

A normal form theorem for first order
formulas and its application to Gaifman's
splitting theorem

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Let L be a first order predicate calculus with equality which has a fixed binary predicate symbol $<$. In this paper, we shall deal with quantifiers $\forall^*x$, $\forall_{x \leq y}$, $\exists x \leq y$ defined by; $\forall^*x A(x)$ is $\forall y \exists x (y \leq x \wedge A(x))$, $\forall_{x \leq y} A(x)$ is $\forall x (x \leq y \supset A(x))$, and $\exists x \leq y A(x)$ is $\exists x (x \leq y \wedge A(x))$. The expressions $\bar{x}$, $\bar{y}$, will be used to denote sequences of those symbols, e.g. $\bar{x}$ is $\langle x_1, x_2, \dots, x_n \rangle$ and $\bar{y}$ is $\langle y_1, y_2, \dots, y_m \rangle$. Also, $\exists \bar{x}$, $\forall \bar{x} \leq \bar{y}$, will be used to denote $\exists x_1 \exists x_2 \dots \exists x_n$, $\forall x_1 \leq y_1 \forall x_2 \leq y_2 \dots \forall x_n \leq y_n$,, respectively. Let X be a set of formulas in L such that; X contains every atomic formulas and is closed under substitutions of free variables and applications of propositional connectives $\neg$ (not), $\wedge$ (and), $\vee$ (or). Then, $\Sigma(X)$ is the set of formulas of the form $\exists \bar{x} B(\bar{x})$, where $B \in X$, and $\Phi(X)$ is the set of formulas of the form;
 $\forall^*x_1 \exists y_1 \dots \forall^*x_n \exists y_n \exists \bar{z} \forall \bar{u} \leq \bar{x} \exists \bar{v} \leq \bar{z} B(\bar{u}, \bar{y}, \bar{v})$, where $B(\bar{x}, \bar{y}, \bar{z}) \in X$. Since X is closed under $\wedge$, $\vee$, two sets $\Sigma(X)$ and $\Phi(X)$ are closed under $\wedge$, $\vee$ in the following sense: for any formulas A and B in $\Sigma(X)$ [$\Phi(X)$], there are formulas in $\Sigma(X)$ [$\Phi(X)$] which are obtained from $A \wedge B$ and $A \vee B$ by prefixing some quantifiers in them in the usual manner.

Let $W(x,y,z)$ be a formula in $\Sigma(X)$ which has no free variables except x,y,z . Then, the theory T_W in L consists of the following sentences;

$$\text{Tr} : \forall x \forall y \leq x \forall z \leq y (z \leq x),$$

$$\text{Ex}(W) : \forall x \forall y \exists z W(x,y,z),$$

$$\text{Un}(W) : \forall x \forall y \forall z \forall w (W(x,y,z) \wedge W(x,y,w) \supset z=w),$$

$$\text{Bn}(W) : \forall x \forall y \forall z (W(x,y,z) \supset z \leq x),$$

and

$$\text{Col}(W) : \forall \bar{w} [\forall u \leq x \exists v A(u,v,\bar{w}) \supset \exists y \forall u \leq x \exists v (W(y,u,v) \wedge A(u,v,\bar{w}))],$$

where $A(x,v,\bar{w})$ is a formula in L .

Since $\text{Col}(W)$ is a schema, T_W is an infinite set of sentences.

A mapping f from a set of formulas in L (domain of f) on to a set of formulas in L (range of f) is called a formula-mapping if $f(A)$ and A have the same set of free variables for each formula A in the domain of f .

In this paper, we shall give a concrete method to construct a formula-mapping f_W whose domain is the set of formulas in L and whose range is a subset of $\Phi(X)$, and prove the following fact.

THEOREM A. For any formula A in L , the formula $A \supset f_W(A)$ is provable from T_W in L , and the formula $f_W(A) \supset A$ is provable in L , i.e. $T_W \vdash_L A \supset f_W(A)$ and $\vdash_L f_W(A) \supset A$.

This theorem shows that any formula A in L is equivalent to $f_W(A)$ in $\Phi(X)$ with respect to the theory T_W , and furthermore the implication from $f_W(A)$ to A is provable logically. This is a normal

form theorem for first order formulas, of a new type.

In §1 below, we shall show some applications of Theorem A above and one of which, Corollary E below, is a generalization of Gaifman's splitting theorem, Corollary G below, in Gaifman [1].

