

On the work of J. ECALLE

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I. Let D be a small disc $\{|z| < \epsilon\}$ in $\mathbb{C}$; write $D^* = D - \{0\}$, $\tilde{D}^*$ = the universal covering of D^* with some fixed base-point $a \in D^*$.

Put $\tilde{\mathcal{O}} = \mathcal{O}(\tilde{D}^*)$, the space of holomorphic functions on $\tilde{D}^*$, and $\tilde{\mathcal{E}} = \tilde{\mathcal{O}}/\mathcal{O}(D)$; one has two well-known morphisms $\tilde{\mathcal{O}} \xrightleftharpoons[\text{var}]{\text{can}} \tilde{\mathcal{E}}$, where "can" is the quotient map, and "var" is defined by $\text{var} \cdot \text{can} = I \cdot \text{id}$; here I is the action of the monodromy on $\tilde{\mathcal{O}}$. The space $\tilde{\mathcal{E}}$ can be considered as a space of microfunctions at $0 \in \mathbb{R}$; on $\tilde{\mathcal{E}}$, the convolution product $(f, g) \mapsto f * g$ is well defined, with all the usual properties.

Now, let Ω be a discrete subgroup of $\mathbb{C}$, for instance $\Omega = \mathbb{Z}$; we suppose $D \cap \Omega = \{0\}$. Definition 1 We denote by $\mathcal{C}(\Omega)$ the set of the $f \in \tilde{\mathcal{E}}$ such that $\text{var } f$ has an analytic continuation to the whole space $\mathbb{C} - \Omega$; here, $\mathbb{C} - \Omega$ denotes the universal covering of $\mathbb{C} - \Omega$ with the same base-point $a \in \tilde{D}^*$ as before.

Theorem 2 (Ecalé). $\mathcal{C}(\Omega)$ is a convolution subalgebra of $\tilde{\mathcal{E}}$.

This convolution algebra is the basic object of Ecalé's theory. A important result is the description of the singularities of $f * g$ ($f, g \in \mathcal{C}(\Omega)$) in terms of the singularities of f and g ; this is done with the introduction of "alien derivations"; for more precise statements, see (E1).

II. Let G be the group of germs of analytic automorphisms $(\mathbb{C}, 0) \rightarrow (\mathbb{C}, 0)$, and let H be the subgroup of germs tangent to identity. One of the works of Ecalé is the classification of conjugacy classes in H ; two methods are given to describe that classification; for the first one, I refer to [E2] or [M]; I will describe briefly the second one.

For $h \in \mathbb{C}[[z]]$, write $h(z) = z + a_2 z^2 + \dots$; using the fact that h can be formally embedded in a one-parameter group, it is easy to prove that the formal conjugacy class of h is determined by two invariants

- i) The pair (n, a_n) , where $n = \inf \{m \mid a_m \neq 0\}$
- ii) The coefficient of $\frac{1}{z}$ in $\frac{1}{h(z)-z}$ (this depends only on $a_n, \dots, a_{2n-1}$)

Then one has to find the analytic invariants corresponding to a given formal class. For simplicity, I will consider only the formal class defined by $a_2 = a_3 = 1$. Then we have

$h(z) = z + z^2 + z^3 + a_4 z^4 + \dots$; write $h_0(z) = \frac{z}{1-z}$. After the change of variable $z = \frac{1}{g}$, $g = \frac{1}{h}$, one has $g(\xi) = \xi - 1 + a_2 \xi^{-2} + a_3 \xi^{-3} + \dots$, and $g_0(\xi) = \xi - 1$; one sees easily that there exist one and only one fixed power series

$\varphi(\xi) = \xi - c_1 \xi^{-1} + c_2 \xi^{-2} + \dots$, which satisfies $\varphi \circ g = g \circ \varphi$.

Let $\tilde{\Phi}$ be the Fourier-Borel transform of φ , i.e. the formal microfunction

$$\tilde{\Phi} = \delta' + \sum_{k \geq 0} c_k \frac{z^{k-1}}{(k-1)!} \gamma \quad (\gamma = \text{the Heaviside function})$$

one sees in fact $\tilde{\Phi} \in \tilde{\mathcal{E}}$; a much stronger result is the following

Theorem 3 (Ecahle, [E2]) If Ω denotes the set $2\pi i \mathbb{Z}$, one has: $\tilde{\Phi} \in \mathcal{C}(\Omega)$

Ecahle prove more precise results; consider for instance the analytic continuation of $\tilde{\Phi}$ at the half-plane $\operatorname{Re} x > 0$, and denote by $\tilde{\Phi}_\eta$ the microfunction that this continuation determines at the point $2\pi i \eta$; then one has, for some $c_n \in \mathbb{C}$ and p_n holomorphic near $2\pi i \eta$: $\tilde{\Phi}_\eta = c_n \delta(x - 2\pi i \eta) + p_n \gamma(x - 2\pi i \eta)$. Ecahle give some functional equation for $\tilde{\Phi}_\eta$ (and other singularities over $2\pi i \mathbb{Z}$ of $\tilde{\Phi}$), in terms of other derivations, and proves also that the c_n give a complete list of analytic invariants.

References

- [E1], [E2] J. Ecahle, Théorie des fonctions résurgentes, ~~2~~ vol 1 and 2, Publications Mathématiques de l'Université d'Orsay, (1981-82)
- [M] B. Malgrange, Travaux d'Ecahle, ed. Martine-Rami, in La systématique des singularités,