

Vanishing theorems with algebraic growth and
algebraic divisible properties .

(Complex analytic De Rham cohomology 1)

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The purpose of the present note is to announce certain
quantitative properties of coherent sheaves and analytic varieties .
Results given here are originally and primarily intended for
applications to differential forms on complex analytic varieties
with arbitrary singularities (c.f. the end of this note). Results
stated here are ,however, of their own interests . Our basic purpose
is to discuss vanishing theorems of certain types where quantitative
properties appear . Details of this note will appear elsewhere .
(of objects considered)
Quantitative properties examined here are as follows : (I) Asymptotic
behaviours w.r.t pole loci. (I I) Divisible properties w.r.t. subvarieties.
Our arguments will be divided to two steps : (i) A step
in which the asymptotic behavior only enters. (ii) A step where
⁽ⁱⁱ⁾ both asymptotic behaviors and divisible properties appear.

Notational remarks : We write linear functions and
monomials as L and M . A couple, denoted by $\sigma = (\sigma_1, \sigma_2)$, is
a couple of positive numbers. For a set $\{\sigma^1, \dots, \sigma^s\}$ of couples

Maps $(L): \{\sigma^1, \dots, \sigma^s\} \rightarrow \sigma'$ and $(M): \{\sigma^1, \dots, \sigma^s\} \rightarrow \xi \in \mathbb{R}$ are

said to be of exponential-algebraic type ((e.a)-type) if

$$\sigma' = \left\{ M_1(\sigma_1^1, \dots, \sigma_1^s) \wedge \exp M_2(\sigma_2^1, \dots, \sigma_2^s), L(\sigma_2^1 + \dots + \sigma_2^s) \right\}, \mathcal{E} = M_1(\sigma_1^1, \dots, \sigma_1^s) \wedge \exp M_2(\sigma_2^1, \dots, \sigma_2^s).$$

(I) We start with a datum $(\Delta(r; P_0), X, D)$ of a polydisc Δ with the center P_0 of radius r in $\mathbb{C}^n$, a variety $V \ni P_0$ in Δ and a divisor $D \ni P_0$ in Δ . We write irreducible decompositions of X and D at P_0 as $X_{P_0} = \bigcup_j X_{P_0 j}$ and $D_{P_0} = \bigcup_j D_{P_0 j}$.

Assume that D contains the singular locus of X and that $X_{P_0 j} \neq D_{P_0 j'}$ for any pair (j, j') . Moreover, consider a coherent sheaf (F) admitting a resolution of the following form.

$$0 \longrightarrow (O)^d_s \xrightarrow{\quad} \dots \xrightarrow{K_2} (O)^d_1 \xrightarrow{K_1} (F(O^d)) \longrightarrow$$

where K 's are matrices whose coefficients are meromorphic functions on X with the pole $D' = D \cap X$. A point P is near P_0 if P is in a small neighbourhood of P_0 . For a point near P_0 , the

* A variety and a function are always complex analytic ones in this note.

intersection $\Delta(r; P) \cap X$ is denoted by $\Delta(r; P, X)$. Moreover,

for a point $Q \in \Delta(r; P, X) - D$, we mean by $N_Q(Q, D)$ the neighbourhood of Q in X defined by $N_Q(Q, D) = \{ Q' : d(Q, Q') \leq \sigma_1 d(Q, D)^{\sigma_2} \}$.

Define a (non locally finite) covering $(A)(r; P, D)$ of $\Delta(r; P, X)$ by $(A)(r; P, D) = \{ N_Q(Q, D) : Q \in \Delta(r; P, X) - D \}$. We formulate our problem in terms of such coverings (A) . A q -cochain $\psi \in C^q(N((A)(r; P, D)), (\mathbb{F}))$; N = nerve; is of algebraic growth $\alpha = (\alpha_1, \alpha_2)$

if

$$|\psi(Q)| \leq \alpha_1 d(Q, D)^{\alpha_2}$$

Then our first result is as follows.

Lemma 1. A vanishing theorem with algebraic growth.

There exists a datum (L_1, L_2, M) depending on (X, D, F)

only such that the following is valid .

For a cocycle $\psi \in Z^q(N((A)(r; P, D), \mathbb{F}))$ of algebraic growth α , there exists a cochain $\psi' \in C^{q-1}(N((A)(r; P, D), \mathbb{F}))$ of growth (α'_1, α'_2) so that

$$(1) \quad \delta_{q-1}(\psi') = \psi$$

$$(2) \quad (\alpha'_1, \alpha'_2) = (\alpha''_1 \times M(r)^{-1}, \alpha''_2), \quad \sigma'_1 \in I_{\sigma_1}(\sigma)$$

and $r' = M(r)$ with $(\alpha''_1, \alpha''_2) = L_1(\alpha)$.

In the equation (1) ψ is regarded as an element in $C^q(N((A)_{\rho}(r;P,D)), (F))$

by taking a refinement and a restriction suitably.

Remark 1. For a domain $\Sigma = \Delta(r;P) \times C^N$ and for a coherent sheaf F' over Σ , a similar result to the lemma 1 is valid

(by changing the distance to D by $\sum_j |x_j|$: (x_j) are coordinates of C^N).

Remark 2. That the datum $((L_1, L_2, M))$ in the lemma 1 is independent of points P shows that the lemma 1 is of semi-global nature.

Remark 3. Cohomology theories with growth conditions have recently been studied by various persons for various purposes (c.f. [1], [3], [4], [5], [6]). Our methods depend on examinations of Cousin integrals and of combinatorial arguments. We proceed along standard methods of discussing vanishing theorems on Stein varieties and have many similarities with works cited above. However our situation as well as our statement are, to the author's knowledge, new. Our notion of 'algebraic growth' was inspired the notion of 'polynomial growth' due to R. Narasimhan (His result is that of $\bar{\partial}$ -estimation of L. Hörmander (see [5])).

() Here we state our basic problem in this note .

We consider a proper subvariety V of X and a set of analytic functions

$(f) = (f_1, \dots, f_t)$. Let $V_{P_0} = \bigcup V_{P_0, j}$ be the irreducible decomposition of V at P_0 . We assume conditions : $V_{P_0, j} \neq X_{P_0, j}$ for any pair (j, j')

, $V_{P_0, j} \neq D_{P_0, j'}$ for any pair (j, j') and $f_i \neq 0$ on $X_{P_0, j}$ for any pair (i, j) . Moreover we assume that V is the zero locus

of (f) on X . Our problem is formulated in terms of (f) .

Let $(\bar{X})^{m, 0}$ be the subsheaf of $(\mathcal{O})_X$ defined to be $(\mathcal{O})_X (f_1^m, \dots, f_t^m)$. This sheaf $(\bar{X})^{m, 0}$ is our basic subject . We associate sheaves $(\bar{X})^{m, s}$ ($s=1, \dots, t-1$)

to $(\bar{X})^{m, 0}$ in the following manner : For a multiindex $I_s = (i_1, \dots, i_s)$ define an element $(f)(I_s, m) \in (\mathcal{O})^{(s-1)}_X (X)$ by

$$(f)(I_s, m)(J_{s-1}) = \begin{cases} 0, & \text{if } J_{s-1} \neq I_s \\ (-1)^{k-1} f_{i_k}^m & \text{if } J_{s-1} = (i_1, \dots, \hat{i}_k, \dots, i_s), \end{cases}$$

(for a vector $(g) \in (\mathcal{O})^{(t)}_X$, $(g)(J_s)$; $J_s = (j_1, \dots, j_s)$ is J_s -component of (g) .)

Using the above elements $(f)(I_s, m)$, define sheaf homomorphisms $K(s, m)$

$(s=1, \dots, t) : (\mathcal{O})^{(t)}_X \rightarrow (\mathcal{O})^{(s-1)}_X$ by

$$K(s, m)(g^s) = \sum_{I_s} (g^s)(I_s) (f)(I_s, m).$$

Note that $(\bar{X})^{m, 0} = K(s, m)(\mathcal{O})^{(t)}_X$. Define $(\bar{X})^{m, s}$ ($s=1, \dots, t-1$) by $(\bar{X})^{m, s} = K(s, m)(\mathcal{O})^{(s+1)}_X$.

If the jacobian condition : $\det \frac{\partial (f_1, \dots, f_t)}{\partial (x_1, \dots, x_t)} \neq 0$ holds at each point

on $X((x_1, \dots, x_t))$ are coordinates of X) then the exact sequence

$$0 \rightarrow (0) \xrightarrow{K(t,m)} (0)^{(t-1)} \rightarrow \dots \rightarrow (0)^t \xrightarrow{K(1,m)} X^m \rightarrow 0$$

holds. In general the above sequence fails. It is, however, found

that associating (X^m, s) to $(X^m, 0)$ is meaningful in our discussion concerned

with divisible properties : We formulate two problems in terms of

sheaves (X^m, s) rather than $(X^m, 0)$ only: (i) A cochain $\psi \in C^q(N(A(r; P, D), X^m, s))$

is of algebraic growth $(\alpha_1, \alpha_2, \alpha_3)$ if ψ has the following property.

If $\bigcap_{i=1}^{q+1} N_{\sigma}(Q_i) \neq \emptyset$, the element $\psi(\bigcap_{i=1}^{q+1} N_{\sigma}(Q_i) : Q)$

is written as

$$(1) \quad \psi(Q) = \sum_{I_s} \psi^s(I_s, m); \quad \psi^s \in (0)^{(t/s)}(\bigcap_{i=1}^N (Q_i))$$

with the estimation

$$(2) \quad |\psi^s| \leq \alpha_1 d(Q, D)^{\alpha_2} d(Q, V)^{\alpha_3}$$

Now our first assertion is as follows.

Theorem 1 . There exists a datum $(L, L_1, M_i; i=1, 2, 3)$

depending on $(X, V, D, (f))$ only with which the following are valid.

For a cocycle $\psi \in Z^q(N(A(r; P, D), (X^m, s)))$ of growth $(\alpha_1, \alpha_2, \alpha_3)$

there exists a cochain $\psi' \in C^{q-1}(N(A)(r, P, D), X^{m, s})$ of growth

$(\alpha'_1, \alpha'_2, \alpha'_3)$ such that the equation

$$(1) \quad \delta(\psi') = \psi,$$

and

$$(2) \quad r' = M_1(r), \sigma' = L(\sigma), (\alpha'_1, \alpha'_2, \alpha'_3) =$$

$$(M_2(\alpha_1, \sigma_1, r) \cdot \exp M_3(\alpha_2, \sigma_2, \alpha_3), L_1(\alpha_2 + \sigma_2), L_2(\alpha_3)),$$

so far as $\alpha_3 \in L_3(m)$.

In the above statement an emphasize is put on a 'middle point' of the asymptotic behavior and the divisible property :

'The independenceness of the order of the pole α'_2 from the divisible index m ' is a key point. This is possible by diminishing the given α_3 to $\alpha_{31} = L_3(\alpha_3)$ and will be made an essential use of in our application of Theorem 1 to differential forms :

The second problem is as follows : Given an element $(g^s) \in (0)^{(\frac{1}{s})}$ so that $K(s, m)(g^s) = 0$ ($s \geq 1$, if $s = 0$ we do not consider an algebraic condition), find $(g^{s+1}) \in (0)^{(\frac{1}{s+1})}$: $K(s+1, m)(g^{s+1}) = g^s$.

Precisely let us consider a proper subvariety V' of V . Instead we do not consider the divisor D in this case. Take a point $P \in V - V'$.

$N(P, V')$ is a neighbourhood of P in X defined to be $\{Q: d(Q, V') \leq r\}$.

An element $(g^s) (s=0, \dots, t-1) \in (0) \binom{t}{s} (N_r(P, V'))$ is a testifying datum with quantitative property (b, α_3) if the following are valid.

$$(A)_1 \quad K(s, m)(g^s) = 0 \quad (s \geq 1),$$

$$(A)_2 \quad |g^s| \leq bd(Q, V)^{\alpha_3}, \quad Q \in N_r(P, V'). \quad (s=0, 1, \dots, t-1).$$

Then our second assertion is follows.

Theorem 2 . . Weak syzygy with quantities .

There exists a datum $(M, M_i (i=1, 2, 3), L_i (i=1, 2, 3), \sigma)$

depending on $(X, V, V', (f))$ only with which the following is true.

(B) For a testifying datum (g^s) with quantitative property (b, α_3) , there exists an element $(g^{s+1}) \in (0) \binom{s+1}{s} (N_r(P, V'))$ so that

$$(B)_1 \quad K(s, m+1)(g^{s+1}) = g^s,$$

$$(B)_2 \quad (g^{s+1}) \text{ is of quantitative property } (b', \alpha'_3),$$

where $b' = M_1(r)^{-1} \cdot M_2(b) \cdot \exp M_3(\alpha_3)$, $\alpha'_3 = L_1(\alpha_3)$ and $r' = M(r)$, holds so far as $\alpha_3 \geq L_2(m)$, $r \leq \sigma_1 d(P, V')^{\sigma_2}$.

Remark . Problems of differential forms which we consider are as follows . Detailed arguments will be given elsewhere . Here we do a sketch of our problems : Start with a datum (X, V, D, P_0)

defined in previous arguments : For differential sheaves $\Omega = \Omega_X$

and $\Omega(*D) = \Omega_X(*D') =$ sheaf of meromorphic forms with the pole D'
 $= D \cap X$, $\widehat{\Omega}$ and $\widehat{\Omega}(*D')$ are completions of Ω and $\Omega(*D')$

along V . Our problems are spoken in terms of the above two rings

$\hat{\Omega}$, $\hat{\Omega}(*D)$. Concerning the ring $\hat{\Omega}(*D)$ our problem is to show the isomorphism

$$(R_1^* \mathbb{C})_{p_0} \cong H^*(\hat{\Omega}(*D)_{p_0}); i \text{ is the inclusion;}$$

$$i: V-D \hookrightarrow V.$$

This is a generalization of a well known theorem of A. Grothendieck [1].

Concerning the ring $\hat{\Omega}$ we ask, under the assumption of $X = \text{smooth variety}$, the exact sequence: $0 \rightarrow \hat{\Omega}^0 \xrightarrow{d} \hat{\Omega}^1 \rightarrow \dots$, and a divisible property of the integration of differential forms.

Precise meaning of the above problems will be discussed, the author plans, in another announcement. Roughly the theorem 1 and the lemma 1 are analytic keys to our problems on differential forms.

References.

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Quantitative properties of analytic varieties in C^∞ -differentiable aspects

(Complex analytic De Rham Cohomology 2.)

This is a continuation of 1°. We state certain (elementary) quantitative properties of real analytic varieties. Our results stated here are intended for differential forms as in 1°. As in 1° two properties ... asymptotic behavior w.r.t. a subvariety and divisible properties w.r.t. a subvariety ... will be studied.

Let V be a real analytic manifold, and let W be a real analytic subvariety of V . We assume that $V-W$ is covered by finite coordinate neighbourhoods U_i . A C^∞ -function φ in $V-W$ will be said to have asymptotic behavior w.r.t. W if the following is true.

For any differential operator $D^i = \frac{\partial^{n-i}}{\partial x_{i,1}^{r_1} \cdots \partial x_{i,i}^{r_i}} \cdot D^i(\varphi)$ is estimated as

$$|D^i(\varphi)| \leq r_{i,1}^{r_1} \cdots r_{i,i}^{r_i} \cdot \delta_i^i(\varphi)$$

where $(x^i) =$ coordinates of U^i .

A C^∞ -form φ in $V-W$ has asymptotic

behavior w.r.t. W if each coefficient φ_j has such a property.

Let (U, V, P) be a datum composed of a domain U in $\mathbb{R}^n$ a subvariety V in U and a point P in V . This datum is fixed throughout this note. Our results are two. We state at first our results in an 'intrinsic' manner (i.e. in terms of varieties in question and of coordinates (x)). The first problem is as follows:

(I) C^∞ thickenings and their quantitative properties.

In this problem we consider a proper subvariety V' of V in addition to the datum (U, V, D) . For a couple σ , we mean by $N_\sigma(V:V')$ the neighbourhood of $V-V'$ defined by $N_\sigma(V:V') =$

$\{N(Q:V'): Q \in V-V'\}$. A neighbourhood N of $V-V'$ is a δ -thickening of $V-V'$ if $H^*(V-V':\mathbb{R}) \cong H^*(N:\mathbb{R})$. Let $\{N_j; j \in \mathbb{Z}^+\}$ be a direct system (w.r.t. the inclusion relation) of C^∞ thickenings of $V-V'$. For a direct system $\{N_j; j \in \mathbb{Z}^+\}$ the following conditions are always assumed.

- (1) For any N_j there exists a couple σ_j such that $N_j \supset N_{\sigma_j}(V:V')$.
- (2) For an arbitrarily given σ , $N_j \subset N_\sigma(V:V')$ for a sufficiently large j .

For a neighbourhood N of $V-V'$, $\Omega(N)$ stands for the ring of C^∞ -differential forms in N . Moreover, we understand by $\Omega(N:V')$ the subring of $\Omega(N)$ composed of those C^∞ -forms having

asymptotic behaviors w.r.t. V' . Given a direct system $\{N_j; j \in \mathbb{Z}^+\}$ of $\tilde{c}$ -thickenings of $V-V'$, we let $\hat{\Omega}(N; V')$ be the direct limit: $\lim. \text{dir. } \Omega(N_j; V')$. This ring $\hat{\Omega}$ is a differential ring in an obvious way. Our first result, which is essentially of elementary nature, is as follows:

Lemma 1. For a fixed datum (U, V, V', P) , there exist a neighbourhood U' of $P : U \supset U'$, and a direct system $\{N_j\}$ of $\tilde{c}$ -thickenings of $V-V'$ (in U') in such a manner that

$$\underline{H^*(V-V'; R)} = (H^*(\hat{\Omega}))$$

holds.

The second problem is as follows.

(I I) Quantitative properties of retraction maps.

In this second situation we start with the datum (U, V, P) .

Consider a subvariety D' of U such that $D' \not\supset V$. Let I be the interval $[0, 1]$. A continuous map $\tau : I \times U \rightarrow U$ is a retraction of (U, V, D') (to P) if the following are satisfied: (i) $\tau(1, Q) = Q : Q \in U$, (i i) $\tau(0, Q) = P : Q \in U$, (i i i) $\tau : I \times V \rightarrow V$ (or $I \times D' \rightarrow V$ (D')). Moreover, τ is $\tilde{c}$ -map outside D' if ψ is $\tilde{c}$ -map in $(0, 1] \times (U - D')$. For a fixed datum (U, V, D', P) , a retraction map τ is assumed to satisfy the above conditions (i), (i i), (i i i) and the

differentiable property mentioned just above . A retraction map γ satisfies quantitative conditions w.r.t. (V, D') if the following conditions are satisfied .

(1) There exist triples $^*(\beta) = (\beta_1, \beta_2, \beta_3)$ and $(\beta') = (\beta'_1, \beta'_2, \beta'_3)$ in such a manner that the following ' distance preserving property ' to V

$$\beta_1 \cdot d(Q, V)^{\beta_2} \cdot \rho^{\beta_3} \leq d(Q, V) \leq \beta'_1 \cdot d(Q, V)^{\beta'_2} \cdot \rho^{\beta'_3},$$

holds for a point $Q \in U$. Here ρ is in $(0, 1]$ and $Q_\rho = \gamma(\rho, Q)$.

(2) For each pair (k, K) ; $k \in \mathbb{Z}^+$, $K \in \mathbb{Z}^{+n}$, there exists a triple $\gamma(k, K)$ with which the inequality

$$\left(\frac{\partial}{\partial \rho} \right)^k \rho^K (x_j(Q_\rho)) \leq \gamma(k, K)_1 d(Q, D') - \gamma(k, K)_2 \cdot \rho^{\gamma(k, K)_3}$$

holds for each point $Q \in U - D'$.

Then our second assertion, which is also elementary,

is as follows.

Lemma 2. Quantitative properties of retractions .

For a given datum (U, V, P) we find a neighbourhood

-* 'Triple' is a triple of positive numbers.

U' of P such that the following is valid .

() There exist varieties D'_j ($j=1, \dots, m$) so that

() $\bigcap_j D'_j$ is a proper subvariety of V ,

and

() for each D'_j there exists a retraction τ_j

of (U, V, D'_j) satisfying quantitative conditions w.r.t. (V, D'_j) .

Remark 1 . In both lemmas 1 and 2 neighbourhoods U' , with which assertions in lemmas are valid , are cofinal in the set of all the neighbourhoods of P .

Remark 2 . In the lemma 2 it seems quite likely that a ~~divisor~~ ^{subvariety} D' is chosen to be the singular locus of V . In our treatment of lemmas we take a 'relative version' which will be explained soon later . Varieties D'_j appear because of the existence of singular loci of maps considered besides the singular locus of V itself .

We quickly indicate in what a manner the above lemmas are related to our original proof of the complex analytic De Rham cohomology¹ : The lemma 1 is a $\tilde{c}^\infty$ -help as well as a $\tilde{c}^\infty$ -analogue of the isomorphism : $(R_{i*} \mathcal{O})_P \cong (H^i(\hat{\Omega}(*D)))$ (c.f. [3]).

The lemma 2 is used to show a 'divisible property' of integrations of differential forms : Start with a datum (U, V, P) . Let V be the zero locus of an analytic function f . An r -times differentiable form $\mathcal{Y}$ is m -times differentiable by f if each coefficient $\mathcal{Y}_j$ of $\mathcal{Y}$ is divisible m -times by f in the category of C^r -functions. Given a C^r closed form $\mathcal{Y}$ in U which is divisible by f m -times ($m \geq 0$). Our problem, whose precise formulation will be given in another publication, is of the following type.

To find a domain $U' : U \supset U' \ni P$, and a C^r -form $\mathcal{Y}'$ in U'

in such a manner that

$$\underline{d\mathcal{Y}' = \mathcal{Y}}, \text{ and } \mathcal{Y}' \text{ is } m'\text{-times divisible by } f \text{ (} m' \geq 0 \text{)}$$

It is not difficult to see that the lemma 2, combined with a standard method of proving the Poincare's lemma (c.f. De Rham [2]), plays a key roll to the above explained problem.

Lemmas 1 and 2 are of intrinsic nature to the given data (U, V, P) and (U, V, V', P) . In our discussions of these lemmas some other materials are introduced. Materials introduced will be explained soon later. Several interesting (to the author) problems arise

through our approach though it is an another thing if our approach gives ~~an~~ 'short ways' to the above lemmas. We attach a series of varieties $\{V^1, V^2, \dots, V^d = V\}$, where $V^1 \subset U^1 \subset \mathbb{R}^1, \dots, V^i \subset U^i \subset \mathbb{R}^i \dots$ to V . Also we attach a series of stratification $S^1, \dots, S^d$ of $V_*,^1 \dots, V^d$. We do not assume the basic conditions A, B of H. Whitney by some technical reasons. We assume that $\pi_{i+1,i}: V^{i+1} \rightarrow V^i$ is (not necessarily surjective) finite ramified map

We will state our problem in terms of these strata. Roughly we impose 'compatibility Conditions' between objects considered and stratifications $\{S^i\}$. In doing our arguments the following condition will be considered for $\{S^i\}$.

(i). If $S^{j'} \in \mathcal{S}^{j'}$ then $\pi_{jj'}(S^{j'}) \in \mathcal{S}^j$ ($j < j'$), and conversely if S^j is in $\mathcal{S}^j$ then $\pi_{jj'}^{-1}(S^j) \cap U^{j'}$ is a union of strata in $\mathcal{S}^{j'}$.

In addition to the series S , sets of analytic functions (F)

$$= \{ f_t^j(y) ; j=1, \dots, n, S \in \mathcal{S}, t=1, \dots, j-\dim(S^j) \}$$
 are considered.

The following conditions for (F) are assumed.

(i)₁ $f_t^j = f_t^j(x_1, \dots, x_{n_j}; x_{n_j+t})$ is a monic polynomial in x_{n_j+t} with variables $(x_1, \dots, x_{n_j}) : n_j = \dim(S^j)$.

(i)₂ $f_t^j(S^j)$ vanishes on S^j and the 'jacobian condition:

$$\det \left[\frac{\partial (g_1, \dots, g_n)}{\partial (x_{n_j+1}, \dots, x_j)} \right] \neq 0 \text{ holds.}$$

Next given (i)₃ If $S_1^{j'} > S_1^j$ then the order of $f_t^j(S_2^{j'})$

= the minimal non-negative integer so that $\partial_t^{k_0} x_{n_j+t} = 0$: is constant on S_1^j .

The condition (i) is imposed in order to control behaviors of $\mathcal{S}_2^{j'}$ and is important in our quantitative discussions. We work around $\mathcal{S}_1^{j'}$ with the data $\{U, V, S, \mathcal{D}\}$. Then corresponding facts to lemmas 1,2 are formulated in term of S^j 's and behaviors of concerned subjects under the projections $\pi_{jj'}$ are examined*. Concerning the lemma 2 we impose conditions to retractions τ of (U^j, V^j) : For each stratum S^j , $\tau_j : U^{S^j} \rightarrow S^j$ and if $j' > j$ then $\pi_{jj'} \cdot \tau_{j'} = \tau_j$.