The construction of f_W requires two auxiliary formula-mapping h and g_W , where h is a formula-mapping whose domain is the set of formulas (denoted by $\Pi_2(X)$) of the form $\forall \bar{x} \exists \bar{y} B$, $B \in X$, and whose range is a subset of $\Phi(X)$, and g_W is a formula-mapping whose domain is the set of formulas in L and whose range is a subset of $\Phi(X)$. Moreover, we can prove;

LEMMA 1. For any formula A in $\Pi_2(X)$, the formula $A \supset h(A)$ is provable from T_W in L and the formula $h(A) \supset A$ is provable in L , i.e. $T_W \vdash_L A \supset h(A)$ and $\vdash_L h(A) \supset A$.

LEMMA 2. For any formula A in L , the formula $A \supset g_W(A)$ is provable from T_W in L and the formula $g_W(A) \supset A$ is provable from $Tr, Ex(W), Un(W)$ in L , i.e. $T_W \vdash_L A \supset g_W(A)$ and $Tr, Ex(W), Un(W) \vdash_L g_W(A) \supset A$.

Although $Tr, Ex(W), Un(W)$ are not formulas in $\Pi_2(X)$, there are formulas $Tr^\circ, Ex(W)^\circ, Un(W)^\circ$ in $\Pi_2(X)$ which are obtained from $Tr, Ex(W), Un(W)$ by prefixing some quantifiers in them, respectively. Therefore, they are equivalent each other. Let $f_W(A)$ be one of formulas in $\Phi(X)$ which are obtained from $g_W(A) \wedge h(Tr^\circ) \wedge h(Ex(W)^\circ) \wedge h(Un(W)^\circ)$ by prefixing some quantifiers in it. Then, Theorem A clearly holds from Lemma 1 and

and Lemma 2. So, in order to prove Theorem A above, it is sufficient to construct h and g_w and prove Lemma 1 and Lemma 2, which will be done in §2 below.

21. Some applications.

In this section, we shall show some applications of Theorem A to normal form theorems and splitting theorems. From Theorem A, we have the following fact, immediately.

COROLLARY B. If T is a theory in L such that $T_W \subseteq T$ for some formula W in $\Sigma(X)$, then for any formula A in L , there is a formula B in $\Phi(X)$ such that $T \vdash_L A \supset B$ and $\vdash_L B \supset A$.

If $T = PA$, the first order axioms of Peano Arithmetic (cf. p.68-69 in Takeuti [4]), $L = L_{PA}$, the logic for PA , and X = the set of open formulas in L , then the assumptions of Corollary B hold by Matijasevic's theorem ([2]). Therefore, we have;

COROLLARY C. For any formula A in L_{PA} , there is a formula B of the form $\forall x_1 \exists y_1 \dots \forall x_n \exists y_n \exists z \forall u \leq x \exists v \leq z C(\bar{u}, \bar{y}, \bar{z})$, where $C(\bar{x}, \bar{y}, \bar{z})$ is an open formula, such that $PA \vdash_L A \supset B$ and $\vdash_L B \supset A$.

A weak form of Corollary C is proved in Motohashi [3] (Theorem F in [3]). Suppose that $\mathcal{A}$ and $\mathcal{A}'$ are two L -structures such that $\mathcal{A}'$ is an extension of $\mathcal{A}$. Then, $\mathcal{A}'$ is an outer extension of $\mathcal{A}$ (denoted by $\mathcal{A} \subseteq_o \mathcal{A}'$) if $\mathcal{A}' \models a \leq b$ and $b \in |\mathcal{A}|$ imply $a \in |\mathcal{A}|$, for any elements a, b in $|\mathcal{A}'|$. $\mathcal{A}'$ is a cofinal extension of $\mathcal{A}$ (denoted by $\mathcal{A} \subseteq_c \mathcal{A}'$) if $\mathcal{A}'$ is a model of the sentence Tr and for any b in $|\mathcal{A}'|$, there is an element a in $|\mathcal{A}|$ such that $\mathcal{A}' \models b \leq a$. Let S be a set

of formulas in L . Then, $\mathcal{U}$ is an S -extension of $\mathcal{M}$ (denoted by $\mathcal{M} \subseteq_S \mathcal{U}$) if $\mathcal{M} \models A[\bar{a}]$ implies $\mathcal{U} \models A[\bar{a}]$, for any formula $A(\bar{x})$ in S and any sequence $\bar{a}$ of elements in $|\mathcal{M}|$. Let Δ_0 be the set of bounded formulas in L (cf. p.133 in [1]) and $\mathfrak{F}_0(X)$ be the set of formulas of the form; $\forall \bar{u} \leq \bar{x} \exists \bar{v} \leq \bar{y} B(\bar{x}, \bar{y}, \bar{u}, \bar{v})$, where $B \in X$. From these definitions, the following facts follow immediately.

