

Ramification of local fields and Fontaine’s property (P_m) : a résumé

By

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Abstract

This is a résumé of our results ([7], [9]) on Fontaine’s property (P_m) which is an effective tool for estimating the ramification of torsion Galois representations.

§ 1. Fontaine’s property (P_m)

Let K be a complete discrete valuation field with perfect residue field k of characteristic $p > 0$, K^{alg} a fixed algebraic closure of K , $\bar{K}$ the separable closure of K in K^{alg} and v_K the valuation on K^{alg} normalized by $v_K(K^\times) = \mathbb{Z}$. We denote by G_K the absolute Galois group of K . Let $G_K^{(m)}$ be the m th upper numbering ramification group in the sense of [3]. Namely, we put $G_K^{(m)} = G_K^{m-1}$, where the latter is the upper numbering ramification group defined in [6].

One of the classical problems in ramification theory is to obtain a ramification bound of torsion geometric Galois representations. Assume $\text{char}(K)=0$ for the moment. Let e be the absolute ramification index of K . Consider a proper smooth variety X_K over K and put $X_{\bar{K}} = X_K \times_K \bar{K}$. In [3], Fontaine conjectured the upper numbering ramification group $G_K^{(m)}$ acts trivially on the r th étale cohomology group $V = H_{\text{ét}}^r(X_{\bar{K}}, \mathbb{Z}/p^n\mathbb{Z})$ for $m > e(n + r/(p-1))$ if X_K has good reduction. This is equivalent to the inequality $u_{L/K} \leq e(n + r/(p-1))$, where L is defined by $G_L = \text{Ker}(G_K \rightarrow \text{Aut}(V))$ and $u_{L/K}$ is the infimum of the real numbers m such that $G_K^{(m)} \subset G_L$. For $e = 1$ and $r < p-1$, this conjecture was proved independently by himself ([4], for $n = 1$) and Abrashkin ([1], for any n). There are also similar ramification bounds if X_K has semi-stable reduction by

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Caruso-Liu ([2]) and Hattori ([5]). All of them used Fontaine's property (P_m) studied in [3] to obtain each ramification bound.

Now, let us go back to the general case where K is of either characteristic. For an algebraic extension E/K , we denote by $\mathcal{O}_E$ the integral closure of $\mathcal{O}_K$ in E . Fontaine's property (P_m) is the following condition for a finite Galois extension L of K and a real number m :

(P_m) For any algebraic extension E/K , if there exists an $\mathcal{O}_K$ -algebra homomorphism $\mathcal{O}_L \rightarrow \mathcal{O}_E/\mathfrak{a}_{E/K}^m$, then there exists a K -embedding $L \hookrightarrow E$,

where $\mathfrak{a}_{E/K}^m = \{x \in \mathcal{O}_E \mid v_K(x) \geq m\}$. We define the greatest upper numbering ramification break $u_{L/K}$ as above. Fontaine proved the following:

Proposition 1.1 ([3], Prop. 1.5). *Let L be a finite Galois extension of K , m a real number and $e_{L/K}$ the ramification index of L/K . Then there are following relations:*

- (i) *If we have $m > u_{L/K}$, then (P_m) is true.*
- (ii) *If (P_m) is true, then we have $m > u_{L/K} - e_{L/K}^{-1}$.*

Remark. The above (i) can be generalized to the imperfect residue field case by Abbes-Saito's ramification theory ([9], Prop. 4.3).

Given a torsion Galois representation V of G_K , we can use the above proposition to bound its ramification as follows: Let L/K be the finite Galois extension defined by $G_L = \text{Ker}(G_K \rightarrow \text{Aut}(V))$. If V is of some geometric origin, it is often possible to verify (P_m) for L/K and a suitable m . Then the above inequality of (ii) gives the upper bound $u_{L/K} < m + e_{L/K}^{-1}$. Thus it will be useful to sharpen the inequality. Indeed, our first main theorem below shows that we can improve the bound to $u_{L/K} \leq m$. This result is actually used in [5], Section 5, Proposition 5.6.

For a finite Galois extension L of K , we put

$$m_{L/K} = \inf\{m \in \mathbb{R} \mid (P_m) \text{ is true for } L/K\}$$

By the above proposition, we have the inequalities

$$u_{L/K} - e_{L/K}^{-1} \leq m_{L/K} \leq u_{L/K}.$$

More precisely, we have the following equality:

Theorem 1.2 ([9], Prop. 3.3). *We have $u_{L/K} = m_{L/K}$.*

An outline of the proof: We can prove easily that (P_m) is not true if $m = u_{L/K}$ (hence $u_{L/K} = m_{L/K}$) in the case where L/K is at most tamely ramified. Hence we may assume L/K is wildly ramified. It suffices to show the inequality $u_{L/K} - (e')^{-1} \leq m_{L/K}$ with an

arbitrarily large integer e' . Take an arbitrary finite tamely ramified Galois extension K' of K and put $L' = LK'$. Then we have $u_{L'/K} = u_{L/K}$ since L/K is wildly ramified. If we apply (ii) of Proposition 1.1 to L'/K , and (i) of Proposition 1.1 to K'/K respectively, then we can prove the inequality $u_{L/K} - e_{L'/K}^{-1} \leq m_{L/K}$.

§ 2. (P_m) at the ramification break

Let L be a finite Galois extension of K and e its ramification index. Assume L/K to be totally and wildly ramified for simplicity. In this section, we completely determine the truth of (P_m) ¹. The equality $u_{L/K} = m_{L/K}$ in Theorem 1.2 gives no information about the truth of (P_m) at the break $m = u_{L/K}$. The behavior of (P_m) at the break depends on the residue field:

Theorem 2.1 ([7], Thm. 1.1). *The property (P_m) is true for $m = u_{L/K}$ if and only if the residue field k has no Galois extension whose degree is divisible by p .*

This theorem can be proved by using the local class field theory of Serre and Hazewinkel. On the other hand, we consider a weaker property (P_m^e) as follows:

(P_m^e) *For any totally ramified extension E/K of degree e , if there exists an $\mathcal{O}_K$ -algebra homomorphism $\mathcal{O}_L \rightarrow \mathcal{O}_E/\mathfrak{a}_{E/K}^m$, then there exists a K -embedding $L \hookrightarrow E$.*

Then we have the following theorem, which is a similar result as Theorem 2.1. The proof employs the notion of a non-Archimedean metric on the set of all Eisenstein polynomials over K .

Theorem 2.2 ([8], Thm. A, Prop. 5.1). *The property (P_m^e) is true for $m = u_{L/K}$ if and only if the residue field k has no Galois extension of degree p .*

Remark. Both (i) and (ii) of Proposition 1.1 remain true for a finite totally and wildly ramified Galois extension L of K if we consider (P_m^e) instead of (P_m) . However, (P_m^e) does not satisfy the equality in Proposition 1.2 in general. In fact, we can check that (P_m^e) is true for $m = u_{L/K}$ if and only if (P_m^e) is true for $m > u_{L/K} - e_{L/K}^{-1}$.

¹The results in this section were obtained by joint work with Takashi Suzuki after the talk in “Algebraic Number Theory and Related Topics 2009”.

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