4)$$

Assuming the existence of the universal L -operator $\mathcal{L}^+(u) \in \text{End}_{\mathbb{F}}(\mathcal{F}_\bullet^{(0,-1)}) \tilde{\otimes} \mathcal{U}$, which coincides with $R_{T^{1/2}}^+(u, v; \mathfrak{z}^*)$ for $\mathcal{U} = \mathcal{F}_v^{(0,-1)}$ and with $L^+(u)$ for $\mathcal{U} = \mathcal{F}_\bullet^{(1,N)}$, one can define a comultiplication Δ as a matrix tensor product of $\mathcal{L}^+(u)$. Then the relations in Proposition 7.2 turn out to be the intertwining relations w.r.t. Δ .

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