- LEMMA 3. (i) If $X \subseteq \Delta_0$, then $\mathfrak{F}_0(X) \subseteq \Delta_0$.
 (ii) If $\mathcal{M} \subseteq_o \mathcal{U}$, then $\mathcal{M} \subseteq_{\Delta_0} \mathcal{U}$.
 (iii) If $\mathcal{M} \subseteq_{\mathfrak{F}_0(X)} \mathcal{U}$ and $\mathcal{M} \subseteq_c \mathcal{U}$, then $\mathcal{M} \subseteq_{\mathfrak{F}(X)} \mathcal{U}$.

From Theorem A and (iii) of Lemma 3, we have:

COROLLARY D. Suppose that $\mathcal{M}$ is a model of T_W and $\mathcal{U}$ is a $\mathfrak{F}_0(X)$ -extension of $\mathcal{M}$. If $\mathcal{U}$ is a cofinal extension of $\mathcal{M}$, then $\mathcal{U}$ is an elementary extension of $\mathcal{M}$.

(Proof). Assume that $\mathcal{M} \models T_W$, $\mathcal{M} \subseteq_{\mathfrak{F}_0(X)} \mathcal{U}$ and $\mathcal{M} \subseteq_c \mathcal{U}$. Let $A(\bar{x})$ be an arbitrary formula in L and $\bar{a}$ a sequence of elements in $|\mathcal{M}|$ such that $\mathcal{M} \models A[\bar{a}]$. Let B be the formula $f_W(A)$. Since $T_W \vdash_L A \supset B$, we have that $\mathcal{M} \models B[\bar{a}]$. By (iii) of Lemma 3, $\mathcal{U}$ is a $\mathfrak{F}(X)$ -extension of $\mathcal{M}$. Hence, $\mathcal{U} \models B[\bar{a}]$, because $B \in \mathfrak{F}(X)$. Since $\vdash_L B \supset A$, we have that $\mathcal{U} \models A[\bar{a}]$. This means that $\mathcal{U}$ is an elementary extension of $\mathcal{M}$. (q.e.d.)

From Corollary D and (i), (ii) of Lemma 3, we have:

THEOREM E. Suppose that $\mathcal{A}$ and $\mathcal{B}$ are models of T_W , and X is a subset of Δ_0 . If $\mathcal{B}$ is a $\Phi_0(X)$ -extension of $\mathcal{A}$, then there is an elementary extension $\mathcal{C}$ of $\mathcal{A}$ such that $\mathcal{A} \subseteq_e \mathcal{C} \subseteq_e \mathcal{B}$.