Our situation in the lemma 1 is as follows : A c -thickening N^B of S^j is an assignment $N^j : S^j \rightarrow N^j(S^j)$ = neighbourhood of S^j . Basic properties of N^j are : For S'^j , $N(N^j(S^j)) = S'^j$, (i) For $S_1^j, \dots, S_t^j$, $H(N(S_1^j), \dots, N(S_t^j)) = H(S_1^j, \dots, S_t^j)$. Consider direct systems N^j of c -thickenings of S^j . A compatible condition with j, j' similar to (i) is imposed to N^j . Subvarieties V^j ($V^j \cap S^j = V'$) are determined by V' and S . For a series $S_1^j, \dots, S_t^j$ rings $(\bigoplus_{i=1}^t N^j(S_i^j) : V^j)$ and their direct limit $(S_1^j, \dots, S_t^j ; V^j)$ are defined in a similar way to $(V - V' : V')$, $(V - V' : V')$. Our first key point is to reduce () to the following 'local version'.

() For each series $S_1^j, \dots, S_t^j$, $H^*(S_1^j, \dots, S_t^j ; V^j) \xrightarrow{\text{dir. lim.}} H^*(N^j(S_1^j, \dots, S_t^j) : V^j)$

The above procedure is similar to that of the residue operator.

Our another key is to associate a finite simple covering $A(N(S^j))$ to each series $S_1^j, \dots, S_t^j$ so that a compatible condition similar to (i)₁ and a quantitative condition isare satisfied. (For the definition of the simple covering , see A. Weil []).

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§ 2-1 C^∞ -thickenings

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n. 1. In this numero, our arguments will be concentrated to problems of expressing the (topological) cohomology group by C^∞ -differentiable forms with suitable asymptotic behavior: We shall fix some notations used here. Let U be a domain in $\mathbb{R}^n$, and let W be a closed set in U . (we do not assume that W is a $\mathbb{K}$ -analytic variety in general.) Moreover, assume that a finite set $\mathcal{S}^*$ of $\mathbb{K}$ -analytic manifolds is given. We assume the following two conditions.

(2.1.1), Each element $S_\lambda^* \in \mathcal{S}^*$ is equidimensional (of dimension $n(\lambda) \leq \dim(U)$).

(2.1.2)₂ W is a disjoint union of S_λ^* , Δ in $\mathcal{S}^*$: $W = \bigcup_\lambda S_\lambda^*$; $S_\lambda^* \in \mathcal{S}$

Let $\bar{S}^*$ be the set theoretical closure of S^* in U . We call $\bar{S}^*$ the closure of S^* (as usual). On the otherhand, $\bar{S}^* - S^*$ is called the

.*

This numero is independent of the other parts of this section.

(set - theoretical) frontier of S^* . Concerning the set $\mathcal{S}^*$, we shall impose the following (frontier condition) only .

(2.1.1)' For each $S^* \in \mathcal{S}^*$, $\text{Fron} (S^*)$ is a disjoint union of lower dimensional strata in $\mathcal{S}^*$.

The following follows easily from (2.1.1)₂ and (2.1.1)'

(2.1.1)'₁ If S^* has a common point with $S^* \in \mathcal{S}^*$ then $S^* \subset \text{Fron} (S)$ holds.

the set $\mathcal{S}^*$ satisfying (2.1.1) and (2.1.1)' will be called a pre-stratification of W , and W is called a pre stratified set.

(Remark) in this numero, we shall not impose basic conditions of R.Thom [] other than (2.1.1)' . Especially we do not assume the basic conditions concerning the existence of retraction maps (c.f. R. Thom []). By weakening conditions as explained in the above , we are led to work with differential forms with weaker conditions.

In this number ~~also~~, we consider not only strata S^* 's but also certain K -analytic functions related to strata S^* 's.

First we consider the following condition.

(2.2.1)'' For each $S^* \in \mathcal{S}^*$, there exists a real analytic functions $g(S^*)$, $h(S^*)$, defined in U , in such a manner that the following relations are valid.

$$(2.2.1)''_{1.1} \quad g(S^*; Q) \sim d(Q, \overline{S^*})$$

for any points Q in a neighbourhood $N_\delta(S^*, \text{Fron}(S^*))$ with a suitable couple (δ) .

$$(2.2.1)''_{1.2} \quad h(S^*, P) \sim d(P, \text{Fron}(S^*))$$

for any points P in S^* .

In order to impose a further condition, define a subset

$W^+(S^*)$ or W attached to each S^* , as follows

$$(W^+)_{2.1} \quad W^+(S^*) = \bigcup_{S'^*} S'^*, \quad \text{where } S'^* \text{'s are elements in } \mathcal{S}$$

so that condition: $S^* \supset S, \dim(S^*) < \dim(U)$ hold.

A further condition which we consider is formulated in the following manner.

(2.1.1)'' For any $S^* \in \mathcal{S}^*$, there exists a real analytic function $g(W(S^*))$ in U in such a way that the relation

$$g(W(S^*); Q_0) \sim d(Q, W(S^*))$$

is valid in a neighbourhood $N_g(S^*, \text{Fron}(S^*))$ of S^* with a suitable (δ') .

Besides the above conditions (2.2.1)', (2.2.1)'', we consider the following two quantitative conditions on $\mathcal{S}^*$.

$$(2.2.1)''_1 \quad d(P, \bar{S}^{*'}) \sim d(P, \bar{S}' \cap \bar{S}^*) : \forall P \in S^* \\ \text{and } \bar{S}' \cap \bar{S}^* \neq \emptyset$$

Note that the conditions (2.2.1)'', (2.1,1) and (2.1,1)'

imply the following

$$(2.2.1)''_2 \quad \text{If } S^* \cap S^* = \emptyset \text{ then } N_g(S^*, \text{Fron}(S^*)) \cap S^* = \emptyset \\ \text{for a small } (\delta').$$

A $\mathbb{K}$ -analytic pre-stratified set W satisfying (2.2.1)'' and endowed

with real analytic functions $\mathcal{F}^*(S^*) = \{g(S^*)\}$,

$h(S^*)$, $g(W^+(S^*))$ $S^* \in \mathcal{S}^*$ } with which

the conditions (2.1.1)'', (2.1.10)'' hold, will be called

(Q - D) - admissible K - pre-stratified set

Henceforth, when we speak of (Q - D) - pre stratified

set W, we assume that a set of functions $\mathcal{F}^*(S^*)$

are fixed. Our argument in this section is mainly

related to reduce our problems spoken in terms of W

to corresponding problems of pre - stratification $\mathcal{S}^*$

We shall begin by introducing certain

notations, which will be used in the later

arguments.

Let $\mathcal{N}: \mathcal{S}^* \rightarrow N(\mathcal{S})$ be an assignment which assigns to each stratum $S^* \in \mathcal{S}^*$ a neighborhood $N(S^*)$ of S^* in the ambient space of $\mathcal{W}^* = \mathcal{U}$. We assume that $N(S^*)$ is of N (N = dimension of the ambient space) dimensional manifold ($\subset U$) and moreover, that the following disjoint conditions are valid.

- (1) Unless $S_\lambda^* \succ S_\lambda^*$ $N(S_\lambda^*) \cap S_\lambda^* = \emptyset$,
- (2) If $N(S_\lambda^*) \cap N(S_{\lambda'}^*) \neq \emptyset$ then one of the inclusion relations $S_\lambda^* \succ S_{\lambda'}^*$ or $S_{\lambda'}^* \succ S_\lambda^*$ holds.

We let $N_{\mathcal{W}^*}(S^*)$ be the intersection: $N(S^*) \cap \mathcal{W}^*$. The set $N_{\mathcal{W}^*}(S^*)$ is a neighbourhood of S^* in $\mathcal{W}^*$. The disjoint conditions (1), (2) lead to the disjoint conditions imposed on $N_{\mathcal{W}^*}(S^*)$ obtained from (1) and (2) by changing $N(S^*)$, $N_{\mathcal{W}^*}(S^*)$ to $N_{\mathcal{W}^*}(S)$, $N_{\mathcal{W}^*}(S)$. For a given series of strata $S_1, \dots, S_t$, we define sets $N(S_1, \dots, S_t)$ (resp. $N_{\mathcal{W}^*}(S_1, \dots, S_t)$) by $N(S_1, \dots, S_t) = \bigcap_{\lambda=1}^t N(S_\lambda)$ (resp. $N_{\mathcal{W}^*}(S_1, \dots, S_t) = \bigcap_{\lambda=1}^t N_{\mathcal{W}^*}(S_\lambda)$). We assume the following basic isomorphisms for the assignment $\mathcal{N}: S \rightarrow N(S)$.

$$(2.1.2)_1 \quad \mathcal{H}^*(N(S^*)) \cong \mathcal{H}^*(N_{\mathcal{W}^*}(S^*)) \cong \mathcal{H}^*(S^*) \quad \text{for each stratum } S.$$

$$(2.1.2)_2 \quad \mathcal{H}^*(N(S_1^*, \dots, S_t^*)) \cong \mathcal{H}^*(N_{\mathcal{W}^*}(S_1^*, \dots, S_t^*))$$

for any series of strata $S_1^* \prec \dots \prec S_t^*$.

If the above two conditions are satisfied, we say that the assignment $\mathcal{N}: \mathcal{S}^* \rightarrow N(\mathcal{S})$ is a C^∞ -thickening of the stratification $\mathcal{S}^*$: The first condition is a desired one in order that the assignment $\mathcal{N}: S \rightarrow N(S)$ is called a C^∞ -thickening of $\mathcal{S}$. On the other hand the second condition, which we call Mayer-Vietoris condition, is imposed as a homological version of incidence conditions through which the results obtained around each stratum can be pieced together. We strongly remark that the second one:

Mayer-Vietoris condition is of algebraic nature and that this condition might be possibly taken up as a suitable homological incidence relations for more abstract situations^(*) where the notion of the stratification is well understood. Given a closed set W^* of W^* which can be expressed as a union of strata of $\mathcal{S}^*$. For a given assignment $N: S^* \rightarrow N(S^*)$, we denote by $N(W^*: \mathcal{S}^*)$ and $N(W-W^*: \mathcal{S}^*)$ the unions $\bigcup_{S^* \subset W^*} N(S^*)$ and $\bigcup_{S^* \subset W-W^*} N(S^*)$ respectively. It is clear that these two sets $N(W^*: \mathcal{S}^*)$ and $N(W-W^*: \mathcal{S}^*)$ are neighbourhoods of W^* and $W-W^*$ respectively. We also remark that $N(W-W^*: \mathcal{S}^*) \cap W^* = \emptyset$ holds. Now we show the following

Proposition 2.1.1 Given a c^∞ -thickening N of the stratification $\mathcal{S}^*$ of W^* and subvariety W^* (expressed as a union of strata of $\mathcal{S}^*$), the pair $(N(W^*, \mathcal{S}^*), N(W^*, \mathcal{S}^*))$ is a c^∞ -thickening of the pair (W^*, W^*) while $N(W-W^*, \mathcal{S}^*)$ is a c^∞ -thickening of $W-W^*$.

We paraphrase the above result simply that a c^∞ -thickening of the stratification $\mathcal{S}^*$ determines c^∞ -thickenings of (W^*, W^*) as well as of $(W-W^*, W-W^*)$.

Proof. Let $\mathcal{S}_{W^*}^* = \{S^* \in \mathcal{S}^* : S^* \subset W^*\}$. From the condition imposed on the subvariety W^* , it is clear that a collection $\mathcal{S}_{W^*}^*$ determines a stratification of W^* . It is obvious that conditions (2.1.2)₁ and (2.1.2)₂ are valid for the strata in $\mathcal{S}_{W^*}^*$. Therefore the assignment

(*) For algebraic varieties with arbitrary singularities over a field of characteristic p , notions of p -adic completion have been developed recently by S. Lubkin and D. Meredoth. It would be meaningful to try the possibilities of translations of proposition 2 to such abstract (and attractively open) subjects especially with connection of finiteness property of p -adic cohomology theory.

$N_{W'}: S^* \in \mathcal{S}_{W'}^* \rightarrow N(S^*)$: determines a c^∞ -thickening of the stratification $\mathcal{S}_{W'}^*$ of W' . Including the case where $W' = \emptyset$, we know that it is sufficient for our assertion to show that $N(W-W': \mathcal{S}^*) = \bigcup_{S^* \in W-W'} N(S^*)$ is a c^∞ -thickening of $W-W'$. We also remark that the case of the neighbourhood $N(W-W', \mathcal{S}^*)$ of $W-W'$ is the unique case which we shall make use of later. Now the verification of the above explained fact is a direct consequence of proposition 2.1.1' which will be stated soon after:

Given a topological space X and two sets of open subsets of X :

$\mathcal{U} = \{U_n\}, \mathcal{U}' = \{U'_n\} (n \in \mathbb{Z}^+)$. We assume the inclusion relation $U_n \supset U'_n$ for each n . Moreover, we assume that the coverings $\mathcal{U}$ and $\mathcal{U}'$ are locally finite coverings of the sets $\bigcup_n U_n$ and $\bigcup_n U'_n$ respectively. Finally we assume that the Mayer-Vietoris condition

$$(2.1.3)_1 \quad H^*(U_{n_1} \cap \dots \cap U_{n_\ell}) \cong H^*(U'_{n_1} \cap \dots \cap U'_{n_\ell})$$

is valid for any indices $(n_1, \dots, n_\ell)$,

and that

Under the above assumptions we show the following.

Proposition 2.1.1' $H^*(\bigcup_n U_n) = H^*(\bigcup_n U'_n)$.

Proof. Given indices $(n_1, \dots, n_\ell)$, we let $U_{n_1, \dots, n_\ell}$ and $U'_{n_1, \dots, n_\ell}$ be intersections $\bigcap_{i=1}^\ell U_{n_i}$ and $\bigcap_{i=1}^\ell U'_{n_i}$ respectively. For each open set U_μ (resp. U'_μ), we define $U_{n_1, \dots, n_\ell}(\mu)$ ($U'_{n_1, \dots, n_\ell}(\mu)$) by $U_{n_1, \dots, n_\ell}(\mu)$ (resp. $U'_{n_1, \dots, n_\ell}(\mu)$) = $U_{n_1, \dots, n_\ell} \cap U_\mu$ (resp. $U'_{n_1, \dots, n_\ell} \cap U'_\mu$).

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We let $\mathcal{U}_{n_1 \dots n_\ell}$ and $\mathcal{U}'_{n_1 \dots n_\ell}$ be finite open coverings of $U_{n_1 \dots n_\ell}$ and $U'_{n_1 \dots n_\ell}$ composed of open sets of the form $U_{n_1 \dots n_\ell}(\mu) (\neq \emptyset)$ and $U'_{n_1 \dots n_\ell}(\mu) (\neq \emptyset)$. It is clear that, for indices $\mu_1, \dots, \mu_t$, the Mayer-Vietoris isomorphism of the following form follows from the

Mayer-Vietoris isomorphism (2.1.3)₂

$$(2.1.3)_2 \quad \mathcal{H}^*(U_{n_1 \dots n_\ell}(\mu_1) \cap \dots \cap U_{n_1 \dots n_\ell}(\mu_t)) \\ \cong \mathcal{H}^*(U'_{n_1 \dots n_\ell}(\mu_1) \cap \dots \cap U'_{n_1 \dots n_\ell}(\mu_t)).$$

We first show the following statement:

$$(2.1.4)_n^m \quad \text{For any indices } (n_1, \dots, n_\ell) \text{ and } U_{n_1 \dots n_\ell}(\mu_s) (U'_{n_1 \dots n_\ell}(\mu_s)) \\ \in \mathcal{U}_{n_1 \dots n_\ell} (\mathcal{U}'_{n_1 \dots n_\ell}) \quad (s=1, \dots, m), \text{ the following isomorphism}$$

$$(2.1.3)' \quad \mathcal{H}^*(\bigcup_{s=1}^m U_{n_1 \dots n_\ell}(\mu_s)) = \mathcal{H}^*(\bigcup_{s=1}^m U'_{n_1 \dots n_\ell}(\mu_s))$$

holds.

The above assertion is proven inductively on m : If $m=1$ then the condition (2.1.3) provides directly an answer. Fix indices $(n_1, \dots, n_\ell)$ and $(\mu_1, \dots, \mu_{m+1})$. Then from the inductive hypothesis it follows that

$$\mathcal{H}^*(\bigcup_{s=1}^m U_{n_1 \dots n_\ell}(\mu_s)) \cong \mathcal{H}^*(\bigcup_{s=1}^m U'_{n_1 \dots n_\ell}(\mu_s)) \text{ holds. On the}$$

otherhand we have the following relations immediately.

$$(\bigcup_{s=1}^m U_{n_1 \dots n_\ell}(\mu_s)) \cap U_{n_1 \dots n_\ell}(\mu_{m+1}) = \bigcup_{s=1}^m U_{n_1 \dots n_\ell, \mu_{m+1}}(\mu_s),$$

$$(\bigcup_{s=1}^m U'_{n_1 \dots n_\ell}(\mu_s)) \cap U'_{n_1 \dots n_\ell}(\mu_{m+1}) = \bigcup_{s=1}^m U'_{n_1 \dots n_\ell, \mu_{m+1}}(\mu_s).$$

Therefore the induction hypothesis (for m) implies the following isomorphism

$$(2.1.3) H^*(\bigcup_{\lambda=1}^m U_{n_1, \dots, n_{\lambda}, \mu_{m+1}}(\mu_{\lambda})) \cong H^*(\bigcup_{\lambda=1}^m U'_{n_1, \dots, n_{\lambda}, \mu_{m+1}}(\mu_{\lambda}))$$

(The Meyer-Vietoris sequence as well as the five lemmas applied to the couples $(U_{n_1, \dots, n_{\lambda}}(\mu_{\lambda}), U_{n_1, \dots, n_{\lambda}}(\mu_{m+1}))$ and $(\bigcup_{\lambda=1}^m U'_{n_1, \dots, n_{\lambda}}(\mu_{\lambda}), U'_{n_1, \dots, n_{\lambda}}(\mu_{m+1}))$ leads easily to the isomorphism $(2.1.3)^{m+1}$).

Now the isomorphism $(2.1.3)^{m+1}$, the Meyer-Vietoris sequence and the five lemma applied to couples

$$(\bigcup_{\lambda=1}^m U, U_{m+1}), (\bigcup_{\lambda=1}^m U', U'_{m+1}) \quad (m=1, \dots, \infty)$$

gives the proof of our proposition.

Now we return to our original problem : Given two

c-thickenings $\mathcal{N}_i : S^* \rightarrow N(S_i^*)$ ($i=1, 2$), we understand by $\mathcal{N}_1 > \mathcal{N}_2$ the inclusion relation of the following form.

$$\mathcal{N}_1(S^*) \supset \mathcal{N}_2(S^*) \quad \text{for each } S^* \in \mathcal{S}^*.$$

It is obvious that the above relation implies the following

$$\text{relation : } \mathcal{N}_1(W : \mathcal{S}^*) \supset \mathcal{N}_2(W : \mathcal{S}^*) \quad \text{between c-thickenings of } W.$$

For a given c-thickening $\mathcal{N}$, we

define differential graded rings $\Omega_L^*(N(S^*)), \Omega_L^*(N(S_1^* \leftarrow S_2^*)), \Omega_L^*(N(S_1^* \leftarrow S_2^* \leftarrow S_3^*))$ to be the ones composed of c-differential forms in $N(S^*), N(S_1^* \leftarrow S_2^*)$ and $\bigcup_{\mu} N(S_{\mu}^*)$.

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Let $\mathcal{F}(V_\mu S_\mu^*)$ be the sub-closed set of W defined by 85

$$\mathcal{F}(V_\mu S_\mu^*) = V_\mu S_\mu^* - U_\mu S_\mu^*; \quad \text{where } S_\mu^* \text{ exhaust all}$$

the strata contained in $\text{Fron}(S_\mu^*)$'s while S_μ^* are at the same time appearing as elements in $\{S_\mu^*\}$.

Define sub rings $\Omega_{\text{asy}}^\infty(S^*, \text{Fron}(S^*))$ of $\Omega(S^*)$, $\Omega_{\text{asy}}^\infty(S_\mu^*, \text{Fron}(S_\mu^*))$ of $\Omega(S_\mu^*)$ and $\Omega_{\text{asy}}^\infty(V_\mu S_\mu^*, \mathcal{F}(S_\mu^*))$ of $\Omega(S_\mu^*)$ by the following asymptotic

behaviors (for notations used here, see)

$$(1.1.5) \quad \begin{cases} (1.1.5)_1 & |y(q)| \leq b_1 \cdot d(q, \text{Fron}(S^*))^{-b_2}, & b_1, b_2 \in \mathbb{R}^+, \\ (1.1.5)_2 & |y(q)| \leq b'_1 \cdot d(q, \text{Fron}(S_\mu^*))^{-b'_2}, & b_1, b_2 \in \mathbb{R}^+, \\ (1.1.5)_3 & |y(q)| \leq b''_1 \cdot d(q, \mathcal{F}(S_\mu^*))^{-b''_2}, & b_1, b_2 \in \mathbb{R}^+ \end{cases}$$

Moreover, we define , for a given closed set W' , a

graded ring $\Omega(W-W')$ and its subring $\Omega_{\text{asy}}(W-W')$ to be the ring of c-forms in $N(W-W')$ and the ring of elements $y \in \Omega(W-W')$ characterized by

$$(1.1.5)'. \quad |y(q)| \leq b_1 \cdot d(q, W')^{-b_2}; \quad b_1, b_2 \in \mathbb{R}^+.$$

Let us assume that a directed set $\{\mathcal{N}_n\}_{n \in \mathbb{Z}}$ of c-

thickenings of S^* is given. In this section when we say a

directed set $\{\mathcal{N}_n\}_{n \in \mathbb{Z}}$, we always assume the following

conditions .

(2.1.6)₁ For a given (S) , the relation $\mathcal{N}_n(S^*) \subset \mathcal{N}_s(S^*, \text{Fron}(S^*))$

holds for each $S^* \in \mathcal{S}$ for a suitable integer n .

(2.1.6)₂ For a given integer n , there exists a couple (S)

in such a manner that the following condition

$$N_n(S^*) \supset N_g(S^*, \text{Form}(S^*))$$

is valid.

For sets of strata $S^*, S_1^*, \dots, S_r^*, \{S_1^*, \dots, S_r^*\}$ define

ring homomorphisms $\rho_{n,i}^*: \Omega(N_n(S)) \rightarrow \Omega(N_n(S_i))$ by the restrictions of

c-forms from $N_n(S)$ to $N_n(S_i)$, $i=1, \dots, r$. Define rings

$$\hat{\Omega}_{asy}^\infty(N_n(S^*), \text{Form}(S^*)), \quad \hat{\Omega}_{asy}^\infty(S_1^*, \dots, S_r^*, \text{Form}(S_i^*)), \quad \text{and} \quad \hat{\Omega}_{asy}^\infty(\bigcup_{i=1}^r S_i^*, \text{Form}(S_i^*))$$

$$\text{proj. lim}_n \hat{\Omega}_{asy}^\infty(S_1^*, \dots, S_r^*, \text{Form}(S_i^*)) \quad \text{and} \quad \text{proj. lim}_n \hat{\Omega}_{asy}^\infty(\bigcup_{i=1}^r S_i^*, \text{Form}(S_i^*)) \quad \text{respectively.}$$

(Strictly speaking, the limit should be understood to be

for the fixed set $\mathcal{W}_n \}_{n \in \mathbb{Z}^+}$. Because there is no

confusion, we omit the chosen $\{\mathcal{W}_n\}_{n \in \mathbb{Z}^+}$ in the sequel.)

The differential operator d is compatible with the limit

process, therefore the rings $\hat{\Omega}_{asy}^\infty$'s are graded differential

rings with the operator d . We use the symbols $\hat{\mathbb{F}}_{asy}^\infty$'s

to mention subrings of $\hat{\Omega}_{asy}^\infty$'s composed of closed forms in $\hat{\Omega}_{asy}^\infty$.

We ask in what a manner the considerations of the ring

$\hat{\Omega}_{asy}^\infty(\mathcal{W}, \mathcal{W})$ is reduced to the problems of $\hat{\Omega}_{asy}^\infty(S_1^*, \dots, S_r^*, \text{Form}(S_i^*))$. Let

us assume that a directed set $\{\mathcal{W}_n\}_{n \in \mathbb{Z}^+}$ is given and is fixed.

Now we show the following

Proposition. 2.1.2 . Assume the following conditions (2.1.7)₁ :

(2.1.7)₁ For each series $S_1^* \dots S_t^* : S_i^* \in W - W'$,

and for any integers n, k , the isomorphisms

$$H^*(N_n(S_1^* \dots S_t^*)) \cong H^*(N_n(S_1^* \dots S_t^*))$$

hold,

(2.1.7)₂ The incidence isomorphisms

$$\text{Proj. lim}_n H^*(N_n(S_1^* \dots S_t^*)) \cong \frac{\hat{\Phi}_{\text{asy}}(S_1^* \dots S_t^*)}{d\hat{\Omega}_{\text{asy}}(S_1^* \dots S_t^*)}$$

holds for each $S_1^* \dots S_t^*$.

Then the isomorphism

$$H(W - W' : 0) \cong \frac{\hat{\Phi}_{\text{asy}}(\hat{W} - \hat{W}', W')}{d\hat{\Omega}_{\text{asy}}(\hat{W} - \hat{W}', W')}$$

holds.

Proof . Given a series of strata : $S_1^* \dots S_t^* ; S_i^* \in W - W'$,

we define a set $\delta^* \langle S_1^*, \dots, S_t^* \rangle$ of strata

S^* 's by the following requirements.

(2.1.8), A stratum $S^* \in \mathcal{J}^m(S_1^*, \dots, S_t^*)$ if and only

if the conditions (i) $S^* \supset S_t^*$ and (ii) $\ell(S^*) \equiv m$

hold. (In the sequel we assume the inequality : m

$\equiv \ell(S_t) - 1$.)

