

On a sufficient condition for p-valently starlikeness

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Let $A(p)$ be the class of functions of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \quad (p \in \mathbb{N} = \{1, 2, 3, \dots\})$$

which are regular in $D = \{z \mid |z| < 1\}$.

A function $f(z)$ in $A(p)$ is said to be p-valently starlike iff

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > 0 \quad \text{in } D.$$

We denote by $S(p)$ the subclass of $A(p)$ consisting of functions which are p-valently starlike in D .

THEOREM. Let $f(z) \in A(p)$ and assume that

$$(1) \quad \left| \arg \frac{f'(z)}{z^{p-1}} \right| < \frac{\pi}{2} \alpha \quad \text{in } D$$

and

$$(2) \quad \left(\operatorname{Im} \frac{f'(z)}{z^{p-1}} \right) \left(\operatorname{Im} e^{-i\beta} z \right) \neq 0$$

for $z \in D(\beta) = \{z \mid |z| < 1, z \neq 0 \text{ and } (\arg z - \beta)(\arg z - \beta - \pi) \neq 0\}$

where α and β are real numbers, $0 < \alpha \leq 1$ and $0 \leq \beta < \pi$.

Then we have

$$\left| \arg \frac{zf'(z)}{f(z)} \right| < \frac{\pi}{2} \alpha \quad \text{in } D$$

and therefore $f(z)$ is p-valently starlike in D or $f(z) \in S(p)$.

PROOF. Applying the same method as in the proof of Ruscheweyh [1, p.142],

we have

$$\frac{f(z)}{zf'(z)} = \int_0^1 \frac{f'(tz)}{f'(z)} dt$$

$$= \frac{z^{p-1}}{f'(z)} \int_0^1 t^{p-1} \frac{f'(tz)}{(tz)^{p-1}} dt, \quad z \in D$$

and it follows that

$$(3) \quad \arg t^{p-1} \frac{f'(tz)}{(tz)^{p-1}} = \arg \frac{f'(tz)}{(tz)^{p-1}}.$$

From the assumption (1) and (2), and from (3), if we have

$$0 < \arg \frac{f'(z)}{z^{p-1}} < \frac{\pi}{2} \alpha,$$

then the integral

$$\int_0^1 t^{p-1} \frac{f'(tz)}{(tz)^{p-1}} dt$$

lies in the same convex sector $\{z \mid 0 < \arg z < \frac{\pi}{2} \alpha\}$ and by the same reason as the above, if we have

$$0 > \arg \frac{f'(z)}{z^{p-1}} > -\frac{\pi}{2} \alpha$$

then we have

$$0 > \arg \left(\int_0^1 t^{p-1} \frac{f'(tz)}{(tz)^{p-1}} dt \right) > -\frac{\pi}{2} \alpha.$$

Therefore we have

$$\left| \arg \frac{zf'(z)}{f(z)} \right| < \frac{\pi}{2} \alpha \quad \text{in } D.$$

This shows that $f(z)$ is p -valently starlike in D .

From the THEOREM, we easily have the following corollaries:

COROLLARY 1. Let $f(z) \in A(p)$ and assume that $0 < \alpha \leq 1$ and

$$\left| \arg \frac{f'(z)}{z^{p-1}} \right| < \frac{\pi}{2} \alpha \quad \text{in } D$$

and $f'(z)/z^{p-1}$ is typically real in D .

Then $f(z)$ belongs to $S(p)$ and

$$\left| \arg \frac{zf'(z)}{f(z)} \right| < \frac{\pi}{2} \alpha \quad \text{in } D.$$

COROLLARY 2. Let $f(z) \in A(p)$ and assume that $\operatorname{Re}(f'(z)/z^{p-1}) > 0$ in D and $f'(z)/z^{p-1}$ is typically real in D . Then $f(z)$ belongs to $S(p)$ or

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > 0 \quad \text{in } D.$$

COROLLARY 3. Let $f(z) \in A(1)$ and assume that $f'(z)$ is typically real in D and satisfies

$$|\arg f'(z)| < \frac{\pi}{2} \alpha \quad \text{in } D.$$

Then $f(z)$ is univalently starlike and

$$|\arg \frac{zf'(z)}{f(z)}| < \frac{\pi}{2} \alpha \quad \text{in } D$$

Reference

- [1] S. Ruscheweyh, Coefficient conditons for starlike functions, Glasgow Math. J., 29(1987), 141-142.