(Proof). Assume that $\mathcal{A} \models T_W$, $\mathcal{B} \models T_W$, $X \subseteq \Delta_0$, and $\mathcal{A} \subseteq_{\Phi_0(X)} \mathcal{B}$. Let C be the set $\{b \in |\mathcal{B}|; \mathcal{B} \models b \leq a \text{ for some } a \text{ in } |\mathcal{A}|\}$. Then, clearly $|\mathcal{A}| \subseteq C$. Suppose that f is an n -ary function symbol in L and $b_1, b_2, \dots, b_n$ are elements in C . Let $a_1, a_2, \dots, a_n$ be elements in $|\mathcal{A}|$ such that $\mathcal{B} \models b_1 \leq a_1, \mathcal{B} \models b_2 \leq a_2, \dots, \mathcal{B} \models b_n \leq a_n$. Since $\mathcal{A} \models T_W$ and $\mathcal{A} \models \forall x_1 \leq a_1 \dots \forall x_n \leq a_n \exists y (f(x_1, \dots, x_n) = y)$, there is an element a in $|\mathcal{A}|$ such that $\mathcal{A} \models \forall x_1 \leq a_1 \dots \forall x_n \leq a_n \exists y \leq a (f(x_1, \dots, x_n) = y)$. Since the formula $\forall x_1 \leq u_1 \dots \forall x_n \leq u_n \exists y \leq v (f(x_1, \dots, x_n) = y)$ belongs to $\Phi_0(X)$, $\mathcal{B} \models \forall x_1 \leq a_1 \dots \forall x_n \leq a_n \exists y \leq a (f(x_1, \dots, x_n) = y)$. Hence, we have that $\mathcal{B} \models \exists y \leq a (f(b_1, \dots, b_n) = y)$, because $\mathcal{B} \models b_1 \leq a_1, \dots, \mathcal{B} \models b_n \leq a_n$. This means that $\mathcal{B} \models f(b_1, \dots, b_n) \leq a$. Therefore, the value of the interpretation of the function symbol f in $\mathcal{B}$ at $b_1, b_2, \dots, b_n$ belongs to the set C . So, the set C is closed under functions which are interpretations of function symbols of L in $\mathcal{B}$. Therefore, there is a substructure $\mathcal{C}$ of $\mathcal{B}$ whose universe is C . By the definition of $\mathcal{C}$, $\mathcal{A} \subseteq_e \mathcal{C} \subseteq_e \mathcal{B}$. By (ii) of Lemma 3, $\mathcal{C} \subseteq_{\Delta_0} \mathcal{B}$. On the other hand, $\Phi_0(X) \subseteq \Delta_0$ by $X \subseteq \Delta_0$ and (i) of Lemma 3. Hence, we have that $\mathcal{A} \subseteq_{\Phi_0(X)} \mathcal{C}$. Therefore, we conclude that $\mathcal{C}$ is an elementary extension of $\mathcal{A}$ by Corollary D. (q.e.d.)

From Theorem E, we have:

COROLLARY F. Suppose that X is a subset of Δ_0 and T is a theory in L such that $T_W \subseteq T$ for some W in $\Sigma(X)$.

If $\mathcal{A}$ and $\mathcal{A}$ are models of T such that $\mathcal{A}$ is an $\Phi_0(X)$ -extension of $\mathcal{A}$, then there is an elementary extension $\mathcal{I}$ of $\mathcal{A}$ such that

$$\mathcal{A} \subseteq_e \mathcal{I} \subseteq_o \mathcal{A}.$$

Let $T = PA$ and $X =$ the set of open formulas. Then, we have the following theorem from Corollary F and Matijasevic's theorem.

COROLLARY G (Gaifman's Splitting Theorem). If $\mathcal{A}$ and $\mathcal{A}$ are models of PA such that $\mathcal{A}$ is an extension of $\mathcal{A}$, then there is an elementary extension $\mathcal{I}$ of $\mathcal{A}$ such that $\mathcal{A} \subseteq_e \mathcal{I} \subseteq_o \mathcal{A}$.

§2. Some proofs. In this section, we shall construct two formula-mappings h and g_W , and prove Lemma 1 and Lemma 2 in the introduction of this paper.

LEMMA 4. $\vdash_L \forall x A(x) \equiv \forall^* x \forall u \leq x A(u)$.

Lemma 4 is an obvious consequence of the definition of $\forall^*$.

LEMMA 5. (i) $\vdash_L \exists \bar{z} \forall \bar{u} \leq \bar{x} \exists \bar{v} \leq \bar{z} A(\bar{u}, \bar{v}) \supset \forall \bar{u} \leq \bar{x} \exists \bar{z} A(\bar{u}, \bar{z})$.

(ii) $\text{Tr}, \text{Bn}(W), \text{Col}(W) \vdash_L \forall \bar{u} \leq \bar{x} \exists \bar{z} A(\bar{u}, \bar{z}) \supset \exists \bar{z} \forall \bar{u} \leq \bar{x} \exists \bar{v} \leq \bar{z} A(\bar{u}, \bar{v})$.