Similarly we define a set $\mathcal{J}^{*m}(W-W')$ of strata S' in $W-W'$ by

(2.1.8)', A stratum S is in $\mathcal{J}^{*m}(W-W')$ if and only if

$S \subset W-W'$ and $\ell(S) \equiv m$ holds. (we assume that m

$\equiv \max_{S \subset W-W'} \ell(S)$ holds in the sequel.)

Let N_n be a C^∞ -thickening of $\mathcal{J}^{*m}(W-W')$. Denote by $\Omega_n^\infty(N_n(\mathcal{J}_n^m(S_1^*, \dots,$

$\dots, S_t^*)$ and $\Omega_n^\infty(N_n(\mathcal{J}_n(W-W')))$ the

rings of C^∞ -differentiable forms in $N(\mathcal{J}_n^m(S_1^*, \dots, S_t^*)$

$= N_n(S_1^*, \dots, S_t^*) \cap \left(\bigcup_S N(S^*) \right) : S \in \mathcal{J}^{*m}(S_1^*, \dots, S_t^*)$

and in $\bigcup_S N_n(S^*) (S^* \in \mathcal{J}^{*m}(W-W'))$ respectively. Define

subclosed set $\mathcal{I}_n = \mathcal{I}_n(W)$ of W and $\mathcal{I}_n(W-W')$ of $W-W'$ by the following equations.

$$\mathcal{I}_n(W) = \bigcup_{S^* \in \mathcal{J}^{*m}} \overline{S^*} ; \ell(S) \geq m+1,$$

$$= W,$$

$$\mathcal{I}_n(W-W') = \bigcup_{S' \in \mathcal{J}^{*m}(W-W')} \overline{S'} : S' \in W-W'.$$

Define also sub closed set $\mathcal{I}_n(S_1, \dots, S_t)$ and

$\mathcal{I}_n(W-W')$ by

$$\mathcal{F}_m \langle s_1^*, \dots, s_t^* \rangle = \bigcup_{s^*} \mathcal{F} : s^* \in \mathcal{F} \text{ and } s^* > s_t^*, \quad 89$$

$$\mathcal{F}_m \langle W - W' \rangle = \bigcup_{s^* \in W} s^* \cup W' : s^* \in W - W' \text{ and } l(s^*) \geq m + 1.$$

Denote by $\Omega_{asy}^\infty (N_n(\mathcal{F}_m \langle s_1^*, \dots, s_t^* \rangle), \mathcal{F}_m)$ and

$\Omega_{asy}^\infty (N_n(\mathcal{F}_m \langle W - W' \rangle), \mathcal{F}_m^{(W)})$ the subrings of $\Omega_{asy}^\infty (N_n(\mathcal{F}_m \langle s_1^*, \dots, s_t^* \rangle))$

and $\Omega_{asy}^\infty (N_n(\mathcal{F}_m \langle W - W' \rangle))$ whose elements are

characterized by the following asymptotic behaviors respectively.

$$(2.1.9)_1 \quad |y| \leq b_1 \cdot d(Q, \mathcal{F}_m)^{-b_2}; \quad b_1, b_2 \in \mathbb{R}^+$$

$$(2.1.9)_2 \quad |y| \leq b'_1 \cdot d(Q, \mathcal{F}_m)^{-b'_2}; \quad b'_1, b'_2 \in \mathbb{R}^+.$$

In view of (2.1.1), it is clear that $d(Q, \mathcal{F}_m) = d(Q,$

$\mathcal{F}_m \langle s_1^*, \dots, s_t^* \rangle)$ holds if Q is in $N_n(\mathcal{F}_m \langle s_1^*, \dots, s_t^* \rangle)$. On the otherhand it is also clear

that, if Q is in $N_n(s_1^*, \dots, s_{m'}^*, s_{m'}^*)$; $s_{m'}^*$ is of length $m' (< m)$, then $d(Q, \text{Fron}(s_{m'}^*)) \leq$

$d(Q, \mathcal{F}_m \cap (\mathcal{F}_m \cup W'))$ holds for $Q \in N_n(s_{m'}^*, \text{Fron}(s_{m'}^*))$. Thus

the relation $d(Q, \mathcal{F}_m \cup W') \sim d(Q, \mathcal{F}_m \cap (\mathcal{F}_m \cup W'))$,

also holds for $Q \in N_n(s_{m'}^*, \text{Fron}(s_{m'}^*))$.

Another obvious fact is that the relation

$$d(Q, \mathcal{F}_m \cup W') \sim d(Q, W - W') \text{ holds.}$$

Therefore varieties $\mathcal{F}_m$ and $\mathcal{F}_m \langle W - W' \rangle$ in (2.1.9)_{1,2}

can be replaced by $\mathcal{F}_m \langle s_1^*, \dots, s_t^* \rangle$ and by $\mathcal{F}_m \langle W - W' \rangle$

respectively. Now we take up sets

$$J_m^* \langle S_1^*, S_2^*, \dots, S_x^* \rangle, \quad J_{m+1}^* \langle S_1^*, \dots, S_x^* \rangle \quad (m+1 \leq l(S_2^*)-1)$$

and

$$J_m^*(W - W'), \quad J_{m+1}^*(W - W') \quad (m+1 \leq n(W-W)-1)$$

Then the following intersection relations

$$\begin{aligned} N_n(J_m^* \langle S_1^*, \dots, S_x^* \rangle) &= \bigcap \left\{ \bigcup_{\mu} N_n(S_1^*, \dots, S_x^*, S_{m+\mu}^*) \right\} \\ &= \begin{cases} N_n(J_m^* \langle S_1^*, \dots, S_x^* \rangle), \\ \bigcup_{\mu} N_n(J_m^* \langle S_1^*, \dots, S_x^*, S_{m+\mu}^* \rangle) \end{cases} \\ N_n(J_m^*(W - W')) &= \bigcap \left\{ \bigcup_{\mu} N_n(S_{m+\mu}^*) \right\} = \begin{cases} N_n(J_m^*(W - W')), \\ \bigcup_{\mu} N_n(J_m^*(S_{m+\mu}^*)) \end{cases} \end{aligned}$$

hold, where $S_{m+\mu}^*$'s and $S_{m+\mu}^*$'s exhaust strata of exactly

length $(m+1)$ in $J_m^* \langle S_1^*, \dots, S_x^* \rangle$ and $J_m^*(W - W')$ respectively.

Then we obtain the resulting Meyer - Vietoris sequences

for every $n \in \mathbb{Z}$ in usual manners:

$$\begin{aligned} (2.1.10) \quad 0 &\rightarrow \Omega_{\mathbb{L}}^{\infty}(N_n(J_m^* \langle S_1^*, \dots, S_x^* \rangle)) \xrightarrow{i_{n,x}} \Omega_{\mathbb{L}}^{\infty}(N_n(J_m^* \langle S_1^*, \dots, S_x^* \rangle)) \\ &\oplus \sum_{\mu} \Omega_{\mathbb{L}}^{\infty}(N_n(S_{m+\mu}^*)) \xrightarrow{f_{n,x}} \sum_{\mu} \Omega_{\mathbb{L}}^{\infty}(N_n(J_m^* \langle S_1^*, \dots, S_x^*, S_{m+\mu}^* \rangle)) \\ &\rightarrow 0 \\ (2.1.10)' \quad 0 &\rightarrow \Omega_{\mathbb{L}}^{\infty}(N_n(J_m^*(W - W'))) \xrightarrow{i_{n,x}} \Omega_{\mathbb{L}}^{\infty}(N_n(J_m^*(W - W'))) \oplus \sum_{\mu} \Omega_{\mathbb{L}}^{\infty}(N_n(S_{m+\mu}^*)) \\ &\rightarrow \sum_{\mu} \Omega_{\mathbb{L}}^{\infty}(N_n(J_m^*(S_{m+\mu}^*))) \rightarrow 0 \end{aligned}$$

Consider subrings

$$\Omega_{\text{asy}}^{\infty}(N_n(J_m^* \langle S_1^*, \dots, S_x^* \rangle), \mathbb{F}_m(W)), \quad \Omega_{\text{asy}}^{\infty}(N_n(J_m^*(W - W')), \mathbb{F}_m(W))$$

$$\Omega_{\text{asy}}^{\infty}(N_n(S_1^*, \dots, S_x^*, S_{m+\mu}^*), \mathbb{F}_m(W)) \quad \text{and} \quad \Omega_{\text{asy}}^{\infty}(N_n(J_m^*(S_{m+\mu}^*)), \mathbb{F}_m(W))$$

of above four rings in (2.1.10). Also consider subrings

$\Omega_{asy}^{\infty} N_n (\mathcal{S}^{m+1} \langle W - W' \rangle, \mathcal{F}_{n+1}^{(W-W')}), \Omega_{asy}^{\infty} (N_n (\mathcal{S}^m (W - W')), \mathcal{F}_{n+1}^{(W-W')})$
 $\Omega_{asy}^{\infty} N_n (S_{m+1, \mu}), \mathcal{F}_{n+1}^{(W-W')})$ and $\Omega_{asy}^{\infty} N_n (\mathcal{S}^m \langle S_{m+1, \mu} \rangle), \mathcal{F}_{n+1}^{(W-W')})$ of
 four rings defined in the sequence (2.1.10). We remark
 that the following isomorphisms are valid.

$$\begin{aligned}
 \Omega_{asy}^{\infty} (N_n \langle S_1, \dots, S_k, S_{m+1, \mu} \rangle, \mathcal{F}_{n+1}^{(W)}) &\cong \Omega_{asy}^{\infty} (N_n \langle S_1, \dots, S_k, S_{m+1, \mu} \rangle, \mathcal{F}_{n+1}^{(W)} (S_{m+1, \mu})), \\
 \Omega_{asy}^{\infty} (N_n (S_{m+1, \mu}), \mathcal{F}_{n+1}^{(W)}) &\cong \Omega_{asy}^{\infty} (N_n (S_{m+1, \mu}), \mathcal{F}_{n+1}^{(W)} (S_{m+1, \mu})), \\
 \Omega_{asy}^{\infty} (N_n (\mathcal{S}^m S_{m+1, \mu}), \mathcal{F}_{n+1}^{(W)}) &\cong \Omega_{asy}^{\infty} (N_n (\mathcal{S}^m S_{m+1, \mu}), \mathcal{F}_{n+1}^{(W)} (S_{m+1, \mu})), \\
 \Omega_{asy}^{\infty} (N_n (\mathcal{S}^m \langle S_{m+1, \mu} \rangle), \mathcal{F}_{n+1}^{(W)}) &\cong \Omega_{asy}^{\infty} (N_n (\mathcal{S}^m \langle S_{m+1, \mu} \rangle), \mathcal{F}_{n+1}^{(W)} (S_{m+1, \mu}))
 \end{aligned}$$

Taking projective limits of the above rings, we

obtain graded differential rings $\hat{\Omega}_{asy} = \hat{\Omega}_{asy} (\mathcal{S}^{m+1} \langle S_1, \dots, S_k \rangle, \mathcal{F}_n)$
 $(= \text{proj. lim}_n \Omega (N_n (\mathcal{S}^{m+1} \langle S_1, \dots, S_k \rangle, \mathcal{F}_{n+1})), \dots \text{etc.} \dots)$

Now we show that the following (Mayer-Vietoris) sequence
 corresponding to (2.1.10), (2.1.10)' are valid.

$$\begin{aligned}
 (2.1.11)_1 \quad 0 \rightarrow \hat{\Omega}_{asy} (\mathcal{S}^{m+1} \langle S_1, \dots, S_k \rangle, \mathcal{F}_{n+1}) &\xrightarrow{\hat{\iota}_*} \hat{\Omega}_{asy} (\mathcal{S}^{m+1} \langle S_1, \dots, S_k \rangle, \mathcal{F}_n) \oplus \\
 &\quad \sum_{\mu} \hat{\Omega}_{asy} (\langle S_1, \dots, S_k, S_{m+1, \mu} \rangle, \mathcal{F}_{n+1}) \xrightarrow{\hat{\iota}_*} \sum_{\mu} \hat{\Omega}_{asy}^* (\mathcal{S}^m \langle S_1, \dots, S_k, S_{m+1, \mu} \rangle, \mathcal{F}_n) \\
 (2.1.11)_2 \quad 0 \rightarrow \hat{\Omega}_{asy} (\mathcal{S}^{m+1} \langle W - W' \rangle, \mathcal{F}_{n+1}^{(W-W')}) &\xrightarrow{\hat{\iota}_*} \hat{\Omega}_{asy} (\mathcal{S}^m \langle W - W' \rangle, \mathcal{F}_n^{(W-W')}) \oplus \\
 &\quad \sum_{\mu} \hat{\Omega}_{asy} (S_{m+1, \mu}, \mathcal{F}_{n+1}^{(W-W')}) \xrightarrow{\hat{\iota}_*} \sum_{\mu} \hat{\Omega}_{asy} (\mathcal{S}^m \langle S_{m+1, \mu} \rangle, \mathcal{F}_n^{(W-W')})
 \end{aligned}$$

Note that, in the above sequences, the 'polar loci'

$\mathcal{F}_{n+1}, \dots, \mathcal{F}_n$ are outside of sets in which differential
 forms in question are defined.

In the above $\hat{\lambda}_*, \hat{\gamma}_*, \hat{\lambda}'_*, \hat{\gamma}'_*$ are defined by taking limits of $\lambda_{n,*}, \dots, \gamma_{n,*}$. It is clear from (2.1.10) (2.1.10') that $\hat{\lambda}_*$ (resp. $\hat{\lambda}'_*$) is injective and that $\hat{\gamma}_* \hat{\lambda}_* = 0$ ($\hat{\gamma}'_* \hat{\lambda}'_* = 0$) holds.

That $\text{Coker } (\hat{\gamma}_*)$ (resp. $\text{Coker } \hat{\gamma}'_*$) =

$\text{Ker.}(\hat{\lambda}_*)$ (resp. $\text{Ker } (\hat{\lambda}'_*)$) is shown

quickly as follows : Given $(\hat{\gamma}, \sum_{\mu} \hat{\gamma}_{\mu})$ (resp. $(\hat{\gamma}', \sum_{\mu} \hat{\gamma}'_{\mu})$)

in $\hat{\Omega}_{\text{asy}}(\mathcal{S}^m \langle S_1, \dots, S_t \rangle, \mathcal{F}_m)$ (resp. $\hat{\Omega}_{\text{asy}}((W-W'), \mathcal{F}_{m-W-W'})$),

let us take a representative $(\varphi_n, \sum \varphi_{n,\mu})$

(resp. $(\varphi'_n, \sum \varphi'_{n,\mu})$) in the ring

$$\hat{\Omega}_{\text{asy}}^{\infty} (N_n(\mathcal{S}^m \langle S_1, \dots, S_t \rangle, \mathcal{F}_m) \oplus \sum_{\mu} \hat{\Omega}_{\text{asy}}^{\infty} (N_n(S_1, \dots, S_t, S_{m+\mu}), \mathcal{F}_{m+\mu}))$$

(resp. in $\hat{\Omega}_{\text{asy}}^{\infty} (N_n(\mathcal{S}^{m-W-W'}), \mathcal{F}_{m-W-W'}) \oplus \sum_{\mu} \hat{\Omega}_{\text{asy}}^{\infty} (N_n(S_{m+\mu}), \mathcal{F}_{m+\mu-W-W'})$)
for a suitable n so that $\gamma_{n,*}(\varphi_n, \sum \varphi_{n,\mu}) = 0$

(resp. $\gamma'_{n,*}(\varphi'_n, \sum \varphi'_{n,\mu}) = 0$) holds.

From the sequences (2.1.10) (2.1.10') there exists the unique

element $\varphi_n \in \hat{\Omega}_{\text{asy}}^{\infty} (N_n(\mathcal{S}^{m+1} \langle S_1, \dots, S_t \rangle))$ (resp. $\varphi_n \in$

$\hat{\Omega}_{\text{asy}}^{\infty} (N_n(\mathcal{S}^{m+1}(W-W'))$) in such a way that $i_n(\varphi_n) = (\varphi_n, \sum \varphi_{n,\mu})$
(resp. $i'_n(\varphi'_n) = (\varphi'_n, \sum \varphi'_{n,\mu})$) holds. Examine asymptotic

behaviors of elements φ_n (resp. φ'_n). (:) That

$i_{n,*}(\varphi_n)$ (resp. $i'_{n,*}(\varphi'_n)$) is equal to $\sum \varphi_{n,\mu}$

$$(\text{resp. } \sum_{\mu} \gamma_{m,\mu}) \in \sum_{\mu} \Omega^{\infty}(N_n(\mathcal{S}^m \langle S_{m+1,\mu} \rangle, \mathcal{F}_{n+1})) (\text{resp. } \Omega^{\infty}(N_n(\mathcal{S}^m \langle S_{m+1,\mu} \rangle, \mathcal{F}_{m+1})))$$

shows that γ_n (resp. γ'_n) have desired asymptotic behaviors.

$$\left(\text{in } \bigcup_{\mu} N_n((S_1^*, \dots, S_x^*, S_{m+1,\mu}^*)) (\text{resp. } \bigcup_{\mu} N_n(S_{m+1,\mu}^*)) \right) \text{ w.r.t. } \mathcal{F}_{m+1} (\text{resp. } \mathcal{F}'_{m+1})$$

(ii) On the otherhand, if Q (resp. Q') ($\in N_n(\mathcal{S}^m \langle S_1^*, \dots, S_x^* \rangle$)

(resp. $N_n(\mathcal{S}^m \langle W - W' \rangle)$) is outside of $\bigcup_{\mu} N_n(S_{m+1,\mu}^*)$

(resp. $\bigcup_{\mu} N_n(S_{m+1,\mu}^*)$) then inequalities of the forms

$$d(Q, \mathcal{F}_{m+1}^{\langle W \rangle}) \leq c \cdot d(Q, \mathcal{F}_{m+1}^{\langle W \rangle})^{c_2}$$

$$d(Q', \mathcal{F}'_{m+1} \langle W - W' \rangle) \leq c' \cdot d(Q, \mathcal{F}_m \langle W - W' \rangle)^{c'_2}$$

are valid with constants (c) , (c') 's.

The above inequalities are sufficient to assure the

inequality of the following forms for $Q \in N_n(\mathcal{S}^m \langle S_1^*, \dots, S_x^* \rangle)$

(resp. $Q \in N_n(\mathcal{S}^m \langle V - V' \rangle)$)

$$|\gamma_n(Q)| < \tilde{b}_1 \cdot d(Q, \mathcal{F}_{m+1}^{\langle W \rangle})^{-\tilde{b}_2}; (\tilde{b}_1, \tilde{b}_2) \in \mathbb{R}^+,$$

$$|\gamma'_n(Q')| < \tilde{b}'_1 \cdot d(Q, \mathcal{F}'_{m+1} \langle W - W' \rangle)^{-\tilde{b}'_2}; (b'_1, b'_2) \in \mathbb{R}^+.$$

It is clear that the above inequalities combined with

(i) assures that $\text{Coker}(\hat{r}) = \text{Ker}(\hat{i})$

(resp. $\text{Coker}(\hat{r}') = \text{Ker}(\hat{i}')$) hold.

Thus the remaining problem is to show that $\hat{r}$ (resp. $\hat{r}'$)

is surjective: Recall that $W(S_{m+1,\mu}^*)$ is defined to be

the closed set : $W^+(S_{m+1, \mu}^*) = \bigcup \bar{S}^* : S^* > S_{m+1, \mu}^*$ and that

$W^+(S_{m+1, \mu}^*)$ is defined to be the union $W^+(S_{m+1, \mu}^*) =$

$\bigcup S^* : S^* > S_{m+1, \mu}^*$. For the closed set $W^+(S_{m+1, \mu}^*)$, we

consider a series of sub-closed set $W_{\lambda}^+(S_{m+1, \mu}^*)$ ($1 \leq \lambda \leq m+1$)

as follows : $W_1^+(S_{m+1, \mu}^*) = W^+(\bar{S}_1^*) : l(\bar{S}_1^*) \geq 1, S_1^* < W^+(S_{m+1, \mu}^*)$

$W_{m+1}^+(S_{m+1, \mu}^*) = \bigcup \bar{S}^* ; l(S^*) \geq 0, S_{\lambda}^* < V(S_{m+1, \mu}^*)$

($1 \leq \lambda \leq m$). $W(S^*) = \bar{S}_{m+1, \mu}^*$ ($1 \leq s \leq m$)

$W(S^*)$ holds. It is clear that $W(S_{m+1, \mu}^*)$

$W(S_{m+1, \mu}^*) \supset \dots \supset W(S_{m+1, \mu}^*)$ holds. Note that, if S^*

($\subset V(S_{m+1, \mu}^*)$) is of length $m' \leq m$, then

$d(Q'_m, V_{m+1}(S_{m+1, \mu}^*)) \sim d(Q'_m, \text{Fron}(S_{m+1, \mu}^*))$ holds

for $Q'_m \in \bar{S}_m^* \cap N_s(S_{m+1, \mu}^*)$. For $S^* > S_{m+1, \mu}^*$, $\text{Fron}(S^*)$

is defined to be the union $\bigcup S^* (S^* < \text{Fron}(S^*))$,

$S^* > S_{m+1, \mu}^*$. Then $\text{Fron}(S_{m+1, \mu}^*)$ is closed in

$N_s(S_{m+1, \mu}^*)$. Note that, in $N_s(S_{m+1, \mu}^*, \text{Fron}(S_{m+1, \mu}^*))$,

$N_s(S_{m+1, \mu}^*, \text{Fron}(S_{m+1, \mu}^*)) \cap V(S_{m+1, \mu}^*) = N_s(S_{m+1, \mu}^*,$

$\text{Fron}(S_{m+1, \mu}^*)) \cap V(S_{m+1, \mu}^*)$, holds. Also note that,

for a set $S^* > S_{m+1, \mu}^*$ and $N_s(S_{m+1, \mu}^*, \text{Fron}(S_{m+1, \mu}^*))$

$$N_n(S^*, \text{Fron}(S^*)) \cap N_s(S_{n+1}^*, \text{Fron}(S_{n+1}^*))$$

$N_s(S_{n+1}^*, \text{Fron}(S_{n+1}^*))$. Combining the above remark with (1), we obtain the following relation.

$$\left\{ N_n(S_{n+1}^*, \text{Fron}(S_{n+1}^*)) \cap \left\{ \bigcup N_n(S_n^*, \text{Fron}(S_n^*)) \right\} \right\} \\ \sim \left\{ N_s(V(S_{n+1}^*), \overline{S_{n+1}^*}) \cap N_s(S_{n+1}^*) \right\}.$$

In the above the left and the right sides stand for

all the open sets $\dagger$ in and in

respectively. Choose an element $\sum_{\mu} \varphi_{\mu} \in \sum_{\mu} \Omega(S^n < S_1, \dots, S_r,$

$S'_{n+1, \mu} >, \varphi_{\mu})$ (resp. $\sum_{\mu} \varphi_{\mu} \in \sum_{\mu} \Omega(S^n < S_{n+1, \mu} >, \varphi_{\mu})$)

and take a representative $\sum_{\mu} \varphi_{n, \mu} \in \sum_{\mu} \Omega(N_n < S^n < S_1, \dots, S_r, S_{n+1, \mu} >$

$, \varphi'_{\mu})$ or $\sum \varphi_{\mu}$ (resp. $\sum_{\mu} \varphi_{\mu}$) for a suitable

n . Take a couple (a, ϵ) of positive numbers

suitably. Then we can assume that the c^{∞} -function

$\chi_a(V(S_{n+1, \mu}), \overline{S_{n+1, \mu}})$ ((c.f) definition) has the

following properties.

(2.1.12) If $\chi_a = \chi_a(V(S_{n+1, \mu}), \overline{S_{n+1, \mu}}) \neq 0$ at a

point Q in $S_{n+1, \mu}^{V \cap}$, then the point Q is

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contained in $N_n(S' : \text{Fron}(S')) (S' \in \mathcal{S}^{< S_{m+1}, \mu > })$.

Thus any form φ_μ , defined in $\Omega(N_n(\mathcal{S}^{< S_1, \dots, S_t, S_{m+1}, \mu > }))$ (resp. $\Omega(N(\mathcal{S}^{< S_{m+1} > }, \text{Fron}(W-W'))$) can

be extendable to a form φ_μ (resp. φ_μ) in $N_n(S_1, \dots,$

$S_t, S_{m+1}, \mu) = S_{m+1}, \mu$ (resp. $N(S_{m+1}, \mu) = S_{m+1}, \mu$) by

$$\varphi_\mu = \chi_a \cdot \varphi_\mu \quad (\text{ resp. } \varphi_\mu = \chi_a \varphi_\mu).$$

It is clear that this form φ (resp. φ) has an

asymptotic behavior w.r.t. $\overline{S}_{m+1, \mu}'$ (in $N(S_1 \dots S_t S_{m+1}, \mu) - \overline{S}_{m+1, \mu}')$

(resp. $N_n(S_{m+1}, \mu) = \overline{S}_{m+1, \mu}'$) and so w.r.t. $\overline{V}_{m+1, \mu}'$.