(Proof). Since (i) is obvious, we prove (ii) only. For the sake of simplicity, we assume that the lengths of $\bar{x}$ and $\bar{z}$ are the same number 2. Let B, C, D, E be the following formulas;

$$B : \forall u_1 \leq x_1 \forall u_2 \leq x_2 \exists z_1 \exists z_2 A(u_1, u_2, z_1, z_2),$$

$$C : \forall u_1 \leq x_1 \exists z_1 \exists z_2 \forall u_2 \leq x_2 \exists v_1 \exists v_2 (W(z_1, u_2, v_1) \wedge W(z_2, u_2, v_2) \wedge A(u_1, u_2, v_1, v_2)),$$

$$D : \exists z_1 \exists z_2 \forall u_1 \leq x_1 \exists w_1 \exists w_2 \forall u_2 \leq x_2 \exists v_1 \exists v_2 (W(z_1, u_1, w_1) \wedge W(z_2, u_1, w_2) \wedge W(w_1, u_2, v_1) \wedge W(w_2, u_2, v_2) \wedge A(u_1, u_2, v_1, v_2)),$$

$$E : \exists z_1 \exists z_2 \forall u_1 \leq x_1 \forall u_2 \leq x_2 \exists v_1 \leq z_1 \exists v_2 \leq z_2 A(u_1, u_2, v_1, v_2).$$

It is sufficient to prove $\text{Tr}, \text{Bn}(W), \text{Col}(W) \vdash_L B \supset E$.

But, this is obvious because $\text{Col}(W) \vdash_L B \supset C$, $\text{Col}(W) \vdash_L C \supset D$,

and $\text{Tr}, \text{Bn}(W) \vdash_L D \supset E$. (q.e.d.)

LEMMA 6. Suppose that A is a formula;

$$\forall u \leq x \forall x_1 \exists y_1 \dots \forall x_n \exists y_n \forall \bar{u} \leq \bar{v} C(\bar{u}, u, x_1, \dots, x_n, y_1, \dots, y_n)$$

and B is a formula;

$$\forall x_1 \exists y_1 \dots \forall x_n \exists y_n \forall u \leq x \forall \bar{u} \leq \bar{v} \exists z_1 \dots \exists z_n (W(y_1, u, z_1) \wedge \\ W(y_2, u, z_2) \wedge \dots \wedge W(y_n, u, z_n) \wedge C(\bar{u}, u, x_1, \dots, x_n, z_1, \dots, z_n)).$$

Then, $\text{Col}(W) \vdash_L A \supset B$ and $\text{Ex}(W), \text{Un}(W) \vdash_L B \supset A$.

(Proof). For each $i=1, 2, \dots, n+1$, let A_i be the formula;

$$\forall x_1 \exists y_1 \dots \forall x_{i-1} \exists y_{i-1} \forall u \leq x \forall x_i \exists y_i \dots \forall x_n \exists y_n \forall \bar{u} \leq \bar{v} \exists z_1 \dots \\ \dots \exists z_{i-1} (W(y_1, u, z_1) \wedge \dots \wedge W(y_{i-1}, u, z_{i-1}) \wedge \\ C(\bar{u}, u, x_1, \dots, x_n, z_1, \dots, z_{i-1}, y_i, \dots, y_n)).$$

Then, A_1 is A , A_{n+1} is B , $\text{Col}(W) \vdash_L A_i \supset A_{i+1}$, and

$\text{Ex}(W), \text{Un}(W) \vdash_L A_{i+1} \supset A_i$, for each $i=1, 2, \dots, n$. Therefore, we have that $\text{Col}(W) \vdash_L A \supset B$ and $\text{Ex}(W), \text{Un}(W) \vdash_L B \supset A$. (q.e.d.)

Now, we define the formula-mapping h and prove Lemma 1.

For each formula A of the form $\forall x_1 \dots \forall x_n \exists \bar{z} B(x_1, \dots, x_n, \bar{z})$, $B \in X$, let $h(A)$ be the formula;

$$\forall x_1 \exists y_1 \dots \forall x_n \exists y_n \exists \bar{z} \forall \bar{u} \leq \bar{x} \exists \bar{v} \leq \bar{z} B(\bar{u}, \bar{v}).$$

By Lemma 4, A is equivalent to the formula $\forall \bar{x} \forall \bar{u} \leq \bar{x} \exists \bar{z} B(\bar{u}, \bar{z})$.

On the other hand, $\text{Tr}, \text{Bn}(W), \text{Col}(W) \vdash_L \forall \bar{x} \forall \bar{u} \leq \bar{x} \exists \bar{z} B(\bar{u}, \bar{z}) \supset h(A)$,

and $\vdash_L h(A) \supset \forall \bar{x} \forall \bar{u} \leq \bar{x} \exists \bar{z} B(\bar{u}, \bar{z})$ by Lemma 5.

Therefore, we have that $T_W \vdash_L A \supset h(A)$ and $\vdash_L h(A) \supset A$.