This form $\varphi = \varphi$ (resp. $\varphi = \varphi$), is zero outside of

$V_{S, \mu}(S)$ (resp.) and the relation

$$\varphi = \varphi \quad (\text{ resp. } \varphi = \varphi), \quad \text{holds in } N_n(\mathcal{S}^{< S_1, \dots, S_t, S_{m+1}, \mu > })$$

(resp. $N_n(\mathcal{S}^{< S_{m+1} > })$) for a sufficiently

large n . In this sense the form $\sum_\mu \varphi$ (resp. $\sum_\mu \varphi$)

represents the limit $\sum \varphi$ (resp. $\sum$). Next take a

$\mathcal{C}^\infty$ -function $\chi_{a'} = \chi_{a'}(S_{m+1}, \mu : \text{Fron}(S_{m+1}, \mu))$

with a suitable couple (a') . Define $\mathcal{C}^\infty$ -forms

$\tilde{\varphi}$ (resp. $\tilde{\varphi}'$) ($i = 1, 2$) by

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$$\tilde{\varphi} = \chi_{a'} \cdot \varphi, \quad \tilde{\varphi}' = (1 - \chi_{a'}) \cdot \varphi.$$

Note that we can assume, by a suitable choice of (a')

$\tilde{\varphi}$ ($\tilde{\varphi}'$) is extendable to $N_n(S_1, \dots, S_t) - \overline{S_{m+1}}$ (resp.

$U - \overline{S_{m+1}}$) by defining $\tilde{\varphi}(\tilde{\varphi}')$ to be zero outside of

$N_n(S_{m+1, \mu})$. On the otherhand, φ can be

extended to $N_n(S_1, \dots, S_t, S_{m+1, \mu})$ ($N_n(S_{m+1, \mu})$) by

defining $\tilde{\varphi}'$ to be zero on $S_{m+1, \mu}$. It is obvious that

that the following relation

$$\varphi + \tilde{\varphi} = \tilde{\varphi} \quad (\text{resp. } \tilde{\varphi}' + \tilde{\varphi}' = \tilde{\varphi}')$$

holds in $N_n(S_1, \dots, S_t, S_{m+1, \mu}) - S_{m+1, \mu}$

(resp. in $N_n(S_{m+1}) - S_{m+1}$) for a suitably

large n .

Recall that forms φ , $\tilde{\varphi}$ are defined in $N_n(S_1, \dots, S_t, S_{m+1, \mu})$ (and in $N_n(S_{m+1, \mu})$) respective;

φ , $\tilde{\varphi}$ are defined in $N_n(S_1, \dots, S_t, S_{m+1, \mu})$ (and in $N_n(S_{m+1, \mu})$) respective;

On the otherhand, restricted forms φ , $\tilde{\varphi}$, defined

in $N_n(S_1, \dots, S_t, S_{m+1, \mu}) - \overline{S_{m+1, \mu}}$ (resp. $U - \overline{S_{m+1, \mu}}$), to open

sets $N_n(\mathcal{S}^n(S_1, \dots, S_{t-1}, S_t))$ and $N_n(\mathcal{S}^n < w - w' >)$

respectively, we regard forms $\mathcal{Y}$, $\mathcal{Y}$ elements

in $\Omega(N_{\mathcal{K}'}(\mathcal{J}^m(S_1, \dots, S_t)))$ and $\Omega(N_{\mathcal{K}'}(\mathcal{J}^m(W - W')))$

Define forms $\mathcal{Y}$, $\mathcal{Y}$, defined in $\Omega(\langle S_1, \dots, S_t \rangle)$

(resp. in $N(\mathcal{J}^m \bar{V} - \bar{V}')$) by

$$\mathcal{Y}_{\mathcal{K}'} = \sum_{\mu} \mathcal{Y}_{\mu}, \quad (\mathcal{Y}_{\mathcal{K}''} = \sum_{\mu} \mathcal{Y}_{\mu})$$

it is clear that $\mathcal{Y}_{\mathcal{K}'} = \mathcal{Y}_{\mathcal{K}''}$ and that $\mathcal{Y}_{\mathcal{K}'} = \mathcal{Y}_{\mathcal{K}''}$ hold

in $N_{\mathcal{K}'}(S_1, \dots, S_t, S_{n+1, \mu}) - S_{n+1, \mu}$. Thus, for $(\mathcal{Y}_{\mathcal{K}'}^1,$

$$\sum_{\mu} \mathcal{Y}_{\mu, \mathcal{K}'} \in \Omega(N_{\mathcal{K}'}(\mathcal{J}^m(S_1, \dots, S_t))) \oplus \sum_{\mu} \Omega(N_{\mathcal{K}'}(S_1, \dots, S_t, S_{n+1, \mu})) \quad (\mathcal{Y}_{\mathcal{K}'}^1, \sum_{\mu} \mathcal{Y}_{\mu}) \in \Omega(N_{\mathcal{K}'}(\mathcal{J}^m(W - W')))$$

$\oplus \sum_{\mu} \Omega(N_{\mathcal{K}'}(S_{n+1, \mu}))$, we obtain the equalities

$$(2.1.13) \quad r_{\mathcal{K}'}(\mathcal{Y}_{\mathcal{K}'}^1, \sum_{\mu} \mathcal{Y}_{\mu}) = \sum_{\mu} \mathcal{Y}_{\mu, \mathcal{K}'}^1,$$

$$r(\mathcal{Y}_{\mathcal{K}'}^1, \mathcal{Y}_{\mu, \mathcal{K}'}^1) = \sum \mathcal{Y}_{\mu, \mathcal{K}'}^1.$$

Quantitative properties of $(\mathcal{Y}, \sum_{\mu} \mathcal{Y}_{\mu})$ (resp.

$(\mathcal{Y}_{\mathcal{K}'}^1, \sum_{\mu} \mathcal{Y}_{\mu, \mathcal{K}'}^1)$) are examined in the following

way ; Recall that forms $\mathcal{Y}_{\mu}$ have asymptotic

behaviors w. r. t. From () then

from the facts (i) $\gamma_\mu = 0$ (resp. $\gamma'_\mu = 0$),

outside of a neighbourhood $N_\delta(S_{m+1,\mu}, \text{Fron}(S_{m+1,\mu}))$

and (ii) γ_n and γ'_n have asymptotic

behaviour w.r.t. $\text{Fron}(S_{m+1,\mu})$, it follows that

γ and γ_k have asymptotic behaviour w.r.t. $\text{Fron}(S_{m+1,\mu})$ and

$(v - v')$ respectively. On the otherhand,

because $\gamma = (\gamma_1 + \gamma_2)$ in each

neighbourhood $N(S_1, \dots, S_t, S_{m+1,\mu})$, $N((v - v'))$

$N(S)$ have asymptotic behavior w.r.t.

S in neighbourhood mentioned just above. Thus,

in $\bigcup_{\mu} N_{\mu'}(S_1, \dots, S_t, S_{m+1,\mu})$, $\bigcup_{\mu} N_{\mu'}(\mathcal{S}^m(S_{m+1,\mu}))$, $\gamma_{\mu'}$, $\gamma'_{\mu'}$

have asymptotic behavior w.r.t. $\mathcal{F}_m \mathcal{F}_n(v - v')$.

The surjectivity of the map r (resp. r') follows

from the above arguments immediately. c.e.

It is easy to deduce the assertion (iii)

from the exact sequences $\dots$, $\dots$.

WE leave details to the readers.

n.9 From now on we take the field $\mathbb{K}$ to be the field of real numbers. By this restriction a quantitative argument (concerning a type of argument related to extensions of a given map. (c.f. p.) becomes easy. Now we shall formulate our problem : Let U be a domain in $\mathbb{R}^N$, and let V be a real analytic variety in U ($\dim V \leq N-1$). we also assume a regular series of stratification

$\mathcal{R}$ and its normalizing data $\mathcal{F}$ is attached to V . The series $\mathcal{R}$ is assumed to be reduced and is of type (1). our arguments in this section will be done for such a fixed pair $(\mathcal{R}, \mathcal{F})$. For the sake of simplicity,

we assume that the domains U are of the form $U = \prod_{i=1}^n I_i$ in $\mathbb{R}^N$. Recall that, for a pair $(\mathcal{R}, \mathcal{F})$, a series $\mathcal{R}'$ and a set of functions $\mathcal{F}'$ were fixed we argue by fixing such data $(\mathcal{R}', \mathcal{F}')$ from now on.

What we work with from now on is, however, neither

$\mathcal{R}$ nor $\mathcal{R}'$. We let $\mathcal{S}^{*s}$ be the set of all the
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connected components $S_{\lambda, \mu}^*$ of S^{δ} . Also we mean by $\mathcal{R}^*$ the series composed of sets $\{S^{\delta i}\}_{i=1, \dots, N}$. In this numero our concern is to investigate simple properties of such a series $\mathcal{R}^*$: it is clear that the following are valid:

$$(2.1.14)_1 \quad \dim S_{\lambda, \mu}^* = \dim S_{\lambda}^*, \quad S_{\lambda}^{\delta} \in S^{\delta}, \quad S_{\lambda}^{\delta} \supset S_{\lambda}^*,$$

$$(2.1.14)_2 \quad U^{\delta} = \bigcup_{\lambda, \mu} S_{\lambda, \mu}^*; \quad S_{\lambda, \mu}^* \in S^{\delta}$$

In the above (and in the sequel) the stratum

S_{λ}^{δ} stands for the one which contains $S_{\lambda, \mu}^*$.

Several notations defined for $\mathcal{R}' = \{S^{\delta i}\}_{i=1, \dots, N}$ will be

translated to the series $\mathcal{R}^* = \{S^{\delta i}\}_{i=1, \dots, N}$. An element $S_{\lambda, \mu}^*$

is of first or of second type according to

whether S_{λ}^{δ} is of first or of second type.

Coordinates $(x_{\lambda, \mu})$ and sets of functions $\{f(S_{\lambda, \mu}^*)\}$

$h(S_{\lambda, \mu}^*)$, $g(V(S_{\lambda, \mu}^*))$ are associated with $S_{\lambda, \mu}^*$

by the following formula.

$$(x_{\lambda, \mu}) = (x_{\lambda}), \quad \{f(S_{\lambda, \mu}^*)\} = \{f(S_{\lambda}^{\delta})\},$$

$$h(S_{\lambda, \mu}^*) = h(S_{\lambda}^{\delta}), \quad g(V(S_{\lambda, \mu}^*)) = g(V(S_{\lambda}^{\delta})).$$

It is clear that, for the pair (x_{λ_i, μ_i}) , $\{f(S_{\lambda_i, \mu_i}^{*i})\}$
 the condition (J) $\det \frac{\partial(f(S_{\lambda_i, \mu_i}^{*i}))}{\partial(x_{\lambda_i, \mu_i})}$ is true on $S_{\lambda_i, \mu_i}^{*i}$.

In the sequel, when the necessity occurs to make
 clear the difference of the types of $S_{\lambda_i, \mu_i}^{*i}$, we
 write an element $S_{\lambda_i, \mu_i}^{*i}$ in $\mathcal{S}^{*i}$ as $S_{\lambda_i', \mu_i'}^{*i}$ or $S_{\lambda_i'', \mu_i''}^{*i}$ according
 to whether $S_{\lambda_i, \mu_i}^{*i}$ is of first or of second type.

(A) We shall make several observations about
 the series $\mathcal{S}^{*i} = \{S_{\lambda_i, \mu_i}^{*i}\}_{i=1, \dots, N}$: Let $S_{\lambda_{i-1}, \mu_{i-1}}^{*i-1}$ be an element in $\mathcal{S}_{\lambda_{i-1}, \mu_{i-1}}^{*i-1}$.
 Consider the inverse image $\pi_{i-1, i}^{-1}(S_{\lambda_{i-1}, \mu_{i-1}}^{*i-1})$ of $S_{\lambda_{i-1}, \mu_{i-1}}^{*i-1}$ in U^i .

Take an element $S_{\lambda_i', \mu_i'}^{*i}$ of first type so that $\pi_{i-1, i}(S_{\lambda_i', \mu_i'}^{*i})$

$(S_{\lambda_i', \mu_i'}^{*i}) = S_{\lambda_{i-1}, \mu_{i-1}}^{*i-1}$ holds (Note that it may happen

that there is no $S_{\lambda_i', \mu_i'}^{*i}$ of first type lying over $S_{\lambda_{i-1}, \mu_{i-1}}^{*i-1}$).

Take a point P^{i-1} on $S_{\lambda_{i-1}, \mu_{i-1}}^{*i-1}$ and take a point $P_{\lambda_i', \mu_i'}^{*i}$

on $S_{\lambda_i', \mu_i'}^{*i}$ so that $\pi_{i-1, i}(P_{\lambda_i', \mu_i'}^{*i}) = P^{i-1}$ holds. We

regard the coordinate $x_j(P_{\lambda_i', \mu_i'}^{*i})$ of $P_{\lambda_i', \mu_i'}^{*i}$ as real

analytic function of $P_{\lambda_{i-1}, \mu_{i-1}}^{*i-1}$. Because of the condition

(1.), $x_j(P_{\lambda_i', \mu_i'}^{*i})$ is a (at least multivalued

real analytic function on $S_{\lambda_{i-1}, \mu_{i-1}}^{*i-1}$. Let $\{P_{\lambda_i', \mu_i'}^{*i}\}_{i=1, \dots, N}$

be all the points in $U_{\lambda_i \mu_i} S_{\lambda_i \mu_i}^*$ lying over $P_{\lambda_i \mu_i}^{*i-1}$.

Because of (1.), coordinated $x_i(P_{\lambda_i \mu_i}^{*i}) \neq x_i(P_{\lambda_i \mu_i}^{*i'})$

if $\sigma \neq \sigma'$. Order coordinates $x_i(P_{\lambda_i \mu_i}^{*i})$ as follows :

$$x_i(P_{\lambda_i \mu_i}^{*i}) < x_i(P_{\lambda_i \mu_i'}^{*i}) < \dots < x_i(P_{\lambda_i \mu_i''}^{*i}) < \dots$$

Take a continuous arc γ^{*i-1} in $S_{\lambda_i \mu_i}^{*i-1}$ whose initial

point is $P_{\lambda_i \mu_i}^{*i-1}$. Denote by $P_{\lambda_i \mu_i}^{*i}$ the end point

of γ^{*i-1} . Consider the analytic continuation of $x_i(P_{\lambda_i \mu_i}^{*i})^*$

along γ^{*i-1} . Note that the condition (1.)

implies that $x_i(P_{\lambda_i \mu_i'}^{*i})$, the analytic continuation of

$x_i(P_{\lambda_i \mu_i}^{*i})$ along γ^{*i-1} , is a real valued. Also note

that the condition (1.) shows that the order

of $x_i(P_{\lambda_i \mu_i}^{*i})$'s are unchanged by the analytic

continuation along γ^{*i-1} . Thus we know that coordinates

$x_i(P_{\lambda_i \mu_i}^{*i})$'s are single valued real analytic f

functions on $S_{\lambda_i \mu_i}^{*i-1}$.

(*) For arguments of similar types, see Margrange [].

Therefore we know that

$(S_i^{*i})^\sim$ For each connected component $S_{\lambda_{i-1}, \mu_{i-1}}^{*i-1}$
 $\pi_{i-1, i}^{-1}(S_{\lambda_{i-1}, \mu_{i-1}}^{*i-1})$ is described by the following formula
 $(S_i^{*i})^\sim \cap \pi_{i-1, \mu_{i-1}}^{-1}(S_{\lambda_{i-1}, \mu_{i-1}}^{*i-1}) = S_{\lambda_i, \mu_i}^{*i} \cup S_{\lambda_i', \mu_i'}^{*i} \cup \pi_{i-1, i}^{-1}(S_{\lambda_i, \mu_i}^{*i})$

$= \pi_{i-1, i}^{-1}(S_{\lambda_i, \mu_i}^{*i}) \cup S_{\lambda_i', \mu_i'}^{*i}$, where $S_{\lambda_i', \mu_i'}^{*i}$'s are real analytically biholo-

-morphic onto $S_{\lambda_{i-1}, \mu_{i-1}}^{*i-1}$, while $S_{\lambda_i, \mu_i}^{*i}$'s are of second type.

It is clear that any elements $S_{\lambda_i, \mu_i}^{*i}, S_{\lambda_i', \mu_i'}^{*i}$ are connected components of $\pi_{i-1, i}^{-1}(S_{\lambda_{i-1}, \mu_{i-1}}^{*i-1})$. On the otherhand, it is

also clear that any connected component $S_{\lambda_i, \mu_i}^{*i}, S_{\lambda_i', \mu_i'}^{*i}$ of S_i^{*i} appears in the

right side of $(S_i^{*i})^\sim$. We give an order for

elements $\{S_{\lambda_i', \mu_i'}^{*i}\}'$ of first type by

$$(2.1.15) \quad S_{\lambda_{i,1}', \mu_{i,1}'}^{*i} \prec S_{\lambda_{i,2}', \mu_{i,2}'}^{*i} \prec \dots \prec S_{\lambda_{i,n}', \mu_{i,n}'}^{*i} \text{ if and only}$$

if $\dots$, at one point $P_{\lambda_{i-1}, \mu_{i-1}}^{*i-1} \in S_{\lambda_{i-1}, \mu_{i-1}}^{*i-1}$, coordinates

$x_{\delta}(\pi_{i-1, i}^{-1}(P_{\lambda_{i-1}, \mu_{i-1}}^{*i-1}))$; $P_{\lambda_{i-1}, \mu_{i-1}}^{*i-1} \in S_{\lambda_{i-1}, \mu_{i-1}}^{*i-1}$ are ordered in the

manner

$$x_{\delta}(\pi_{i-1, i}^{-1}(P_{\lambda_{i-1}, \mu_{i-1}}^{*i-1})) < \dots < x_{\delta}(\pi_{i-1, i}^{-1}(P_{\lambda_{i-1}, \mu_{i-1}}^{*i-1})) < \dots$$

Clearly the above order is independent of a point $P_{\lambda_{i-1}, \mu_{i-1}}^{*i-1}$.

Fix a stratum $S_{\lambda_{i-1}, \mu_{i-1}}^{*i-1}$, and consider the difference

$$\pi_{i,\delta}^{-1}(S_{\lambda_{i-1}, \mu_{i-1}}^{*i-1}) - \bigcup_{\lambda_i, \mu_i} S_{\lambda_i, \mu_i}^{*i}, \quad \text{where } S_{\lambda_i, \mu_i}^{*i} \text{ exhaust}$$

all the elements $S_{\lambda_i, \mu_i}^{*i}$ of first type/.

Define a connected manifold $S_{\lambda_i, \mu_i}^{*i}$ by

$$(2.1.16) \quad \pi_{i-1,\delta}^{-1}(S_{\lambda_{i-1}, \mu_{i-1}}^{*i-1}) = S_{\lambda_{i-1}, \mu_{i-1}}^{*i-1} \quad \text{and} \quad \text{for a}$$

point $P_{\lambda_{i-1}, \mu_{i-1}}^{*i-1} \in S_{\lambda_{i-1}, \mu_{i-1}}^{*i-1}$, the inverse $\pi_{i,\delta}^{-1}(S_{\lambda_{i-1}, \mu_{i-1}}^{*i-1}) \cap S_{\lambda_i, \mu_i}^{*i}$ is given by

by

$$\{ \alpha_i : x_i(P_{\lambda_{i-1}, \mu_{i-1}}^{*i-1}) < x_i < x_i(P_{\lambda_i, \mu_i}^{*i}) \}$$

For $t=0$ or $t=t_i$, the above inequality ()

is replaced by one inequality in an obvious way.

Also if $\pi_{i,\delta}^{-1}(S_{\lambda_{i-1}, \mu_{i-1}}^{*i-1}) \cap V^{\delta} = \emptyset$ then $S_{\lambda_i, \mu_i}^{*i}$ is

defined to be the inverse image $\pi_{i,\delta}^{-1}(S_{\lambda_{i-1}, \mu_{i-1}}^{*i-1})$. Now it is

clear that $S_{\lambda_i, \mu_i}^{*i}$'s exhaust all the connected components

of $\pi_{i,\delta}^{-1}(S_{\lambda_{i-1}, \mu_{i-1}}^{*i-1}) - \bigcup_{\lambda_i, \mu_i} S_{\lambda_i, \mu_i}^{*i}$. An order is

given to the elements $\{S_{\lambda_i, \mu_i}^{*i}\}$ (of second type)

, so that $\pi_{i,\delta}^{-1}(S_{\lambda_i, \mu_i}^{*i}) = S_{\lambda_{i-1}, \mu_{i-1}}^{*i-1}$ holds, by

$$(2.1.17) \quad S_{\lambda_i, \mu_i}^{*i} < S_{\lambda_i', \mu_i'}^{*i} \text{ if and only if}$$

and $g (V^{*i} (S_{\lambda_i \mu_i}^{*i}))$, the comparison relations

$(2.1.1)$ are valid with these functions.

Propositions 2.1.~2.1 imply the following facts.

$(2.1.18)$, The collection $\mathcal{S}^{*i}$ is a pre-stratification

of U^i .

$(2.1.18)_2$ Each closed set of U^i , which is

expressed as a disjoint union of $S_{\lambda_i \mu_i}^{*i}$; $S_{\lambda_i \mu_i}^{*i}$ is in $\mathcal{S}^{*i}$,

is $(Q - D)$ — admissible data .

Given a series of directed set of c^* -thickenings

$\mathcal{T}(\mathcal{S}) = \{ \mathcal{T}_{\ell_i}^i(\mathcal{S}^{*i}) \}_{\ell_i=1,2,\dots}$ of $\mathcal{S}^{*i}$, the collection $\mathcal{T}(\mathcal{R}^*) =$

$\{ \mathcal{T}_{\ell_i}^i(\mathcal{S}^{*i}) \}_{\ell_i=1,2,\dots}$ will be called a directed set of

c^* -thickenings of $\mathcal{R}^*$, if the following

conditions are satisfied.

$(2.1.19)$ If $\pi_{\ell_i, \ell_j} (S_{\lambda_i \mu_i}^{*i}) = S_{\lambda_{i-1} \mu_{i-1}}^{*i-1}$

holds, then $\pi_{\ell_i, \ell_j} (\mathcal{T}_{\ell_i}^i (S_{\lambda_i \mu_i}^{*i})) = \mathcal{T}_{\ell_j}^{i-1} (S_{\lambda_{i-1} \mu_{i-1}}^{*i-1})$

In the sequel of this section, when we speak

of a directed set $\mathcal{T}(\mathcal{R}^*)$, we always assume that,

for each directed set $\mathcal{T}^i(\mathcal{S}^*)$, the conditions $(2.1.19)$ is valid.

In the above and in the sequel we denote $\tilde{c}$ -thickenings used in this section by $T(S)_{\tilde{c}}$ rather than by $N(S)_{\tilde{c}}$. We formulate our problems in terms of $\tilde{c}$ -thickenings as follows

Proposition 2.1. 3 . There exists a directed system

$T(\mathcal{Q})$ in such a manner that the following conditions are satisfied .

(2.1.20), There exists series of retract maps $\tau_{\lambda_i, \mu_i}^{\delta} : T(S_{\lambda_i, \mu_i}^{\delta}) \rightarrow T(S_{\lambda_{i+1}, \mu_{i+1}}^{\delta})$ for

any series $S_{\lambda_1, \mu_1}^{\delta} < \dots < S_{\lambda_n, \mu_n}^{\delta}$ so that $\tau_{\lambda_{i+1}, \mu_{i+1}}^{\delta} \circ \tau_{\lambda_i, \mu_i}^{\delta} = \tau_{\lambda_{i+1}, \mu_{i+1}}^{\delta-1}$ hold, if $\tau_{\lambda_{i+1}, \mu_{i+1}}^{\delta-1}(S_{\lambda_i, \mu_i}^{\delta}) = S_{\lambda_{i+1}, \mu_{i+1}}^{\delta-1}$

$S_{\lambda_{i+1}, \mu_{i+1}}^{\delta-1}$ are true,

(2.1.20)_{1,1} For each series $S_{\lambda_1, \mu_1}^{\delta}, \dots, S_{\lambda_n, \mu_n}^{\delta}$ there

exists a finite simple covering $\mathcal{U}_n(S_{\lambda_1, \mu_1}^{\delta}, \dots, S_{\lambda_n, \mu_n}^{\delta})$ of $T_n(S_{\lambda_1, \mu_1}^{\delta}, \dots, S_{\lambda_n, \mu_n}^{\delta})$ so that the following are valid.

(2.1.20)_{1,2} $\mathcal{U}_n(S_{\lambda_1, \mu_1}^{\delta}, \dots, S_{\lambda_n, \mu_n}^{\delta})$ has asymptotic behavior w.r.t. $\tilde{c}$

(2.1.20)_{1.3} For each $\mathcal{U}_n(S_{\lambda_1 \mu_1}^i, \dots, S_{\lambda_k \mu_k}^i)$ there exist a partition of unity $\mathcal{U}_{n, \mu}^i$ subordinate to $\mathcal{U}_n(S_{\lambda_1 \mu_1}^i, \dots, S_{\lambda_k \mu_k}^i)$ so that $\mathcal{U}_{n, \mu}^i(S_{\lambda_1 \mu_1}^i, \dots, S_{\lambda_k \mu_k}^i)$ has quantitative properties w.r.t. From $(S_{\lambda_1 \mu_1}^i)$.

(2.1.20)₁₄ By a suitable choice of indices, the following relations are valid for each $S_{\lambda_1 \mu_1}^i, \dots, S_{\lambda_k \mu_k}^i$

(2.1.20)₁₄ For each $A_{\lambda_1 \mu_1, \dots, \lambda_k \mu_k}^i \in \mathcal{U}_n(S_{\lambda_1 \mu_1}^i, \dots, S_{\lambda_k \mu_k}^i)$, $\pi_{\lambda_1 \mu_1}^i(A_{\lambda_1 \mu_1, \dots, \lambda_k \mu_k}^i)$ holds, and vice versa.

(2.1.20)₁₄ For each (n, μ) and each series $S_{\lambda_1 \mu_1}^i, \dots, S_{\lambda_k \mu_k}^i$ the set $\mathcal{U}_n(S_{\lambda_1 \mu_1}^i, \dots, S_{\lambda_k \mu_k}^i)$ is the restriction of $\mathcal{U}_n(S_{\lambda_1 \mu_1}^i, \dots, S_{\lambda_k \mu_k}^i)$ to $\mathcal{U}_n(S_{\lambda_1 \mu_1}^i, \dots, S_{\lambda_k \mu_k}^i)$. Now we summarize the above propositions in following

Lemma. 2.1.1. (c^∞ - thickenings and c^∞ - forms

with quantities) For a series $\mathcal{R}^*$ in propant

2.1 there exists a series of c^∞ - thickenings $\mathcal{T}(\mathcal{R}^*)$ so

that the following condition is true.

(2.1.21) For each j , and a pair

$V_i^* \subseteq V^*$ of closed sets (= disjoint union of strata

in $\mathcal{S}_{\tilde{\sigma}}^*$, the isomorphism

$$(2.1.21)' \quad H(V^* - V'^* : \mathbb{R}) \cong \frac{\mathbb{H}(V^* - V'^* : V'^*)}{d\Omega_{asy}(V^* - V'^* : V'^*)}$$

It is clear that the propositions 2.1.3 ~ 2.1.

imply the above lemma 2.1.1 : From propositions 2.1.3, 2.1

we showed that $\mathcal{S}_{\tilde{\sigma}}^*(i=1, \dots, N)$ is $(\mathcal{Q} - D)$ pre-stratified

set. On the otherhand, propositions 2.1

show that conditions (2.1.2), (2.1.2)₂ are satisfied by $\mathcal{T}(\mathcal{R}^*)$

Remark 1. We note that the existence of a

simple covering $\mathcal{U}_{\tilde{\sigma}}(S_{\lambda_1}^{*i}, S_{\lambda_2}^{*i})$ of $\mathbb{P}_{\tilde{\sigma}}(S_{\lambda_1}^{*i}, \dots, S_{\lambda_n}^{*i})$ contains more

informations than required in lemma 3.1.1. Actually

It would be of interest to ask the existence of

a finite simple covering $\mathcal{U}_{\tilde{\sigma}}(V^*)$ of $V^* - V'^*$ with asymptotic

behaviors w.r.t. V'^* . Actually the authors

first intension was to show the existence of $\mathcal{U}_{\tilde{\sigma}}(V^*)$ as

above directly. when the author began the present

work in 1971 without enough knowledge of works

of R. Thom, Lojasiewicz and H. Whitney.

It is found that to work with the notion of stratifications is quite basic.

Remark 2.. It would be of interest to pursue

our methods here from two reasons : (i)

From a computation of Betti groups of K -

analytic varieties . (i i) From a point of

view of a possible generalization of the notion

of residue. (c.f. P.A. Griffith ()

J. Leray ())

n.3₁ Remaining parts of this section will be devoted to verifications of propositions and lemmas mentioned above. Our arguments will be done in the following devices.

(A) We show the implication

(B) We verify propositions 2.1.3, 2.1.4

and 2.1.5 inductively on j . Argument of this

part will be divided into several steps. In each

step, our argument is of completely elementary nature.

Arguments will be done by dividing the type of

elements S^{*i} in $\mathcal{A}^{*i}$. Some cares should be taken

further according to natures of elements S^{*i} . We

will argue, in each step, the typical case and

we indicate how to translate our argument to other cases

Because the difference between the typical case and

the other cases is quite small, there is no lack

in our way of discussing problems.

because the steps (I) involve considerable details we shall outline each step taken in the second step.

In the first place (I), we settle a set of neighbourhood to each stratum $S_{\lambda_i \mu_i}^{*j}$. This is done by dividing the strata into first and second type, and this part is of essential (but of quite simple and elementary) nature in the remaining parts of this section. Once we fix a set of neidhbourhoods to each stratum $S_{\lambda_i \mu_i}^{*j}$, we investigate intersection relations of neighbourhoods for a series $S_{\lambda_i \mu_i}^{*j} \rightarrow S_{\lambda_i \mu_i}^{*j}$. This part will be divided into three cases : (i) The case in which the series is composed of strata of first type only . (i i) the case where the series is composed of strata of second type only and finally (i i i) the mixed case where the series contains strata of both types. If the intersections takes a simple form which is commodiate for our purpose, we say that the associated

neighbourhoods are suitable,

we ask quantitative conditions

in order that the associated neighbourhoods are

suitable . This will be done by dividing the cases

into three cases (i) , (ii) , and (iii) , and is

also of quite elementary nature. In (iii)

the condition (2.136) is quickly shown for

the neighbourhoods which are suitable. Finally

in (iv) we show the condition ().

Our arguments in the last two problems will be

done easily on basis of explicit expressions of

intersections of neighbourhoods.

Finally we note that, in the case of $\gamma = 1$

U^i is expressed as a disjoint finite union of

points and one dimensional open segments . Therefore

to verify all the desired conditions in proposition 2.15 is

an obvious matter, and we do not enter into.

Remark . In discussing our problems in (I)

~ (IV), we make the following remarks . These

remarks are chiefly of notational nature . An

element $S_{\lambda, \mu}^{*i} \in \mathcal{S}^{*i}$ will be called simply an element

before we check conditions (2.1.1) ((2.1.1))

After examining the conditions (2.1.1) ((2.1.1)') we

call $S_{\lambda, \mu}^{*i}$ stratum . (i i) To discuss the

inductive steps , it is found to be commodate to divide

considerations into the horizontal and vertical

directions . Especially neighbourhoods of first type

of first type will be designated by $T_{\bar{\sigma}, \bar{\sigma}'}(S_{\lambda, \mu}^{*i})$

rather than $T_{\sigma}(S_{\lambda, \mu}^{*i})$. (i i i) Thirdly we

should make clear uses of distances whether to

the analytical or the topological frontier .

The above remarked points will be taken care of

in the arguments from now on .

№ 3.2. From now on our task will be concentrated to

do discussions mentioned in № 3. (I) First we start with fixing neighbourhoods to each element S_{λ_i, μ_i}^i . This will be done, as was mentioned previously, by dividing into two cases: (i) the element S_{λ_i, μ_i}^i is of first type, (ii) The element S_{λ_i, μ_i}^i is of second type. The stratum of the first type will be written as S_{λ_i, μ_i}^i while the stratum of second type will be written as $S_{\lambda_i, \mu_i}^{i'}$. The elements in S^{i-1} will be written as $S_{\lambda_{i-1}, \mu_{i-1}}^{i-1}$. For a stratum S_{λ_i, μ_i}^i we mean by $S_i^i \{ S_{\lambda_i, \mu_i}^{i-1} \}$ the set of all the strata S_{λ_i, μ_i}^i of first kind lying over $S_{\lambda_{i-1}, \mu_{i-1}}^{i-1}$ while by $S_i^i \{ S_{\lambda_i, \mu_i}^{i-1} \}$ we mean the set of all the elements of second type lying over $S_{\lambda_{i-1}, \mu_{i-1}}^{i-1}$. Some notations introduced in № 1 will be used here without essential changes: Neighbourhoods $N_s (S_{\lambda_i, \mu_i}^{i-1}, \text{From}^{i-1} (S_{\lambda_i}^{i-1}))$ and $T_{s, r} (S_{\lambda_i, \mu_i}^{i-1}, \text{From}^{i-1} (S_{\lambda_i}^{i-1}))$ denote the unions $\bigcup_{p \in S_{\lambda_i, \mu_i}^{i-1}}$

$N_i(P, \text{Fron}^{an}(S_{\lambda_i}^{*i}))$, and $\bigcup_{P \in S_{\lambda_i}^{*i}} T_{\sigma_i}^i(P, \text{Fron}^{an}(S_{\lambda_i}^{*i}))$

respectively. Note that, in view of the existence of the set of functions T^* , we know the following equivalence relation

$$\{N_{\delta}(S_{\lambda_i}^{*i}, \text{Fron}(S_{\lambda_i}^{*i}))\}_{\delta} = \{T_{\sigma_i}^i(S_{\lambda_i}^{*i}, \text{Fron}(S_{\lambda_i}^{*i}))\}_{\delta}$$

For a stratum of first type, $\bar{T}_{\sigma_i}^i$ -level set $\bar{T}_{\sigma_i}^i(S_{\lambda_i}^{*i}, \text{Fron}(S_{\lambda_i}^{*i}))$ was defined in the following way.

$$\bar{T}_{\sigma_i}^i(P^i, \text{Fron}(S_{\lambda_i}^{*i})) = \{Q^i : x_{\mu}(Q^i) = x_{\mu}(P^i) (\mu=1, \dots, M_i) \\ \dim S_{\lambda_i}^{*i}, x_j(Q^i) = x_j(P^i), \quad |x_{\mu+\tau}(Q^i) - x_{\mu+\tau}(P^i)| \leq \delta d(\tau, \text{Fron}(S_{\lambda_i}^{*i})) \\ (\tau=1, \dots, i-1, \mu_i)\}$$

$$\text{Also, for } Q^i \in \hat{T}_{\sigma_i}^i(P^i, \text{Fron}^{an}(S_{\lambda_i}^{*i})), \hat{T}_{\sigma_i}^i(Q^i, \text{Fron}^{an}(S_{\lambda_i}^{*i})) \\ = \{R^i : x_{\tau}(Q^i) = x_{\tau}(R^i) (\tau=1, \dots, i-1), |x_{\tau}(Q^i) - x_{\tau}(R^i)| < \delta d(\tau, \text{Fron}(S_{\lambda_i}^{*i}))\}$$

On the otherhand $\bar{T}_{\sigma_i}^i(S_{\lambda_i}^{*i}, \text{Fron}^{an}(S_{\lambda_i}^{*i}))$ is the union $\bigcup_{P^i} T_{\sigma_i}^i(S_{\lambda_i}^{*i}, \text{Fron}^{an}(S_{\lambda_i}^{*i})) (P \in S_{\lambda_i}^{*i})$, $T_{\sigma_i}^i(S_{\lambda_i}^{*i}, \text{Fron}^{an}(S_{\lambda_i}^{*i})) = \bigcup_{Q^i \in T_{\sigma_i}^i} T_{\sigma_i}^i(Q^i, \text{Fron}(S_{\lambda_i}^{*i}))$

It is easy to see that $\{T_{\sigma_i}^i(S_{\lambda_i}^{*i}, \text{Fron}^{an}(S_{\lambda_i}^{*i}))\}_{\sigma_i} \sim \{T_{\sigma_i}^i(S_{\lambda_i}^{*i}, \text{Fron}^{an}(S_{\lambda_i}^{*i}))\}_{\sigma_i}$. Moreover, for a series of

couples (c^i) , $U_{\sigma, c^i}^i(P^i)$ is defined by (1.1)

and $U_{\sigma, c^i}^i(S_{\lambda, \mu}^{*i})$ is defined to be the union $\bigcup_{P \in S_{\lambda, \mu}^{*i}} U_{\sigma, c^i}^i(P^i)$

$S_{\lambda, \mu}^{*i}$). We shall make quite simple observations used here.

In T_{σ, c^i}^i (or $\bar{T}_{\sigma, c^i}^i$), the projection $\pi_{\lambda, \mu}^i : T_{\sigma, c^i}^i$ (or

$\bar{T}_{\sigma, c^i}^i$) onto $S_{\lambda, \mu}^{*i}$ is defined by $\pi_{\lambda, \mu}^i(R^i) =$

$P_{\lambda, \mu}^i$ where P^i is the uniquely determined point

on $S_{\lambda, \mu}^{*i}$ so that $T_{\sigma, c^i}^i(P_{\lambda, \mu}^i) \supset P^i$ holds.

Let $S_{\lambda, \mu}^{*i-1}$ be a stratum of $\mathcal{S}^{*i-1}$. Then, from

(4.1.1), we know the existence of

positive numbers $(a_{\lambda, \mu, 1}^{*i-1}, a_{\lambda, \mu, 2}^{*i-1})$ in such a way that

the following statement is valid.

(2.1.22) Let $P_{\lambda, \mu}^{*i-1}$ be a point in $S_{\lambda, \mu}^{*i-1}$, and let

$P_{\lambda, \mu}^{*i} < \dots < P_{\lambda, \mu}^{*i}$ be points in $\bigcup S_{\lambda, \mu}^{*i}$ ($S_{\lambda, \mu}^{*i} \in \mathcal{S}^{*i}$)

lying over $P_{\lambda, \mu}^{*i-1}$. Then the inequalities

(2.1.22)' $d(P_{\lambda, \mu}^{*i}, P_{\lambda, \mu}^{*i-1}) \geq a_{\lambda, \mu}^{*i-1} d(P_{\lambda, \mu}^{*i-1}, \text{Front}(S_{\lambda, \mu}^{*i-1}))$

holds for each pair $(P_{\lambda, \mu}^{*i}, P_{\lambda, \mu}^{*i-1})$.

From the above statement, we know that, for

small couples $(\bar{\sigma}_{\lambda, \mu}^{*i}, (\hat{\sigma}_{\lambda, \mu}^{*i}))$ ($i=1, \dots, t_0$) the following

disjoint condition is valid.

$$(2.1.22)' \quad T_{\pi}^{j-1} \left(S_{\lambda_1 \mu_1}^{*j-1}, \text{Fron}^{2n} \left(S_{\lambda_2 \mu_2}^{*j-1} \right) \right) \cap T_{\pi}^{j-1} \left(S_{\lambda_2 \mu_2}^{*j-1}, \text{Fron}^{2n} \left(S_{\lambda_1 \mu_1}^{*j-1} \right) \right) = \emptyset, \text{ if } s \neq \frac{1}{2}.$$

Next take strata $S_{\lambda_2 \mu_2}^{*j-1} \supset S_{\lambda_1 \mu_1}^{*j-1}$ of $\mathcal{S}^{*j-1}$. Take a

sufficiently large n and consider a neighbour

hood $T_{\pi}^{j-1} \left(S_{\lambda_1 \mu_1}^{*j-1} \right)$ of $S_{\lambda_1 \mu_1}^{*j-1}$. Then the conditions (2.1.)_{1,2} are valid in view of the

induction hypothesis for $j-1$. Assume that $\pi_{j-1, j} : S_{\lambda_2 \mu_2}^{*j}$

$\rightarrow S_{\lambda_2 \mu_2}^{*j-1}$ is (real analytically) biholomorphic.

Denote by $S_{\lambda_2 \mu_2}^{*j}(\lambda_1 \mu_1)$ the inverse image of

$S_{\lambda_2 \mu_2}^{*j-1} \cap T_{\pi}^{j-1} \left(S_{\lambda_1 \mu_1}^{*j-1} \right)$ in $S_{\lambda_2 \mu_2}^{*j}$. The map $\pi_{j-1, j}$,

restricted to $S_{\lambda_2 \mu_2}^{*j}(\lambda_1 \mu_1)$, $\pi_{j-1, j} : S_{\lambda_2 \mu_2}^{*j}(\lambda_1 \mu_1) \rightarrow S_{\lambda_2 \mu_2}^{*j-1} \cap T_{\pi}^{j-1} \left(S_{\lambda_1 \mu_1}^{*j-1} \right)$ is

biholomorphic. Take a point $Q_{\lambda_2 \mu_2}^{*j-1}$ in $T_{\pi}^{j-1} \left(S_{\lambda_1 \mu_1}^{*j-1} \right)$

$\cap S_{\lambda_2 \mu_2}^{*j-1}$, and let $P_{\lambda_1 \mu_1}^{*j-1}$ be the point on $S_{\lambda_1 \mu_1}^{*j-1}$

so that $\pi_{j-1, j}^{j-1} \left(Q_{\lambda_2 \mu_2}^{*j-1} \right) = P_{\lambda_1 \mu_1}^{*j-1}$ holds. Join these

two points $Q_{\lambda_2 \mu_2}^{*j-1}$ and $P_{\lambda_1 \mu_1}^{*j-1}$ by an arc $\gamma_{j-1, j}$ in

$T_n^{i-1}(S_{\lambda, \mu_1}^{*i-1}) \cap T_{\sigma_{\lambda_2 \mu_2}^{i-1}}^{i-1}(P_{\lambda, \mu_1}^{*i-1}, \text{Fron}(S_{\lambda, \mu_1}^{*i-1}))$. The point

$Q_{\lambda_2 \mu_2}^{*i}$ stands for the point on $S_{\lambda_2 \mu_2}^{*i}$ so that

$\pi_{i, \delta}(Q_{\lambda_2 \mu_2}^{*i}) = Q_{\lambda_2 \mu_2}^{*i-1}$ is valid. The arc $\gamma_{12}^{*i-1} =$

$\gamma_{1,2}^{*i-1}(P_{\lambda, \mu_1}^{*i-1})$ is lifted to the arc $\tilde{\gamma}_{12}^{*i}$ and the

arc $\tilde{\gamma}_{12}^{*i}$ has an uniquely determined point P_{λ, μ_1}^{*i}

as its boundary ($\neq Q_{\lambda, \mu_1}^{*i}$). It is clear that

P_{λ, μ_1}^{*i} is in V^{δ} and $\pi_{i, \delta}(P_{\lambda, \mu_1}^{*i}) = P_{\lambda, \mu_1}^{*i-1}$ holds.

Let S_{λ, μ_1}^{*i} be the uniquely determined element

of $\mathcal{S}^{*i}$ of first type which contains the

point P_{λ, μ_1}^{*i} . In a little while our argument will

be done for a fixed point $Q_{\lambda, \mu_1}^{*i-1}$ and for an arc

$\gamma_{1,2}^{*i-1}$. We shall use the following terminology for

the relation between $S_{\lambda_2 \mu_2}^{*i}$ of first type and S_{λ, μ_1}^{*i}

of first type.

(2.1.33) The element $S_{\lambda_2 \mu_2}^{*i}$ converges to S_{λ, μ_1}^{*i} along

$\gamma_{1,2}^{*i-1} (\ni Q_{\lambda, \mu_1}^{*i-1})$.

We shall simply express the above fact by

the following manner,

$$(2.1.23)' \quad S_{\lambda_2 \mu_2}^{*i} \xrightarrow[\substack{\gamma_{12}^{*i-1} \\ \pi_{12}}]{\gamma_{12}^{*i}} S_{\lambda_1 \mu_1}^{*i}$$

We shall continue simple observations : It is clear that the following inclusion relation is valid in view of (1.).

$$(2.1.24) \quad S_{\lambda_2 \mu_2}^{*i}(\lambda_1', \mu_1') \subset \bigcup_{\substack{\lambda_1' \mu_1' \\ \pi_{12}^{*i-1} \gamma_{12}^{*i}}} \mathbb{T}^{*i} (S_{\lambda_1' \mu_1'}^{*i}, \text{Fron}^{2n} (S_{\lambda_1'}^{*i}))$$

, where $S_{\lambda_1' \mu_1'}^{*i}$ exhaust all the elements in

$$\{\pi_{12}^{*i-1} (S_{\lambda_1' \mu_1'}^{*i})\}.$$

We remark that the righthand of (2.1.24) is

disjoint union . Because $S_{\lambda_2 \mu_2}^{*i}$ is connected we

know that the set $S_{\lambda_2 \mu_2}^{*i}(\lambda_1', \mu_1')$ is contained in a

neighbourhood $\mathbb{T}^{*i} (S_{\lambda_1' \mu_1'}^{*i}, \text{Fron}^{2n} (S_{\lambda_1'}^{*i}))$ for an

uniquely determined element $S_{\lambda_1' \mu_1'}^{*i}$. It is clear that

for such an uniquely determined element $S_{\lambda_1' \mu_1'}^{*i}$,

the converging relation (2.1.23) is valid for any

point $Q_{\lambda_2 \mu_2}^{*i-1} \in \mathbb{T}_n (S_{\lambda_1 \mu_1}^{*i-1})$ and for an arc γ_{12}^{*i-1} joining

$Q_{\lambda_2 \mu_2}^{*i}$ and $P_{\lambda_2 \mu_2}^{*i} = \pi_{12}^{*i} (Q_{\lambda_2 \mu_2}^{*i})$. The preparations

done hitherto will be used in the sequel .

(I) , After the above preparations , we shall

fix neighbourhoods of certain types for $S_{\lambda, \mu}^{*i}$

of first type. These neighbourhoods will

be written as $T_{n, \hat{\sigma}_{\lambda, \mu}^{*i}}(S_{\lambda, \mu}^{*i})$, and we consider

$T_{n, \hat{\sigma}_{\lambda, \mu}^{*i}}(S_{\lambda, \mu}^{*i})$ parametrized by $(n, \hat{\sigma}_{\lambda, \mu}^{*i})$. In the course

of our arguments to fix $T_{n, \hat{\sigma}_{\lambda, \mu}^{*i}}(S_{\lambda, \mu}^{*i})$, the condition

(2.1.) will be shown. In this part (I),

we assume that n is chosen large enough so

that the inclusion relation

$$\pi_{i+1, i} (T_{\hat{\sigma}_{\lambda, \mu}^{*i}}(S_{\lambda, \mu}^{*i}) \text{ Fron}^{an}(S_{\lambda, \mu}^{*i+1})) \supset T_n(S_{\lambda, \mu}^{*i+1})$$

holds, where $S_{\lambda, \mu}^{*i+1} = \pi_{i+1, i}(S_{\lambda, \mu}^{*i})$.

For such an integer n and for a series of

couples $(\hat{\sigma}_{\lambda, \mu}^{*i})$, define a neighbourhood $T_{n, \hat{\sigma}_{\lambda, \mu}^{*i}}(S_{\lambda, \mu}^{*i})$ by

$$(2.1.25) \quad T_{n, \hat{\sigma}_{\lambda, \mu}^{*i}}(S_{\lambda, \mu}^{*i}) = \bigcup_{Q_{\lambda, \mu}^{*i}} T_{\hat{\sigma}_{\lambda, \mu}^{*i}}(Q_{\lambda, \mu}^{*i} \text{ Fron}^{an}(S_{\lambda, \mu}^{*i}))$$

$$Q_{\lambda, \mu}^{*i} \in \pi_{i+1, i}^{-1}(T_n(S_{\lambda, \mu}^{*i+1})) \cap T_{\hat{\sigma}_{\lambda, \mu}^{*i}}(S_{\lambda, \mu}^{*i}, \text{ Fron}^{an}(S_{\lambda, \mu}^{*i}))$$

From now on , we always assume the following condition

for neighbourhoods $T_{n, \hat{\sigma}_{\lambda, \mu}^{*i}}(S_{\lambda, \mu}^{*i})$.

(2.1.25), For each point $Q_{\lambda, \mu}^{*i}$ appearing in (2.1.25)

the inclusion relation : $U_{C_{\lambda, \mu}^{*i}}(Q_{\lambda, \mu}^{*i}) \subset T_{\sigma_{\lambda, \mu}^{*i}}(Q_{\lambda, \mu}^{*i}, \text{Fron}^{2n}(S_{\lambda, \mu}^{*i}))$

$$(2.1.25)_2 \quad (\sigma_{\lambda, \mu}^{*i}) < (\sigma_{\lambda, \mu}^{*j})$$

in a little while we consider neighbourhoods $T_{\sigma_{\lambda, \mu}^{*i}}(S_{\lambda, \mu}^{*i}, \text{Fron}^{2n}(S_{\lambda, \mu}^{*i}))$ parametrized by a pair $(n, \sigma_{\lambda, \mu}^{*i})$. From the

induction hypothesis we have the following relation

$$\left\{ T_{\sigma_{\lambda, \mu}^{*i}}(S_{\lambda, \mu}^{*i}, \text{Fron}^{2n}(S_{\lambda, \mu}^{*i})) \right\} \sim \left\{ N_{\sigma_{\lambda, \mu}^{*i}}(S_{\lambda, \mu}^{*i}, \text{Fron}^{2n}(S_{\lambda, \mu}^{*i})) \right\}$$

thus, to show (2.1.), it is enough to show

the following

$$(2.1)' \quad T_{\sigma_{\lambda, \mu}^{*i}}(S_{\lambda, \mu}^{*i}, \text{Fron}^{2n}(S_{\lambda, \mu}^{*i})) \cap S_{\lambda, \mu}^{*j} = \emptyset$$

, if $\overline{S}_{\lambda, \mu}^{*i} \cap S_{\lambda, \mu}^{*j} \neq \emptyset$, for a suitable pair $(n, \sigma_{\lambda, \mu}^{*i})$.

(I) _{1.1} We show the conditions (2.1.) (or (2.1)') .

First consider the case where the both elements

$S_{\lambda, \mu}^{*i}$ and $S_{\lambda, \mu}^{*j}$ are of first type so that

$$(i) \quad S_{\lambda, \mu}^{*i} \not\subset S_{\lambda, \mu}^{*j} \quad \text{and} \quad (ii) \quad \overline{S}_{\lambda, \mu}^{*i} \cap S_{\lambda, \mu}^{*j} \neq \emptyset \quad \text{hold.}$$

Denote by $S_{\lambda, \mu}^{*i-1}$ projections $\pi_{\sigma, \delta}(S_{\lambda, \mu}^{*i})$.

Divide cases into the following three cases.