This completes our proof of Lemma 1.

In order to construct g_W and prove Lemma 2, we require some preliminaries.

A quasi $\mathfrak{Q}(X)$ -formula A of degree k ($k=0,1,2,\dots$) is a formula of the form;

$$\forall x_1 \exists y_1 \dots \forall x_n \exists y_n \forall x_{n+1} \exists y_{n+1} \dots \forall x_{n+k} \exists y_{n+k} \forall u_1 \leq x_1 \dots \forall u_n \leq x_n \\ B(u_1, \dots, u_n, x_{n+1}, \dots, x_{n+k}, y_1, \dots, y_{n+k}),$$

where $B(x_1, \dots, x_{n+k}, y_1, \dots, y_{n+k}) \in \Sigma(X)$.

If A is a quasi $\mathfrak{Q}(X)$ -formula of degree k of the above form and k is a positive integer, let $j_W(A)$ be the formula;

$$\forall x_1 \exists y_1 \dots \forall x_n \exists y_n \forall x_{n+1} \exists y_{n+1} \forall x_{n+2} \exists y_{n+2} \dots \forall x_{n+k} \exists y_{n+k} \\ \forall u_1 \leq x_1 \dots \forall u_n \leq x_n \forall u_{n+1} \leq x_{n+1} \exists z_{n+1} \dots \exists z_{n+k} (\\ W(y_{n+1}, u_{n+1}, z_{n+1}) \wedge \dots \wedge W(y_{n+k}, u_{n+k}, z_{n+k}) \wedge \\ B(u_1, \dots, u_n, u_{n+1}, x_{n+2}, \dots, x_{n+k}, y_1, \dots, y_n, z_{n+1}, \dots, z_{n+k})).$$

Then, $j_W(A)$ is a quasi $\mathfrak{Q}(X)$ -formula of degree $k-1$ because $\Sigma(\bar{X})$ is closed under $\wedge$ (cf. introduction of this paper).

If A is a quasi $\mathfrak{Q}(X)$ -formula of degree 0, then A has the form;

$$\forall x_1 \exists y_1 \dots \forall x_n \exists y_n \forall u_1 \leq x_1 \dots \forall u_n \leq x_n \exists \bar{z} C(\bar{u}, \bar{y}, \bar{z}), \quad C(\bar{x}, \bar{y}, \bar{z}) \in X.$$

Let $j_e(A)$ be the formula;

$$\forall x_1 \exists y_1 \dots \forall x_n \exists y_n \exists \bar{z} \forall \bar{u} \leq \bar{x} \exists \bar{v} \leq \bar{z} C(\bar{u}, \bar{y}, \bar{v}).$$

Then, $j_W(A)$ is a formula in $\mathfrak{Q}(X)$.

From Lemma 4, Lemma 5, and Lemma 6, we can obtain the following lemma.

LEMMA 7. $T_W \vdash_L A \supset j_W(A)$ and $\text{Tr}, \text{Ex}(W), \text{Un}(W) \vdash_L j_W(A) \supset A$.

Now, we can define g_W by the following; For each formula A in L , let A° be one of formulas which are equivalent to A and have the form $\forall x_1 \exists y_1 \dots \forall x_n \exists y_n B(\bar{x}, \bar{y})$, where B is an open formula. Then, A° is a quasi- $\Phi(X)$ -formula of degree n . Let $g_W(A)$ be the formula $j_W^{n+1}(A^\circ)$, where $j_W^0(A^\circ)$ is A° and $j_W^{i+1}(A^\circ)$ is $j_W(j_W^i(A^\circ))$ for each $i=0,1,\dots$. Then, clearly g_W is a formula-mapping whose domain is the set of formulas in L and whose range is a sub-set of $\Phi(X)$. Moreover, Lemma 2 holds by Lemma 7. This completes our proofs of Lemma 1 and Lemma 2.

References

- [1] H.Gaifman, A note on models and submodels of Arithmetic,
Conference in Mathematical Logic London'70, Springer Lecture
Note No.255, 128-144.
- [2] Ju.V.Matijasevic, Diophantine representation of enumerable
predicates, Mathematics of the USSR-Izvestija, vol.5(1971),
1-28.
- [3] N.Motohashi, Preservation Theorem and Relativization Theorem
for Cofinal Extensions, to appear.
- [4] G.Takeuti, Proof Theory, North-Holland, Amsterdam, 1975.