(i), $\overline{S}_{\lambda_2 \mu_2}^{*i-1} \cap S_{\lambda_1}^{*i-1} = \phi$ (ii) $\overline{S}_{\lambda_2 \mu_2}^{*i-1} \not\subset S_{\lambda_1}^{*i-1}$ (iii)

$S_{\lambda_2 \mu_2}^{*i-1} \neq S_{\lambda_1 \mu_1}^{*i-1}$, $\overline{S}_{\lambda_2 \mu_2}^{*i-1} \cap S_{\lambda_1 \mu_1}^{*i-1} \neq \phi$. Consider the first

case. The inequality (2.1.22)' shows that $\overline{S}_{\lambda_2 \mu_2}^{*i} \cap S_{\lambda_1 \mu_1}^{*i}$

$= \phi$ holds. Also, for $T_{n, \sigma_{\lambda_1 \mu_1}}^i(S_{\lambda_1 \mu_1}^{*i})$, the inequality (2.1.22)

shows that the condition (2.1.) is valid. Next

consider the second case: From the condition

$\overline{S}_{\lambda_2 \mu_2}^{*i-1} \cap S_{\lambda_1 \mu_1}^{*i-1} \neq \phi$ and from the induction hypothesis,

(2.1.) holds for $T_{n, \sigma_{\lambda_1 \mu_1}}^i(S_{\lambda_1 \mu_1}^{*i})$. Our consideration

of the third case is divided into further two

cases: (iii)₁, $\overline{S}_{\lambda_2 \mu_2}^{*i} \cap S_{\lambda_1 \mu_1}^{*i} \neq \phi$ and (iii)₂,

$\overline{S}_{\lambda_2 \mu_2}^{*i} \cap S_{\lambda_1 \mu_1}^{*i} = \phi$. First consider the case (iii)₂. Then

we know that (2.1.23), (2.1.24) and (2.1.22)' imply

the condition (2.1.) for $T_{n, \sigma_{\lambda_1 \mu_1}}^i(S_{\lambda_1 \mu_1}^{*i})$. On the

otherhand, in the case of (iii)₁, (2.1.23)₁ shows

that $\overline{S}_{\lambda_2 \mu_2}^{*i} \supset S_{\lambda_1 \mu_1}^{*i}$ holds. Also from (2.1.24) it follows

easily that the condition (2.1.) and (2.1.) hold

for $T_{n, \sigma_{\lambda_1 \mu_1}}^i(S_{\lambda_1 \mu_1}^{*i})$ (I). Let us consider the case where

both elements $S_{\lambda_1 \mu_1}^{*i}$, $S_{\lambda_2 \mu_2}^{*i}$ in question are of first

type. Then the above observation leads easily to (2.1.)

In this connection, we remark the following : The

variety V^i is expressed as a disjoint union

$V^i = \bigcup_{\lambda, \mu} S_{\lambda, \mu}^{*i}$, where $S_{\lambda, \mu}^{*i}$ exhaust all the

elements of first type. From (2.1.)_{1,1} we know

that the conditions (2.1.) are valid for elements

$S_{\lambda, \mu}^{*i}$ of first type, and so the conditions (2.1.)

are valid for $V^i = \bigcup_{\lambda, \mu} S_{\lambda, \mu}^{*i}$ with sets of functions

$\mathcal{H}(S_{\lambda}^{*i})$, $\mathcal{H}(S_{\lambda}^{*i})$ and $\mathcal{H}(S_{\lambda}^{*i})$.

(I)₂ Here in (I)₂, we consider the

condition (2.1.) for the case where one of elements $S_{\lambda_1, \mu_1}^{*i}$

is of first type and the other element $S_{\lambda_2, \mu_2}^{*i}$ is of second

type. Remark that the relation $S_{\lambda_1, \mu_1}^{*i} \not\supset S_{\lambda_2, \mu_2}^{*i}$ does never

happens. What we show is the conditions (2.1.) and

(2.1.), and arguments will be divided into three

cases.

(I)₂. Let $S_{\lambda', \mu'}^{i+j}$ be an element of second type. Our problem here is to associate neighbourhoods of certain types with $S_{\lambda', \mu'}^{i+j}$. By $S_{\lambda, \mu}^{i+j}$ we mean the projection $\pi_{i-1, i} (S_{\lambda', \mu'}^{i+j})$ and by $S_{\lambda', \mu'}^{i+j}$ we mean walls of $S_{\lambda', \mu'}^{i+j}$. Our arguments will be done for the case where the element $S_{\lambda', \mu'}^{i+j}$ has two walls. Other cases will be discussed by modifying arguments for $S_{\lambda', \mu'}^{i+j}$ with two walls easily. Before we fix neighbourhoods of $S_{\lambda', \mu'}^{i+j}$, we remark the following: For a point $p_{\lambda', \mu'}^{i+j} \in S_{\lambda', \mu'}^{i+j}$, the comparison relation $d (p_{\lambda', \mu'}^{i+j}, \text{Fron}^{an} (S_{\lambda', \mu'}^{i+j})) \sim d (p_{\lambda', \mu'}^{i+j}, \text{Fron} (S_{\lambda', \mu'}^{i+j}))$ holds, where we let $S_{\lambda'}^{i+j}$ be strata of $\mathcal{J}^{i+j}$ containing $S_{\lambda'}^{i+j}$ respectively. We work with the distance measured to $\text{Fron} (S_{\lambda', \mu'}^{i+j})$, rather than the distance to $\text{Fron}^{an} (S_{\lambda', \mu'}^{i+j})$. In arguments of this part in (II)₃ we shall fix couples $(\overrightarrow{\sigma}_{\lambda', \mu'}^{i+j}), (\overleftarrow{\sigma}_{\lambda', \mu'}^{i+j})$ to elements $S_{\lambda', \mu'}^{i+j}$ of first kind in such a manner that the conditions (1,)

(2.1.25)₁ and (2.1.25)₂ are true. Note that

$\pi_{i,\delta} (S_{\lambda,\mu}^{*i}) = \pi_{i,\delta} (S_{\lambda,\mu'}^{*i})$ holds. Take an integer n

so large that $\pi_{i,\delta} (\pi_{\sigma_{\lambda,\mu}^{*i-1}} (S_{\lambda,\mu'}^{*i-1} , \text{Fron} (S_{\lambda,\mu'}^{*i-1}))) \supset$

$\pi_n^{*i-1} (S_{\lambda,\mu}^{*i-1})$ holds. By $\pi_n^{*i-1} (S_{\lambda,\mu'}^{*i-1})$ the inverse image

of $\pi_n^{*i-1} (S_{\lambda,\mu}^{*i-1})$ by $\pi_{i-1,\delta}$ in $\pi_{\sigma_{\lambda,\mu}^{*i-2}}^{*i-1} (S_{\lambda,\mu'}^{*i-2} , \text{Fron} (S_{\lambda,\mu'}^{*i-2}))$

is meant. Because of the biholomorphic property of $\pi_{i-1,\delta}$

on $\pi_{\sigma_{\lambda,\mu'}^{*i-2}}^{*i-1} (S_{\lambda,\mu'}^{*i-2} , \text{Fron} (S_{\lambda,\mu'}^{*i-2}))$ it follows that

the map $\pi_{i-1,\delta} : \pi_n^{*i-1} (S_{\lambda,\mu'}^{*i-1}) \rightarrow \pi_n^{*i-1} (S_{\lambda,\mu}^{*i-1})$ is biholomorphic

(and surjective). For any point $Q_{\lambda,\mu}^{*i-1} \in \pi_n^{*i-1} (S_{\lambda,\mu}^{*i-1})$

we mean by $Q_{\lambda,\mu'}^{*i-1}$ the (uniquely determined)

points on $\pi_n^{*i-1} (S_{\lambda,\mu'}^{*i-1})$ so that $\pi_{i-1,\delta} (Q_{\lambda,\mu'}^{*i-1}) = Q_{\lambda,\mu}^{*i-1}$

holds. Let $(c_{\lambda,\mu'}^{i,\delta})$ be a couple of positive numbers

so that the inclusion condition $(2.135)_2$ is true

between $(c_{\lambda,\mu'}^{i,\delta})$, $(\sigma_{\lambda,\mu'}^{i,\delta})$ and $(\hat{\sigma}_{\lambda,\mu'}^{i,\delta})$. Also we

assume that $(c_{\lambda,\mu'}^{i,\delta}) < (c_{\lambda,\mu'}^{i,\delta})$ is true. From now

on we always assume that the above two conditions

are valid with fixed $(\hat{\sigma}_{\lambda,\mu'}^{i,\delta})$, $(\sigma_{\lambda,\mu'}^{i,\delta})$ and

$(\sigma_{\lambda,\mu'}^{i,\delta})$ throughout this section. For a point $Q_{\lambda,\mu}^{*i-1} \in \pi_n^{*i-1} (S_{\lambda,\mu}^{*i-1})$

define $\hat{\pi}_{\lambda,\mu'}^{*i-1} (Q_{\lambda,\mu}^{*i-1})$, which is an open segment, as

follows

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$$\begin{aligned}
 (2.1.26)' \quad \hat{T}_{n, c_{\lambda, \mu}^{*i}}^{\delta} (Q_{\lambda, \mu}^{*i}) &= \{ Q_{\lambda, \mu}^{\delta} \in U^{\delta} : \pi_{\lambda, \mu}^{\delta} (Q_{\lambda, \mu}^{\delta}) \\
 &= Q_{\lambda, \mu}^{*i}, x_{n, c_{\lambda, \mu}^{*i}} (Q_{\lambda, \mu}^{*i}) = c_{\lambda, \mu, i} \cdot d (Q_{\lambda, \mu}^{*i}, \pi_{\lambda, \mu}^{\delta} (Q_{\lambda, \mu}^{\delta})) \\
 &\leq x_{n, c_{\lambda, \mu}^{*i}} (Q_{\lambda, \mu}^{*i}) \leq 1 \times (Q_{\lambda, \mu}^{*i}) = c_{\lambda, \mu}^{\delta} \cdot d (Q_{\lambda, \mu}^{*i}, \pi_{\lambda, \mu}^{\delta} (Q_{\lambda, \mu}^{\delta})) \}
 \end{aligned}$$

A neighbourhood $\hat{T}_{n, c_{\lambda, \mu}^{*i}}^{\delta} (S_{\lambda, \mu}^{*i})$ of $S_{\lambda, \mu}^{*i}$ is defined by

$$(2.1.26) \quad \hat{T}_{n, c_{\lambda, \mu}^{*i}}^{\delta} (S_{\lambda, \mu}^{*i}) = \bigcup_{Q_{\lambda, \mu}^{\delta}} \hat{T}_{n, c_{\lambda, \mu}^{*i}}^{\delta} (Q_{\lambda, \mu}^{\delta}) : Q_{\lambda, \mu}^{\delta} \in$$

$\hat{T}_{n, c_{\lambda, \mu}^{*i}}^{\delta} (S_{\lambda, \mu}^{*i})$
Neighbourhoods considered here are of the above

types $\hat{T}_{n, c_{\lambda, \mu}^{*i}}^{\delta} (S_{\lambda, \mu}^{*i})$. In a little while we consider

$\hat{T}_{n, c_{\lambda, \mu}^{*i}}^{\delta} (S_{\lambda, \mu}^{*i})$ as parametrized by pairs $(n, (c_{\lambda, \mu}^{*i}))$.

Examine simple properties of such neighborhoods $\hat{T}_{n, c_{\lambda, \mu}^{*i}}^{\delta} (S_{\lambda, \mu}^{*i})$.

First we show the following relation

$$(2.1.26)'' \quad \left\{ \hat{T}_{n, c_{\lambda, \mu}^{*i}}^{\delta} (S_{\lambda, \mu}^{*i}) \right\}_{n, c_{\lambda, \mu}^{*i}} \sim \left\{ \hat{T}_{\sigma, \delta}^{\delta} (S_{\lambda, \mu}^{*i}) \right\}_{\sigma, \delta},$$

where $(n, c_{\lambda, \mu}^{*i})$ exhaust all the pair so that (2.1.25)_{i,2}

are valid, while (σ, δ) 's are chosen arbitrarily.

The above equivalence relation is shown by dividing

our arguments as follows : (i) Outside of $\hat{T}_{n, c_{\lambda, \mu}^{*i}}^{\delta} (S_{\lambda, \mu}^{*i})$,
From $(S_{\lambda, \mu}^{*i})$ and (ii) In $\hat{T}_{n, c_{\lambda, \mu}^{*i}}^{\delta} (S_{\lambda, \mu}^{*i})$, From $(S_{\lambda, \mu}^{*i})$

First consider the case (i). Let $Q_{\lambda', \mu'}^{*i}$ be a point in $\hat{T}_{n, c_{\lambda', \mu'}^{*i}}^{\delta}(S_{\lambda', \mu'}^{*i}) = \bigcup_{\substack{\sigma_{\lambda', \mu'}^{*i, \delta} \\ \sigma_{\lambda', \mu'}^{*i, \delta}}} T_{\sigma_{\lambda', \mu'}^{*i, \delta}}^{\delta}(S_{\lambda', \mu'}^{*i}, \text{Fron}(S_{\lambda', \mu'}^{*i}))$. Then, for any element $S_{\lambda', \mu'}^{*i}$ of first type satisfying the condition:

$$\pi_{\delta^{-1}, \delta}(S_{\lambda', \mu'}^{*i}) = \pi_{\delta^{-1}, \delta}(S_{\lambda', \mu'}^{*j}), \quad \text{the relation}$$

$$(2.1.27) \quad Q_{\lambda', \mu'}^{*i} \notin \bigcup_{\substack{\sigma_{\lambda', \mu'}^{*i, \delta} \\ \sigma_{\lambda', \mu'}^{*i, \delta}}} T_{\sigma_{\lambda', \mu'}^{*i, \delta}}^{\delta}(S_{\lambda', \mu'}^{*i}, \text{Fron}(S_{\lambda', \mu'}^{*i}))$$

holds in view of (2.1.22)'.

Combining the above relation with the induction hypothesis we know the following fact.

$$(2.1.27)' \quad Q_{\lambda', \mu'}^{*i} \notin \bigcup_{\substack{\sigma_{\lambda', \mu'}^{*i, \delta} \\ \sigma_{\lambda', \mu'}^{*i, \delta}}} T_{\sigma_{\lambda', \mu'}^{*i, \delta}}^{\delta}(S_{\lambda', \mu'}^{*i}, \text{Fron}(S_{\lambda', \mu'}^{*i})),$$

where $S_{\lambda', \mu'}^{*i}$'s exhaust all the elements in $\mathcal{S}^{*i}$ of first kind: $\pi_{\delta^{-1}, \delta}(S_{\lambda', \mu'}^{*i}) \subset \hat{S}_{\lambda}^{*i-1}$.

Recall that the analytical frontier $\text{Fron}^{an}(S_{\lambda', \mu'}^{*i})$ is expressed as a disjoint union of elements

in the following way.

$$(2.1.27)'' \quad \text{Fron}^{an}(S_{\lambda', \mu'}^{*i}) = \pi_{\delta^{-1}, \delta}^{-1}(\text{Fron}^{an}(\hat{S}_{\lambda}^{*i}))$$

$$\bigcup \left\{ \bigcup_{\lambda', \mu'} S_{\lambda', \mu'}^{*i} \right\}. \quad \text{where } S_{\lambda', \mu'}^{*i} \text{ are the}$$

elements of first type satisfying the condition

$$\pi_{\delta^{-1}, \delta}(S_{\lambda', \mu'}^{*i}) \subset \hat{S}_{\lambda}^{*i-1}.$$

Because the frontier of $S_{\lambda', \mu'}^{*i}$ (both in topological

and analytical senses) are contained in $\text{Fron}^{2n}(\mathbb{R}^n)$
the relations (2.1.27), (2.1.27)' lead to the following
comparison relation

$$(2.1.27)'' \quad d \left(Q_{\lambda, \mu}^{*i}, \text{Fron}^{2n} (S_{\lambda, \mu}^{*i}) \right) \sim d \left(Q_{\lambda, \mu}^{*i}, \right. \\ \left. \frac{\pi_{\lambda, \mu}^{-1} \left(\text{Fron}^{2n} (S_{\lambda, \mu}^{*i}) \right)}{\pi_{\lambda, \mu}^{-1} \left(\text{Fron}^{2n} (S_{\lambda, \mu}^{*i}) \right)} \right),$$

so far as $Q_{\lambda, \mu}^{*i}$'s are in $\frac{\pi_{\lambda, \mu}^{-1} (S_{\lambda, \mu}^{*i})}{\pi_{\lambda, \mu}^{-1} (S_{\lambda, \mu}^{*i})}$ but are
not in $\bigcup_{\lambda, \mu} \frac{\pi_{\lambda, \mu}^{-1} (S_{\lambda, \mu}^{*i})}{\pi_{\lambda, \mu}^{-1} (S_{\lambda, \mu}^{*i})}$.

Combining the above relation (2.1.27)'' with the
obvious relation $d \left(Q_{\lambda, \mu}^{*i}, \frac{\pi_{\lambda, \mu}^{-1} \left(\text{Fron}^{2n} (S_{\lambda, \mu}^{*i}) \right)}{\pi_{\lambda, \mu}^{-1} \left(\text{Fron}^{2n} (S_{\lambda, \mu}^{*i}) \right)} \right) = d \left(\right. \\ \left. \pi_{\lambda, \mu}^{-1} (Q_{\lambda, \mu}^{*i}), \text{Fron}^{2n} (S_{\lambda, \mu}^{*i}) \right)$, we know easily the follo-
-wing facts.

(2.1.28), For an arbitrary given (n, $(c_{\lambda, \mu}^{*i})$), choosing $\sigma_{\lambda, \mu}^{\bar{i}}$ small enough leads to
the inclusion relation

$$\frac{\pi_{\lambda, \mu}^{-1} (S_{\lambda, \mu}^{\bar{i}}, \text{Fron}^{2n} (S_{\lambda, \mu}^{*i}))}{\pi_{\lambda, \mu}^{-1} (S_{\lambda, \mu}^{\bar{i}}, \text{Fron}^{2n} (S_{\lambda, \mu}^{*i}))} \subset \frac{\pi_{\lambda, \mu}^{-1} (S_{\lambda, \mu}^{*i}, \text{Fron}^{2n} (S_{\lambda, \mu}^{\bar{i}}))}{\pi_{\lambda, \mu}^{-1} (S_{\lambda, \mu}^{*i}, \text{Fron}^{2n} (S_{\lambda, \mu}^{\bar{i}}))},$$

and conversely,

(2.1.28)₂ for an arbitrary given ($\sigma_{\lambda, \mu}^{\bar{i}}$),

choosing n sufficiently large and choosing ($c_{\lambda, \mu}^{*i}$)

so that (2.1) holds, lead to the inclusion relation

$$\begin{aligned} T_{n, C_{\lambda', \mu}^{+i}}(S_{\lambda', \mu}^{+i}, \text{Fron}^{2n}(S_{\lambda', \mu}^{+i})) &= V T_{\frac{\sigma_{\lambda', \mu}^{+i}}{\sigma_{\lambda', \mu}^{+i}}}(\frac{\sigma_{\lambda', \mu}^{+i}}{\sigma_{\lambda', \mu}^{+i}}, \text{Fron}(S_{\lambda', \mu}^{+i})) \\ &\subset T_{\sigma_{\lambda', \mu}^{+i}}(S_{\lambda', \mu}^{+i}, \text{Fron}^{2n}(S_{\lambda', \mu}^{+i})) = T_{\frac{\sigma_{\lambda', \mu}^{+i}}{\sigma_{\lambda', \mu}^{+i}}}(S_{\lambda', \mu}^{+i}, \text{Fron}(S_{\lambda', \mu}^{+i})) \end{aligned}$$

Next we shall make our arguments in $T_{\frac{\sigma_{\lambda', \mu}^{+i}}{\sigma_{\lambda', \mu}^{+i}}}(S_{\lambda', \mu}^{+i}, \text{Fron}(S_{\lambda', \mu}^{+i}))$. Note that, in view of (2.1.27)', we

can assume that the following relation is valid.

$$(2.1.28)_2, d(Q_{\lambda', \mu}^{+i}, \text{Fron}^{2n}(S_{\lambda', \mu}^{+i})) = d(Q_{\lambda', \mu}^{+i}, S_{\lambda', \mu}^{+i}),$$

so far as $Q_{\lambda', \mu}^{+i}$ is in $T_{\frac{\sigma_{\lambda', \mu}^{+i}}{\sigma_{\lambda', \mu}^{+i}}}(S_{\lambda', \mu}^{+i}, \text{Fron}(S_{\lambda', \mu}^{+i}))$.

First fix a pair $(n, (C_{\lambda', \mu}^{+i}))$. By the inductive condition (2.1), we can choose $\sigma_{\lambda', \mu}^{+i}(n)$ in

such a way that the inclusion relation

$$T_{n, C_{\lambda', \mu}^{+i}}(S_{\lambda', \mu}^{+i}) \subset T_{\frac{\sigma_{\lambda', \mu}^{+i}(n)}{\sigma_{\lambda', \mu}^{+i}}}(S_{\lambda', \mu}^{+i}, \text{Fron}(S_{\lambda', \mu}^{+i}))$$

is valid.

After fixing $(\sigma_{\lambda', \mu}^{+i})$ so that the above inclusion

relation is valid, choose a series of couples $(\sigma_{\lambda', \mu}^{+i})$, associated with the element $S_{\lambda', \mu}^{+i}$, small enough.

More precisely, from (2.1.28)₂ we know the

existence of maps $\mathcal{L}_{\lambda', \mu}^1$, $\mathcal{L}_{\lambda', \mu}^2$ and $\mathcal{L}_{\lambda', \mu}^3$ or $(e-2)$

types depending on $S_{\lambda', \mu'}^{*j}$ only in such a manner that the following statement is valid.

(9.1.29) Inequalities of the following forms

$$\max (\sigma_{\lambda', \mu'}^{*j}, \sigma_{\lambda', \mu'}^{*j}) < \min_{\lambda', \mu'} \{ (c_{\lambda', \mu'}^{*j})^{-1}, (c_{\lambda', \mu'}^{*j}) \}$$

$$\max_{\lambda', \mu'} \mathcal{L}_{\lambda', \mu'}^2 (\sigma_{\lambda', \mu'}^{*j}, \sigma_{\lambda', \mu'}^{*j}) < \min_{\lambda', \mu'} \mathcal{L}_{\lambda', \mu'}^3 (\sigma_{\lambda', \mu'}^{*j}, \sigma_{\lambda', \mu'}^{*j})$$

imply the inclusion relation

$$\mathcal{P}_{\lambda', \mu'}^j (S_{\lambda', \mu'}^{*j}, \text{Fron}^{an} (S_{\lambda', \mu'}^{*j})) \cap \mathcal{P}_{\lambda', \mu'}^j (S_{\lambda', \mu'}^{*j})$$

$$\subset \mathcal{P}_{\lambda', \mu'}^j (S_{\lambda', \mu'}^{*j}) \cap \mathcal{P}_{\lambda', \mu'}^j (S_{\lambda', \mu'}^{*j})$$

(Remark) Maps $\mathcal{L}_{\lambda', \mu'}^i$ ($i = 1, 2, 3$) are, for example, given explicitly as follows.

$$\mathcal{L}_{\lambda', \mu'}^1 (c_1^{-1}, c_2^{-1}) = ((c_{\lambda', \mu'}^1)^{-1}, (c_{\lambda', \mu'}^2)^{-1}), \quad \mathcal{L}_{\lambda', \mu'}^2 (\sigma_{\lambda', \mu'}^{*j}, \sigma_{\lambda', \mu'}^{*j}) =$$

$$(\sigma_{\lambda', \mu'}^{*j}, (\sigma_{\lambda', \mu'}^{*j})^{-1}, \sigma_{\lambda', \mu'}^{*j}, \sigma_{\lambda', \mu'}^{*j}) \quad \mathcal{L}_{\lambda', \mu'}^3 (\sigma_{\lambda', \mu'}^{*j}, \sigma_{\lambda', \mu'}^{*j}) =$$

$$(\sigma_{\lambda', \mu'}^{*j}, (b_{\lambda', \mu'}^{*j})^{-1}, b_{\lambda', \mu'}^{*j}, \sigma_{\lambda', \mu'}^{*j})$$

Here $(b_{\lambda', \mu'}^{*j}, b_{\lambda', \mu'}^{*j})$ are appearing from the follo-

-wing situation : For any point $P_{\lambda', \mu'}^{*j} \in S_{\lambda', \mu'}^{*j}$, $d (P_{\lambda', \mu'}^{*j}, \pi_{\lambda', \mu'}^{-1} (\text{Fron}^{an} (S_{\lambda', \mu'}^{*j}))) > b_{\lambda', \mu'}^{*j} d (P_{\lambda', \mu'}^{*j}, \text{Fron} (S_{\lambda', \mu'}^{*j}))$.

Conversely we start with a given $(\sigma_{\lambda', \mu'}^{*j})$. Choose $(\sigma_{\lambda', \mu'}^{*j})$ and $(c_{\lambda', \mu'}^{*j})$ so that the inequalities

$$\min (\sigma_{\lambda', \mu'}^{*j}, \sigma_{\lambda', \mu'}^{*j}) > \max_{\lambda', \mu'} \mathcal{L}_{\lambda', \mu'}^1 ((c_{\lambda', \mu'}^{*j})^{-1}, (c_{\lambda', \mu'}^{*j})^{-1})$$

$$\min_{\lambda', \mu'} \mathcal{L}_{\lambda', \mu'}^2 (\sigma_{\lambda', \mu'}^{*j}, \sigma_{\lambda', \mu'}^{*j}) > \max_{\lambda', \mu'} \mathcal{L}_{\lambda', \mu'}^3 (\sigma_{\lambda', \mu'}^{*j}, \sigma_{\lambda', \mu'}^{*j})$$

and the inclusion relation

$$T_n^{i-1}(S_{\lambda, \mu}^{*i-1}) \subset T_{\sigma_{\lambda, \mu}^{i-1}}^{i-1}(S_{\lambda, \mu}^{*i-1}, \text{Fron}^n(S_{\lambda, \mu}^{*i-1}))$$

imply the following inclusion relation

$$T_{n, c_{\lambda, \mu}^{i-1}}^i(S_{\lambda, \mu}^{*i}) \subset T_{\sigma_{\lambda, \mu}^{i-1}}^i(S_{\lambda, \mu}^{*i}).$$

Thus the equivalence relation (2.1.26)'' is assured.

(II)' Once the above equivalence relation (2.1.26)'' is shown, it is quite easy to verify desired properties of $T_{n, c_{\lambda, \mu}^{i-1}}^i(S_{\lambda, \mu}^{*i})$: Here, concerning relations between $(n, (c_{\lambda, \mu}^{i-1}))$, the inclusion relation (2.1) only is assumed. Let $(S_{\lambda, \mu}^{*i}, S_{\lambda', \mu'}^{*i})$ be a pair of elements $S_{\lambda, \mu}^{*i}$ of second type and $S_{\lambda', \mu'}^{*i}$ of first type. From the definition of $T_{n, c_{\lambda, \mu}^{i-1}}^i(S_{\lambda, \mu}^{*i})$ itself, the following relation

$$(2.1.30), \quad T_{n, c_{\lambda, \mu}^{i-1}}^i(S_{\lambda, \mu}^{*i}) \cap S_{\lambda, \mu}^{*i} = \emptyset$$

follows immediately.

Next let $S_{\lambda, \mu}^{*i} (\neq) S_{\lambda', \mu'}^{*i}$ be two elements of second type. As in (II), divide our considerations into three cases: (i) $\Pi_{i, \lambda}^i(S_{\lambda, \mu}^{*i}) = \Pi_{i, \lambda'}^i(S_{\lambda', \mu'}^{*i})$,

(ii) $\pi_{\delta, \delta'}(S_{\lambda', \mu'}^{*i}) \not\subset \pi_{\delta, \delta'}(\bar{S}_{\lambda', \mu'}^{*i})$ (iii) $\pi_{\delta, \delta'}(S_{\lambda', \mu'}^{*i}) \neq \pi_{\delta, \delta'}(\bar{S}_{\lambda', \mu'}^{*i})$

$S_{\lambda', \mu'}^{*i} \subset \overline{\pi_{\delta, \delta'}(S_{\lambda', \mu'}^{*i})}$. In the cases of (i)

and (ii), the definition of $T_{\delta, \delta'}^i(S_{\lambda', \mu'}^{*i})$ and the

induction hypothesis imply the relation

$$(2.1.30)_2 \quad T_{\delta, \delta'}^i(S_{\lambda', \mu'}^{*i}) \cap S_{\lambda', \mu'}^{*i} = \phi.$$

consider the third case : Strata $S_{\lambda, \mu}^{*i-1}$, $S_{\lambda_1, \mu_1}^{*i-1}$ denote the

projections $\pi_{\delta, \delta'}(S_{\lambda', \mu'}^{*i})$ and $\pi_{\delta, \delta'}(S_{\lambda', \mu'}^{*i})$ respectively.

From (2.1.22), $\bar{S}_{\lambda', \mu'}^{*i} \cap S_{\lambda', \mu'}^{*i} \neq \phi$ is valid if and

only if

$$(2.1.30)_3' \quad S_{\lambda', \mu'}^{*i} \supseteq S_{\lambda, \mu}^{*i-1} \supseteq \bar{S}_{\lambda', \mu'}^{*i} \supseteq \bar{S}_{\lambda', \mu'}^{*i}$$

holds. Here $S_{\lambda', \mu'}^{*i}$'s are those elements lying over

$S_{\lambda, \mu}^{*i-1}$ so that $S_{\lambda', \mu'}^{*i} \rightarrow S_{\lambda', \mu'}^{*i-1} (S_{\lambda', \mu'}^{*i-1} \rightarrow S_{\lambda, \mu}^{*i-1})$ holds

(in this place we discuss the case where both

elements $S_{\lambda', \mu'}^{*i}$, $S_{\lambda', \mu'}^{*i-1}$ have two walls. Other cases are

discussed similarly and we shall omit tedious arguments.)

From the definition of $T_{\delta, \delta'}^i(S_{\lambda', \mu'}^{*i})$ and (2.1.), we know

the following.

$$(2.1.30)_{3,1} \quad \text{If } S_{\lambda', \mu'}^{*i} \cap \bar{S}_{\lambda', \mu'}^{*i} = \phi \text{ then } T_{\delta, \delta'}^i(S_{\lambda', \mu'}^{*i}) \cap S_{\lambda', \mu'}^{*i} = \phi.$$

On the other hand, if $S_{\lambda, \mu}^{*i} \cap \bar{S}_{\lambda', \mu'}^{*i}$ is non-empty, then

from (2.1.30)' we know that

$$(2.1.30)_{3,2} \quad T_{n, C_{\lambda, \mu}^{*i}}(S_{\lambda, \mu}^{*i}) \cap S_{\lambda', \mu'}^{*i} = \bigcup_{Q_{\lambda', \mu'}^{*i-1}} \hat{T}_{C_{\lambda', \mu'}^{*i}}(Q_{\lambda', \mu'}^{*i-1}) \quad \text{holds,}$$

where points $Q_{\lambda', \mu'}^{*i-1}$ are in $T_n(S_{\lambda, \mu}^{*i-1}) \cap \bar{S}_{\lambda', \mu'}^{*i-1}$.

From the above expression, each 'fiber' $T_{C_{\lambda, \mu}^{*i}}(Q_{\lambda', \mu'}^{*i-1})$ is

an open segment. Thus the validity of the conditions

(2.1) are obvious. Now we summarize the above

arguments in (1). Then we know easily that the proposition

2.1 is true. Also we know that the conditions ()

are valid for neighbouring $T_{n, C_{\lambda, \mu}^{*i}}(S_{\lambda, \mu}^{*i})$ and $T_{n, C_{\lambda', \mu'}^{*i}}(S_{\lambda', \mu'}^{*i})$

(II)_{1,2} let $S_{\lambda_1, \mu_1}^{+i} < S_{\lambda_2, \mu_2}^{+i}$ be two strata in $\mathcal{S}^{+i}$ of second types. For neighbourhoods $T_{n, c_{\lambda_1, \mu_1}}^i(S_{\lambda_1, \mu_1}^{+i})$ ($i = 1, 2$) we ask the intersection relation : $T_{n, c_{\lambda_1, \mu_1}}^i(S_{\lambda_1, \mu_1}^{+i}) \cap T_{n, c_{\lambda_2, \mu_2}}^i(S_{\lambda_2, \mu_2}^{+i})$.

Strata $S_{\lambda_i, \mu_i}^{+i}$ ($i = 1, 2$) mean walls of $S_{\lambda_i, \mu_i}^{+i}$ ($i = 1, 2$).

Here we will be concerned with the case where $S_{\lambda_i, \mu_i}^{+i}$ ($i = 1, 2$) have two walls. Other cases are dealt in similar ways, and will be touched in the last part of

(II)_{1,2}. Strata $S_{\lambda_i, \mu_i}^{+i-1}$ ($i = 1, 2$) denote projections of $S_{\lambda_i, \mu_i}^{+i}$ ($i = 1, 2$) while $S_{\lambda_i, \mu_i}^{+i}$ denotes the stratum lying over $S_{\lambda_i, \mu_i}^{+i-1}$ so that $S_{\lambda_i, \mu_i}^{+i-1} \rightarrow S_{\lambda_i, \mu_i}^{+i}$ ($S_{\lambda_1, \mu_1}^{+i-1} \rightarrow S_{\lambda_1, \mu_1}^{+i}$) holds. If $T_{n, c_{\lambda_1, \mu_1}}^i(S_{\lambda_1, \mu_1}^{+i}) \cap T_{n, c_{\lambda_2, \mu_2}}^i(S_{\lambda_2, \mu_2}^{+i}) \neq \emptyset$, then the relations $S_{\lambda_1, \mu_1}^{+i-1} \rightarrow S_{\lambda_1, \mu_1}^{+i}$ ($S_{\lambda_2, \mu_2}^{+i-1} \rightarrow S_{\lambda_2, \mu_2}^{+i}$) holds. Because we can assume the relation: $d(Q_{12}^{+i-1}, P_{\lambda_1, \mu_1}^{+i-1}) < d(Q_{12}^{+i-1}, P_{\lambda_2, \mu_2}^{+i-1})$, we know from (2.1) the existence of maps $m_{\lambda_1, \mu_1, \lambda_2, \mu_2}^i$ of (e-l) - type depending on $S_{\lambda_1, \mu_1}^{+i}$, $S_{\lambda_2, \mu_2}^{+i}$ only, with which the following are valid.

(2.1.33) If couples $(c_{\lambda_i, \mu_i}^{+i})$ ($i = 1, 2$) satisfy

the condition : $(c_{\lambda_2, \mu_2}^{+i}) < m_{\lambda_1, \mu_1, \lambda_2, \mu_2}^i(c_{\lambda_1, \mu_1}^{+i})$, then the

following intersection relation holds.

$$(2.1.33)_{11} \quad T_{n, c_{\lambda_1, \mu_1}}^i(S_{\lambda_1, \mu_1}^{+i}) \cap T_{n, c_{\lambda_2, \mu_2}}^i(S_{\lambda_2, \mu_2}^{+i}) = \bigcup_{Q_{12}^{+i-1} \in n, S_{\lambda_1, \mu_1}^{+i-1}} \hat{T}_{n, c_{\lambda_1, \mu_1}}^i(Q_{12}^{+i-1})$$

where points $Q_{i,2}^{x_{i-1}}$ are in $T_{\lambda}^{i-1}(S_{\lambda, \mu_1}^{i-1}, S_{\lambda, \mu_2}^{i-1})$.

We say, in a similar way to (2.1.31), that $T_{\lambda, \mu_1}^{i-1}(S_{\lambda, \mu_1}^{i-1}, S_{\lambda, \mu_2}^{i-1})$

($i = 1, 2$) are suitable if the intersection of them

is expressed in the form of (2.1.33). The conditions (2.1.33)

give a sufficient condition in order that T_{λ, μ_1}^{i-1} are

suitable.

(Remark) The above arguments were done for the

case where both strata have two walls and walls

of S_{λ, μ_1}^{i-1} converge to walls of S_{λ, μ_2}^{i-1} . In other cases

situations are as follows : (i) If no walls of S_{λ, μ_1}^{i-1}

converge to walls of S_{λ, μ_2}^{i-1} , we do not require other

conditions than (2.1.35)_{i,2}. In this case the intersection

of neighbourhoods of S_{λ, μ_i}^{i-1} ($i = 1, 2$) are given by (2.1.33)_{i,1}

(i i) if one wall of S_{λ, μ_1}^{i-1} only converges to a

wall of S_{λ, μ_2}^{i-1} (without loss of generality, we assume

$S_{\lambda, \mu_1}^{i-1} \rightarrow S_{\lambda, \mu_1}^{i-1}$ ($S_{\lambda, \mu_2}^{i-1} \rightarrow S_{\lambda, \mu_1}^{i-1}$)) we require (2.1.33)_i for S_{λ, μ_2}^{i-1}

besides (2.1.33). Then the intersection of neighbourhoods

are given by (2.1.33). In both cases of (i), (ii),

if the intersection of $T_{\lambda, \mu_i}^{i-1}(S_{\lambda, \mu_1}^{i-1})$ is given by (2.1.33)

we say that such neighbourhoods are suitable.

(I I I)_{1,3} Thirdly we consider the case in which one of

the strata is of first type while the other is

of second type: Take a series $S_{\lambda_1 \mu_1}^{*i} \prec S_{\lambda_2 \mu_2}^{*i}$. There are two

possibilities: $\pi_{i,j}(S_{\lambda_1 \mu_1}^{*i}) \neq \pi_{i,j}(S_{\lambda_2 \mu_2}^{*i})$ Here we consider

the first case only. The second case is discussed in an

analogous way. In the first case, for $S_{\lambda_1 \mu_1}^{*i-1} = \pi_{i,j}(S_{\lambda_1 \mu_1}^{*i})$

$S_{\lambda_2 \mu_2}^{*i-1} = \pi_{i,j}(S_{\lambda_2 \mu_2}^{*i})$ and walls $S_{\lambda_2 \mu_2}^{*i}$, the following

relation

$$(2.1.34) \quad S_{\lambda_2 \mu_2}^{*i-1} \geq S_{\lambda_1 \mu_1}^{*i} \geq S_{\lambda_2 \mu_2}^{*i}$$

is valid, where $S_{\lambda_2 \mu_2}^{*i} \rightarrow S_{\lambda_2 \mu_2}^{*i-1}$ ($S_{\lambda_1 \mu_1}^{*i} \rightarrow S_{\lambda_1 \mu_1}^{*i-1}$).

We assume that $S_{\lambda_2 \mu_2}^{*i-1} = S_{\lambda_1 \mu_1}^{*i}$ holds. Other cases are

mentioned after discussions of this case. As in

(I I I)_{1,2}, take a point Q_{12}^{*i-1} in $T_{\pi}^{i-1}(S_{\lambda_1 \mu_1}^{*i-1}, S_{\lambda_2 \mu_2}^{*i-1})$. Then

points $P_{\lambda_2 \mu_2}^{*i-1}$ ($i=1,2$) $P_{\lambda_1 \mu_1}^{*i}$, $P_{\lambda_1 \mu_1}^{*i}$ are defined in the

same way as in (I I I)_{1,2}. Also $Q_{12}^{*i-1}(\lambda_2'', \mu_2'')$ is

defined as in (I I I)₂ while $Q_{12}^{*i}(\lambda_1'', \mu_1'')$ is in

$T_{\pi}^{i-1}(S_{\lambda_1 \mu_1}^{*i-1})$ so that the relation $\pi_{i,j}(Q_{12}^{*i}(\lambda_1'', \mu_1'')) = Q_{12}^{*i-1}$

is valid. The inequality

$$(2.1.34)_1 \text{ d } (Q_{12}^{*i}(\lambda_1'', \mu_1''), Q_{12}^{*i}(\lambda_2'', \mu_2'')) < c_{\lambda_1}^{*i} d(Q_{12}^{*i}, P_{\lambda_1 \mu_1}^{*i})$$

holds.

in a completely similar manner to arguments in (1 I)_{1,2},

we know the following:

$$(2.1.35) \quad \text{Inequalities} \quad : \quad C_{\lambda'_1 \mu'_1}^{i*} (\bar{S}_1^{i*})^{C_{\lambda'_2 \mu'_2}^{i*}} < \bar{S}_1^{i*}, \quad C_{\lambda'_1 \mu'_1}^{i*} \bar{S}_2^{i*} > \bar{S}_1^{i*}.$$

and the inclusion relation : $T_n^{i-1}(S_{\lambda_i \mu_i}^{i-1}) \subset \Pi_{\lambda_i \mu_i} T_{\lambda_i \mu_i}^i(S_{\lambda_i \mu_i}^{i-1})$

imply the following inclusion relation.

$$(2.1.35)' \quad T_{\lambda'_1 \mu'_1}^{i-1}(S_{\lambda'_1 \mu'_1}^{i-1}) \cap T_{\lambda'_2 \mu'_2}^{i-1}(S_{\lambda'_2 \mu'_2}^{i-1}) = \bigcup_{Q_{12}^{i-1}} T_{\lambda'_1 \mu'_1}^{i-1}(Q_{12}^{i-1}),$$

where Q_{12}^{i-1} 's are in $T_n^{i-1}(S_{\lambda'_1 \mu'_1}^{i-1}, S_{\lambda'_2 \mu'_2}^{i-1})$.

In accordance with (2.1.), (2.1.) if the above intersection relation (2.1.) is true, then neighbourhoods

$T_{\lambda'_1 \mu'_1}^{i-1}(S_{\lambda'_1 \mu'_1}^{i-1}), T_{\lambda'_2 \mu'_2}^{i-1}(S_{\lambda'_2 \mu'_2}^{i-1})$ are suitable.

(Remark) The above arguments are done for the

case where two walls $S_{\lambda'_1 \mu'_1}^{i-1}$ converge to $S_{\lambda'_1 \mu'_1}^{i-1}$.

In other cases, situations are as follows: If

one wall, for example $S_{\lambda'_2 \mu'_2}^{i-1}$ of $S_{\lambda'_2 \mu'_2}^{i-1}$, converges to $S_{\lambda'_2 \mu'_2}^{i-1}$

then we replace the inequalities in (2.1.35) by one

inequality only. If no walls of $S_{\lambda'_2 \mu'_2}^{i-1}$ converge to $S_{\lambda'_2 \mu'_2}^{i-1}$

then the intersection relations (2.1.25)₁, (2.1.25)₂ are

sufficient. In both cases the intersection relations

are expressed in the following manner.

$$(2.1.35)_1 \quad T_{\sigma_{\lambda_1 \mu_1}^{*i}}(S_{\lambda_1 \mu_1}^{*i}) \cap T_{\sigma_{\lambda_2 \mu_2}^{*i}}(S_{\lambda_2 \mu_2}^{*i}) = \bigcup_{Q_{12}^{*i-1} \in \sigma_{\lambda_1 \mu_1}^{*i-1}, \sigma_{\lambda_2 \mu_2}^{*i-1}} T_{\sigma_{\lambda_1 \mu_1}^{*i-1}}(Q_{12}^{*i-1}),$$

where Q_{12}^{*i-1} 's are in $T_n^{*i-1}(S_{\lambda_1 \mu_1}^{*i-1}, S_{\lambda_2 \mu_2}^{*i-1})$ while the

fiber $\hat{T}_{\sigma_{\lambda_1 \mu_1}^{*i-1}}^{*i}(Q_{12}^{*i-1})$ is expressed in the following way

$$(2.1.35)_1' \quad \hat{T}_{\sigma_{\lambda_1 \mu_1}^{*i-1}}^{*i}(Q_{12}^{*i-1}) = \{x_i : x_i(Q_{12}^{*i}) - \sigma_{\lambda_1 \mu_1}^{*i} d(Q_{12}^{*i})\}$$

$$\text{From } (S_{\lambda_1 \mu_1}^{*i}) \leq x_i \leq x_i(Q_{12}^{*i}) - c_{\lambda_2 \mu_2}^{*i} d(Q_{12}^{*i}, P_{12}^{*i})$$

In the above Q_{12}^{*i} , Q_{12}^{*i-1} are points on $\bar{T}_{\sigma_{\lambda_1 \mu_1}^{*i}}(S_{\lambda_1 \mu_1}^{*i})$ or

$\bar{T}_{\sigma_{\lambda_2 \mu_2}^{*i}}(S_{\lambda_2 \mu_2}^{*i})$ so that $\pi_{i,i}(Q_{12,i}^{*i}) = Q_{12}^{*i-1} (i=1,2)$ holds.

If no walls of $S_{\lambda_2 \mu_2}^{*i}$ converges to $S_{\lambda_1 \mu_1}^{*i}$, then the intersection is of the following form is suitable.

$$(2.1.35)_2 \quad T_{\sigma_{\lambda_1 \mu_1}^{*i}}(S_{\lambda_1 \mu_1}^{*i}) \cap T_{\sigma_{\lambda_2 \mu_2}^{*i}}(S_{\lambda_2 \mu_2}^{*i}) = \bigcup_{Q_{12}^{*i} \in \sigma_{\lambda_1 \mu_1}^{*i}, \sigma_{\lambda_2 \mu_2}^{*i}} \hat{T}_{\sigma_{\lambda_1 \mu_1}^{*i}}^{*i}(Q_{12}^{*i})$$

If $\pi_{i,i}(S_{\lambda_1 \mu_1}^{*i}) = \pi_{i,i}(S_{\lambda_2 \mu_2}^{*i})$ is true, then the following

$$(2.1.35)'' \quad c_{\lambda_1 \mu_1, 1}^{*i}(\bar{\sigma}_{\lambda_2 \mu_2}^{*i}) < \hat{\sigma}_{\lambda_1 \mu_1}^{*i}, \quad c_{\lambda_1 \mu_1, 2}^{*i}(\bar{\sigma}_{\lambda_2 \mu_2}^{*i}) > \hat{\sigma}_{\lambda_1 \mu_1}^{*i}, \quad T_n^{*i}(S_{\lambda_1 \mu_1}^{*i}) \cap T_n^{*i}(S_{\lambda_2 \mu_2}^{*i})$$

is sufficient in order that the intersection $T_{\sigma_{\lambda_1 \mu_1}^{*i}}(S_{\lambda_1 \mu_1}^{*i})$

$\cap T_{\sigma_{\lambda_2 \mu_2}^{*i}}(S_{\lambda_2 \mu_2}^{*i})$ is expressed as in (2.1.35.1).

If the intersection is expressed as in (2.1.35),

then we say that $T_{\sigma_{\lambda_1 \mu_1}^{*i}}(S_{\lambda_1 \mu_1}^{*i})$ and $T_{\sigma_{\lambda_2 \mu_2}^{*i}}(S_{\lambda_2 \mu_2}^{*i})$

are suitable. Henceforth, we write the fiber $\hat{T}_{\sigma_{\lambda_1 \mu_1}^{*i}}^{*i}$ of

$T_{\sigma_{\lambda_1 \mu_1}^{*i}}(S_{\lambda_1 \mu_1}^{*i}) \cap T_{\sigma_{\lambda_2 \mu_2}^{*i}}(S_{\lambda_2 \mu_2}^{*i})$ as $\hat{T}_{\sigma_{\lambda_1 \mu_1}^{*i}}^{*i}$ if $T(\sigma)$'s are suitable.

(I I) ₂ Now we combine the above results in the following manner. Assume that neighbourhoods $\{T_{n, c_{\lambda, \mu}^{*j}}(S_{\lambda, \mu}^{*j})\}_{\lambda, \mu}$ and $\{T_{n, c_{\lambda, \mu}^{*j}}(S_{\lambda, \mu}^{*j})\}_{\lambda, \mu'}$ are associated with strata $S_{\lambda, \mu}^{*j}$ of first type and $S_{\lambda, \mu'}^{*j}$ of second type. We start with a situation in which conditions (2.1.35), and (2.1.35)₂ for $(\hat{\sigma}_{\lambda, \mu}^{*j}, (c_{\lambda, \mu}^{*j}))$ are valid. For neighbourhoods $\{T_{n, c_{\lambda, \mu}^{*j}}(S_{\lambda, \mu}^{*j})\}_{\lambda, \mu} \cup \{T_{n, c_{\lambda, \mu'}^{*j}}(S_{\lambda, \mu'}^{*j})\}_{\lambda, \mu'}$, we generalize the notion of the suitability introduced in (I I)_{1,23}. Divide our consideration into the following three cases according to the nature of series : $S_{\lambda_1, \mu_1}^{*j} \rightarrow S_{\lambda_2, \mu_2}^{*j}$. (2.1.36), strata in series are composed of strata of first type only : $S_{\lambda_1, \mu_1}^{*j} \rightarrow \dots \rightarrow S_{\lambda_k, \mu_k}^{*j}$. In this case, let us consider situations in which the intersection of neighbourhoods of strata are expressed in the following way.

$$(2.1.36)_1' \quad T_{n, c_{\lambda_1, \mu_1}^{*j}}(S_{\lambda_1, \mu_1}^{*j}) \cap T_{n, c_{\lambda_2, \mu_2}^{*j}}(S_{\lambda_2, \mu_2}^{*j}) = \bigcup_{\lambda_1, \mu_1, \dots, \lambda_k, \mu_k} \hat{T}_{n, c_{\lambda_1, \mu_1}^{*j}}(Q_{\lambda_1, \mu_1, \dots, \lambda_k, \mu_k}^{*j})$$

where $Q_{\lambda_1, \mu_1, \dots, \lambda_k, \mu_k}^{*j}$ exhaust all the points in the set

$$\bigcap_{i=1}^k T_{n, c_{\lambda_i, \mu_i}^{*j}}(S_{\lambda_i, \mu_i}^{*j}) \cap T_{n, c_{\lambda_{k+1}, \mu_{k+1}}^{*j}}(S_{\lambda_{k+1}, \mu_{k+1}}^{*j}) \dots \cap T_{n, c_{\lambda_k, \mu_k}^{*j}}(S_{\lambda_k, \mu_k}^{*j}).$$

(2.1.36)₂ strata in series are composed of strata

of second type only : $S_{\lambda_1 \mu_1}^{*i} < \dots < S_{\lambda_k \mu_k}^{*i}$: In this case

the situation considered here is the case in

which the intersection is expressed in the following way

$$(2.1.36)_2' \quad T_{n, \lambda_1 \mu_1}^{*i}(S_{\lambda_1 \mu_1}^{*i}) \cap \dots \cap T_{n, \lambda_k \mu_k}^{*i}(S_{\lambda_k \mu_k}^{*i}) = \bigcup_{Q_{\lambda_1 \mu_1, \dots, \lambda_k \mu_k}^{*i-1}} T_{n, \lambda_1 \mu_1}^{*i-1}(Q_{\lambda_1 \mu_1, \dots, \lambda_k \mu_k}^{*i-1}),$$

where $Q_{\lambda_1 \mu_1, \dots, \lambda_k \mu_k}^{*i-1}$'s are points in $T_n^{*i-1}(S_{\lambda_1 \mu_1}^{*i-1}, \dots, S_{\lambda_k \mu_k}^{*i-1})$.

(2.1.36)₃ Mixed cases : $S_{\lambda_1 \mu_1}^{*i} < S_{\lambda_2 \mu_2}^{*i} < \dots < S_{\lambda_{i-1} \mu_{i-1}}^{*i} < S_{\lambda_i \mu_i}^{*i} < \dots < S_{\lambda_k \mu_k}^{*i}$. In this

case, situations are as follows : The intersection

$$\{T_{n, \lambda_1 \mu_1}^{*i}(S_{\lambda_1 \mu_1}^{*i}) \cap \dots \cap T_{n, \lambda_{i-1} \mu_{i-1}}^{*i}(S_{\lambda_{i-1} \mu_{i-1}}^{*i})\} \cap \{T_{n, \lambda_i \mu_i}^{*i}(S_{\lambda_i \mu_i}^{*i}) \cap \dots \cap T_{n, \lambda_k \mu_k}^{*i}(S_{\lambda_k \mu_k}^{*i})\} \text{ is}$$

expressed in the following fashion.

(2.1.36)_{3.1} If two walls $S_{\lambda_1 \mu_1}^{*i}$ of $S_{\lambda_i \mu_i}^{*i}$ converge to $S_{\lambda_1 \mu_1}^{*i}$, then

$$\text{then } \{T_{n, \lambda_1 \mu_1}^{*i}(S_{\lambda_1 \mu_1}^{*i}) \cap \dots \cap T_{n, \lambda_{i-1} \mu_{i-1}}^{*i}(S_{\lambda_{i-1} \mu_{i-1}}^{*i})\} \cap \{T_{n, \lambda_i \mu_i}^{*i}(S_{\lambda_i \mu_i}^{*i}) \cap \dots \cap T_{n, \lambda_k \mu_k}^{*i}(S_{\lambda_k \mu_k}^{*i})\} = \bigcup_{Q_{\lambda_1 \mu_1, \dots, \lambda_k \mu_k}^{*i-1}} T_{n, \lambda_1 \mu_1}^{*i-1}(Q_{\lambda_1 \mu_1, \dots, \lambda_k \mu_k}^{*i-1})$$

$$(Q_{\lambda_1 \mu_1, \dots, \lambda_k \mu_k}^{*i-1} \in T_n^{*i-1}(S_{\lambda_1 \mu_1}^{*i-1}, \dots, S_{\lambda_k \mu_k}^{*i-1}))$$

(2.1.36)_{3.2} If one wall $S_{\lambda_i \mu_i}^{*i}$ only converges to $S_{\lambda_i \mu_i}^{*i}$, then

$$\begin{aligned} & \{T_{n, \lambda_1 \mu_1}^{*i}(S_{\lambda_1 \mu_1}^{*i}) \cap \dots \cap T_{n, \lambda_{i-1} \mu_{i-1}}^{*i}(S_{\lambda_{i-1} \mu_{i-1}}^{*i})\} \cap \{T_{n, \lambda_i \mu_i}^{*i}(S_{\lambda_i \mu_i}^{*i}) \cap \dots \cap T_{n, \lambda_k \mu_k}^{*i}(S_{\lambda_k \mu_k}^{*i})\} \\ &= \bigcup_{Q_{\lambda_1 \mu_1, \dots, \lambda_k \mu_k}^{*i-1}} T_{n, \lambda_1 \mu_1}^{*i-1}(Q_{\lambda_1 \mu_1, \dots, \lambda_k \mu_k}^{*i-1}) \quad (Q_{\lambda_1 \mu_1, \dots, \lambda_k \mu_k}^{*i-1} \in T_n^{*i-1}(S_{\lambda_1 \mu_1}^{*i-1}, \dots, S_{\lambda_k \mu_k}^{*i-1})). \end{aligned}$$

(2.1.36)_{3.3} If no wall of $S_{\lambda_i \mu_i}^{*i}$ converge to $S_{\lambda_i \mu_i}^{*i}$, then

the intersection

$$T_{n, \lambda_1 \mu_1}^{*i}(S_{\lambda_1 \mu_1}^{*i}) \cap \dots \cap T_{n, \lambda_k \mu_k}^{*i}(S_{\lambda_k \mu_k}^{*i}) = \bigcup_{Q_{\lambda_1 \mu_1, \dots, \lambda_k \mu_k}^{*i-1}} T_{n, \lambda_1 \mu_1}^{*i-1}(Q_{\lambda_1 \mu_1, \dots, \lambda_k \mu_k}^{*i-1}) \quad (Q_{\lambda_1 \mu_1, \dots, \lambda_k \mu_k}^{*i-1} \in T_n^{*i-1}(S_{\lambda_1 \mu_1}^{*i-1}, \dots, S_{\lambda_k \mu_k}^{*i-1}))$$

TO be sure we see this fact quickly .

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Note the following obvious facts For series

$S_{\lambda', \mu'}^{*j} < \dots < S_{\lambda'', \mu''}^{*j}$ composed of first type only

or series $S_{\lambda', \mu'}^{*j} < \dots < S_{\lambda'', \mu''}^{*j}$ composed of second type

only , the intersection relations $\bigcap_{\lambda'=1}^{\lambda''} T_{n, C_{\lambda', \mu'}}(S_{\lambda', \mu'}^{*j}) =$

$\bigcap_{\lambda'=1}^{\lambda''} T_{n, C_{\lambda', \mu'}}(S_{\lambda', \mu'}^{*j}) \cap T_{n, C_{\lambda'', \mu''}}(S_{\lambda'', \mu''}^{*j})$ and $\bigcap_{\lambda'=1}^{\lambda''} T_{n, C_{\lambda', \mu'}}(S_{\lambda', \mu'}^{*j}) = \bigcap_{\lambda'=1}^{\lambda''} T_{n, C_{\lambda', \mu'}}(S_{\lambda', \mu'}^{*j})$

$\cap T_{n, C_{\lambda'', \mu''}}(S_{\lambda'', \mu''}^{*j})$ are valid. Then the exist-

-ence of series $L^j(s^j), M^j(s^j)$ couples $\{\sigma_0^j\}, \{\sigma_0^{j'}\}$

with which the following statement is valid,

follows from (2.132), (2.133) immediately.

(2.1.38)' For assignments $\mathcal{F}^j, \mathcal{C}^j$ of type $\{L^j, M^j\}$ satisfying conditions $\hat{\sigma}_0^j < \hat{\sigma}_0^{j'}$, $C_0^j > C_0^{j'}$, there exist

an integer $n' = n'(\mathcal{F}^j, \mathcal{C}^j, \hat{\sigma}_0^j, C_0^j)$ so that intersection

of neighbourhoods of series, composed of strata of

first (or of second) type only , are suitable.

Assume that assignments $\mathcal{F}^j, \mathcal{C}^j$ of type $\{L^j, M^j\}$ are given

so that the conditions $\hat{\sigma}_0^j < \hat{\sigma}_0^{j'}$, $C_0^j > C_0^{j'}$ hold. Fix an

integer n in such a manner that the above

condition (2.1.38)' is true. Next , for the assign-

-ments $\mathcal{F}^j, \mathcal{C}^j$, choose n' sufficiently large

Let $l_{x_i}^i$ and $l_{x_i}^{i'}$ be integers . $l_{x_i}^{i'} = \max_{S_{x_i}^{i'} \in V} l(S_{x_i}^{i'}), l_{x_i}^{i'} = \max_{S_{x_i}^{i'} \in V} l_{x_i}^{i'}$ 143

Given series $\mathcal{L}^{i'} = \{l_1^{i'}, \dots, l_{l_{x_i}^{i'}}^{i'}\}$ and $\mathcal{M}^{i'} = \{m_1^{i'}, \dots, m_{l_{x_i}^{i'}}^{i'}\}$ of

maps of $(e-l)$ - types . , assignments $\hat{\sigma}^{i'}: \{1, \dots, l_{x_i}^{i'}\} \rightarrow$

$x_i' \rightarrow \sigma_i^{i'}$ and $\hat{c}^{i'}: \{1, \dots, l_{x_i}^{i'}\} \rightarrow \{1, \dots, l_{x_i}^{i'}\}$ are said

to be of type $\mathcal{L}^{i'}$ and $\mathcal{M}^{i'}$ respectively if the following conditions are valid.

$$(2.1.37)_1 \quad \hat{\sigma}_1^{i'} \leq \hat{\sigma}_2^{i'}, \dots, \hat{\sigma}_{l_{x_i}^{i'}}^{i'} \leq \hat{\sigma}_x^{i'}$$

$$(2.1.37)_2 \quad (c_1^{i'}) \leq m_1^{i'}(c_2^{i'}), \dots, c_{l_{x_i}^{i'}}^{i'} \leq m_{l_{x_i}^{i'}}^{i'}(c_x^{i'}) .$$

For a given assignments $\hat{\sigma}^{i'}$, $\hat{c}^{i'}$, define $\hat{\sigma}_{x_i}^{i'}(S_{x_i}^{i'})$

$c^{i'}(S_{x_i}^{i'})$ by $\hat{\sigma}_{x_i}^{i'}(S_{x_i}^{i'}) = \hat{\sigma}^{i'}$ and $c_{x_i}^{i'}(S_{x_i}^{i'}) = c^{i'}$. Now we show quickly

the existence of series $\mathcal{L}^{i'}(S^{i'}) = \{l_1^{i'}, \dots, l_{l_{x_i}^{i'}}^{i'}\}$

and $\mathcal{M}^{i'}(S^{i'}) = \{m_1^{i'}, \dots, m_{l_{x_i}^{i'}}^{i'}\}$ depending on $S^{i'}$

only and also the existence of cou

$(\hat{\sigma}_a^{i'}, (c_a^{i'}))$ so that the following statements

are true.

(2.1.38) For any assignments $\hat{\sigma}^{i'}$ and $\hat{c}^{i'}$ of types

$\mathcal{M}^{i'}(S^{i'})$ so that $\hat{\sigma}_1^{i'} \leq \hat{\sigma}_0^{i'}$, $c_1^{i'} > c_0^{i'}$ hold, there exists an

integer $n = n(\hat{\sigma}_0^{i'}, \hat{c}_0^{i'}, \hat{\sigma}_1^{i'}, c_1^{i'})$ with which

neighbourhoods $\{T_{n, \sigma_{x_i}^{i'}}\}_{x_i, i'}$ and $\{T_{n, c_{x_i}^{i'}}\}_{x_i, i'}$ are suitable.

The above statement follows easily from (III I) (12.3):

so that the condition (2.1.35) is true for

any pair $S_{\lambda', \mu'}^{*j} \prec S_{\lambda'', \mu''}^{*j}$ (Remark that (2.1.35)'s

shows that the mentioned fact is possible for

$S_{\lambda', \mu'}^{*j} \prec S_{\lambda'', \mu''}^{*j}$ by choosing n'' large enough)

It is clear that , for such data .

the assertion (2.1.38) is assured .

The above arguments finish this part of argument
to associate neighbourhoods to S .

(III) Assume that neighbourhoods

$T_{n, c_{\lambda', \mu'}}^j (S_{\lambda', \mu'}^{*j})$ $T_{n, c_{\lambda'', \mu''}}^j (S_{\lambda'', \mu''}^{*j})$ are fixed so

that these neighbourhoods are suitable .

(Recall that the assertion (2.1.38) assures that

we can choose suitable neighbourhoods arbitrarily

small .)

For each series $S_{\lambda'_1 \mu'_1}^{*j} < S_{\lambda'_2 \mu'_2}^{*j} < S_{\lambda'_3 \mu'_3}^{*j}$ the

intersection of neighbourhoods $\bigcap_{d=1}^{t_1} \mathbb{P}_{n, \sigma_{\lambda'_d \mu'_d}^{*j}}^i (S_{\lambda'_d \mu'_d}^{*j}) \cap_{d=1}^{t_2} \mathbb{P}_{n, \sigma_{\lambda'_d \mu'_d}^{*j}}^i (S_{\lambda'_d \mu'_d}^{*j})$

is written as $\mathbb{P}_{n, \sigma_{\lambda'_1 \mu'_1}^{*j}}^i (S_{\lambda'_1 \mu'_1}^{*j}, \dots, S_{\lambda'_2 \mu'_2}^{*j})$.

The intersection of $\mathbb{P}_{n, \sigma_{\lambda'_1 \mu'_1}^{*j}}^i (S_{\lambda'_1 \mu'_1}^{*j}, \dots, S_{\lambda'_2 \mu'_2}^{*j})$ with V is

written as $\mathbb{P}_{n, \sigma_{\lambda'_1 \mu'_1}^{*j}}^i (S_{\lambda'_1 \mu'_1}^{*j}, \dots, S_{\lambda'_2 \mu'_2}^{*j})_V$. We show the

existence of maps $\tau_{n, \sigma_{\lambda'_1 \mu'_1}^{*j}}^i (\lambda'_1 \mu'_1, \dots, \lambda'_2 \mu'_2) : \mathbb{P}_{n, \sigma_{\lambda'_1 \mu'_1}^{*j}}^i (S_{\lambda'_1 \mu'_1}^{*j}, \dots, S_{\lambda'_2 \mu'_2}^{*j})$

$\rightarrow \mathbb{P}_{n, \sigma_{\lambda'_1 \mu'_1}^{*j}}^i (S_{\lambda'_1 \mu'_1}^{*j}, \dots, S_{\lambda'_2 \mu'_2}^{*j})_V$ so that $\pi_{i, \delta} \cdot \tau_{n, \sigma_{\lambda'_1 \mu'_1}^{*j}}^i =$

$\pi_{i, \delta}^{-1} \cdot \pi_{i, \delta}$ is valid. Arguments are divided into two cases

cases : (I) the case where series

$S_{\lambda'_1 \mu'_1}^{*j} < \dots < S_{\lambda'_2 \mu'_2}^{*j}$ is composed of strata of

first type only. First define a retraction

map $\tau_{\lambda'_1 \mu'_1, \dots, \lambda'_2 \mu'_2}^{i, \delta} : \mathbb{P}_{n, \sigma_{\lambda'_1 \mu'_1}^{*j}}^i (S_{\lambda'_1 \mu'_1}^{*j}, \dots, S_{\lambda'_2 \mu'_2}^{*j}) \rightarrow \mathbb{P}_{n, \sigma_{\lambda'_1 \mu'_1}^{*j}}^i (S_{\lambda'_1 \mu'_1}^{*j}, \dots, S_{\lambda'_2 \mu'_2}^{*j})$ so that

$\pi_{i, \delta} \cdot \tau_{\lambda'_1 \mu'_1, \dots, \lambda'_2 \mu'_2}^{i, \delta} = \pi_{i, \delta}^{-1} \cdot \pi_{i, \delta}$

$\pi_{i, \delta} \cdot \tau_{\lambda'_1 \mu'_1, \dots, \lambda'_2 \mu'_2}^{i, \delta} = \pi_{i, \delta}^{-1} \cdot \pi_{i, \delta}$ holds. After that define a

retraction map $\tau_{\lambda'_1 \mu'_1, \dots, \lambda'_2 \mu'_2}^{i, \delta} : \mathbb{P}_{n, \sigma_{\lambda'_1 \mu'_1}^{*j}}^i (\lambda'_1 \mu'_1, \dots, \lambda'_2 \mu'_2) (S_{\lambda'_1 \mu'_1}^{*j}, \dots, S_{\lambda'_2 \mu'_2}^{*j})$

$\rightarrow \mathbb{P}_{n, \sigma_{\lambda'_1 \mu'_1}^{*j}}^i (\lambda'_1 \mu'_1, \dots, \lambda'_2 \mu'_2) (S_{\lambda'_1 \mu'_1}^{*j}, \dots, S_{\lambda'_2 \mu'_2}^{*j})$ by the

following equation

$$2 - \frac{1}{3}$$

$$\pi_{\delta-1, \delta} \cdot \mathcal{E}_{\lambda_1' \dots \mu_k'}^{* \delta} = \mathcal{E}_{\lambda_1' \dots \mu_k'}^{* \delta-1}.$$

Define a map $\tau_{\lambda_1' \dots \mu_k'}^{\delta}$ by $\tau_{\lambda_1' \dots \mu_k'}^{\delta}(n, \sigma_{\lambda_1' \mu_1'}^{* \delta}, \dots, \sigma_{\lambda_k' \mu_k'}^{* \delta}) = \tau_{\lambda_1' \dots \mu_k'}^{\delta-1}(n, \sigma_{\lambda_1' \mu_1'}^{* \delta-1}, \dots, \sigma_{\lambda_k' \mu_k'}^{* \delta-1}) \circ \tau_{\lambda_1' \dots \mu_k'}^{\delta}$. Then the commutativity

$$\pi_{\delta-1, \delta} \circ \tau_{\lambda_1' \dots \mu_k'}^{\delta}(n, \sigma_{\lambda_1' \mu_1'}^{* \delta}, \dots, \sigma_{\lambda_k' \mu_k'}^{* \delta}) = \tau_{\lambda_1' \dots \mu_k'}^{\delta-1}(n, \sigma_{\lambda_1' \mu_1'}^{* \delta-1}, \dots, \sigma_{\lambda_k' \mu_k'}^{* \delta-1}) \circ \pi_{\delta-1, \delta}$$

holds.

From the relation (2.1), the map $\tau_{\lambda_1' \dots \mu_k'}^{\delta}(n, \sigma_{\lambda_1' \mu_1'}^{* \delta}, \dots, \sigma_{\lambda_k' \mu_k'}^{* \delta})$

is a retraction of $T_{n, \sigma, \tau}^{\delta}(S_{\lambda_1' \mu_1'}^{* \delta}, \dots, S_{\lambda_k' \mu_k'}^{* \delta})$ onto

$$T_{n, \sigma, \tau}^{\delta}(S_{\lambda_1' \mu_1'}^{* \delta-1}, \dots, S_{\lambda_k' \mu_k'}^{* \delta-1}) \cap \bar{S}_{\lambda_1' \mu_1'}^{* \delta-1}.$$

Thus, in this case,

our assertion (2.1) is shown.

(IV)₂ Next we consider our assertion

in which all the strata $S_{\lambda_1' \mu_1'}^{* \delta} \prec \dots \prec S_{\lambda_k' \mu_k'}^{* \delta}$

in series are of second type. We consider the

case where the stratum $S_{\lambda_1' \mu_1'}^{* \delta}$ has two walls.

Other cases are treatable in a similar way.

Denote by $S_{\lambda_2 \mu_2}^{* \delta-1}$ the projection $\pi_{\delta-1, \delta}(S_{\lambda_2 \mu_2}^{* \delta})$.

Obviously the relation $S_{\lambda_1 \mu_1}^{* \delta-1} \prec S_{\lambda_2 \mu_2}^{* \delta-1}$ holds.

For a point $Q_{1 \dots k}^{* \delta-1} \in T_n(S_{\lambda_1 \mu_1}^{* \delta-1}, \dots, S_{\lambda_k \mu_k}^{* \delta-1})$, $x_{\delta}^{\pm}(Q_{1 \dots k}^{* \delta-1})$

denotes x_{δ} - coefficients of points $Q_{1 \dots k}^{* \delta-1}$ on $S_{\lambda_1' \mu_1'}^{* \delta}$.

so that $\pi_{\delta-1, \delta} (Q_{1-t}^{*j}) = Q_{1-t}^{*j-1}$ holds. Coordinates

$x_{\delta}^{*j}(Q_{1-t}^{*j})$ is holomorphic in the variable

Q_{1-t}^{*j-1} . Define a map $\tau_{\lambda_1 \dots \lambda_t} (n, \sigma(n)) : I \times T_{n, \mu} (S_{\lambda_1 \mu_1}^{*j-1}, \dots, S_{\lambda_t \mu_t}^{*j-1})$

$\rightarrow T_{n, \mu} (S_{\lambda_1 \mu_1}^{*j}, \dots, S_{\lambda_t \mu_t}^{*j})$ by the following requirements

(2.1.39) For a point $R_{1-t}^{*j} \in T_{n, \mu} (S_{\lambda_1 \mu_1}^{*j-1}, \dots, S_{\lambda_t \mu_t}^{*j-1})$,

(2.1.39)₁ $R_{1-t}^{*j}, e = \bigcup_{\lambda_1 \dots \lambda_t} (n, \sigma(n)) (e, R_{1-t}^{*j})$ is

defined by the following conditions.

(2.1.39)_{1.1} $\pi_{\delta-1, \delta} (R_{1-t}^{*j}, e) = Q_{1-t}^{*j-1}, e$

(2.1.39)_{1.2} $\{x_{\mu_j} (R_{1-t}^{*j}) - x_{\mu_j} (Q_{1-t}^{*j})\} : \{x_{\mu_j} (Q_{1-t}^{*j}) - x_{\mu_j} (Q_{1-t}^{*j-1})\}$
 $= \{x_{\mu_j} (R_{1-t}^{*j}) - x_{\mu_j} (Q_{1-t}^{*j})\}$
 $: \{x_{\mu_j} (R_{1-t}^{*j}) - x_{\mu_j} (Q_{1-t}^{*j})\}$

where $Q_{1-t}^{*j}, e = \tau_{\lambda_1 \dots \lambda_t} (e, \pi_{\delta-1, \delta} (R_{1-t}^{*j}))$.

It is clear that the commutativity condition (2.1)

for $\tau_{\lambda_1 \dots \lambda_t}$ is valid. Also from (2.1),

the retraction condition (2.1) is valid.

(IV) Thirdly consider the series of the

mixed type : $S_{\lambda_1 \mu_1}^{*j} < \dots < S_{\lambda_k \mu_k}^{*j} < S_{\lambda_{k+1} \mu_{k+1}}^{*j} < \dots < S_{\lambda_t \mu_t}^{*j}$

Divide this case into further following

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cases : (i) $\pi_{\delta-1,\delta} (S_{\lambda',\mu'}^{*i}) \neq \pi_{\delta-1,\delta} (S_{\lambda'',\mu''}^{*i})$ and (ii)

$$\pi_{\delta-1,\delta} (S_{\lambda',\mu'}^{*i}) = \pi_{\delta-1,\delta} (S_{\lambda'',\mu''}^{*i}) . \quad \text{In the case (i)}$$

define a map $\tau_{n,\sigma,c}^i = \tau_{n,\sigma,c}^i (\lambda'_1, \mu'_1, \dots, \lambda'_k, \mu'_k) : T_{n,\sigma,c}^i = T_n^i$

$(S_{\lambda'_1, \mu'_1}^{*i}, \dots, S_{\lambda'_k, \mu'_k}^{*i}) \rightarrow T_{n,\sigma,c}^i (S_{\lambda'_1, \mu'_1}^{*i}, \dots, S_{\lambda'_k, \mu'_k}^{*i})$ by the following

requirements

$$(2.139)_{2.1} \quad \pi_{\delta-1,\delta} \cdot \tau_{n,\sigma,c}^i = \tau_{n,\sigma,c}^{i-1} \cdot \pi_{\delta-1,\delta}$$

$$(2.139)_{2.2} \quad \text{The coordinates } \mathcal{X}_{\delta} (R_{\lambda',\mu',c}^{\delta})$$

$$: R_{\lambda',\mu',c}^{\delta} = \tau_{n,\sigma,c}^i (\rho, R_{\lambda',\mu',c}^{\delta-1}) \quad \text{and} \quad R_{\lambda',\mu',c}^{\delta} = R_{\lambda'_1, \mu'_1, \dots, \lambda'_k, \mu'_k, c}^{\delta}$$

is defined by

$$(2.139)_{2.3} \quad \{ x_{\delta} (\bar{R}_{\lambda',\mu',c}^{\delta}) - x_{\delta} (\bar{R}_{\rho}^{\delta}) \} : \{ x_{\delta} (\bar{R}_{\rho}^{\delta}) - x_{\delta} (\bar{R}_{\rho}^{\delta-1}) \} = \{ x_{\delta} (R^{\delta}) - x_{\delta} (R^{\delta-1}) \} : \{ x_{\delta} (R^{\delta}) - x_{\delta} (R_{\rho}^{\delta}) \} .$$

Here $R^{\pm\delta}$ is defined by the conditions (i) $\pi_{\delta-1,\delta} (R^{\pm\delta})$

$$= \pi_{\delta-1,\delta} (R^{\pm\delta-1}) = \mathbb{R} (1,1) \quad \bar{R}^{\pm\delta} \quad \text{are in the}$$

boundary of the 'fiber' $T_{n,\sigma,c}^i (R^{\pm\delta})$ and (ii) $x_{\delta} (R^{\pm\delta}) > x_{\delta} (R^{\pm\delta-1})$ hold. Explicit expressions

of $\bar{R}^{\pm\delta}$ are easily given according to the

cases divided in () . Then it is clear

that, for the value $c = 0$, $\tau_{n,\sigma,c}^i (0, T_{n,\sigma,c}^{+i}) = \pi_{n,\sigma}^{-1} (\bar{u}_0, \bar{t}_{n,\sigma})$

$$\cap (\pi_{n,\sigma,c} (S_{\lambda'_1, \mu'_1}^{*i}, \dots, S_{\lambda'_k, \mu'_k}^{*i})) .$$

Because of the relation (2.1.36)₃, the condition
 () is assured. In the second case the
 situation is completely similar: Namely let

$\tau_n^{i-1}(S_{\lambda\mu_1}^{i-1}, \dots, S_{\lambda\mu_k}^{i-1})$ be the retraction map $\tau_n^{i-1}(S_{\lambda\mu_1}^{i-1}, \dots, S_{\lambda\mu_k}^{i-1}) \rightarrow$
 $T_n^{i-1}(S_{\lambda\mu_1}^{i-1}, \dots, S_{\lambda\mu_k}^{i-1})$. Then, from (2.1.36)₃, the map is
 liftable to a map $\tau_n^i(S_{\lambda\mu_1}^i, \dots, S_{\lambda\mu_k}^i)$ satisfying ().

(Observe that the fiber is an open segment).

Therefore the condition (2.1.) is assured.

The above assertion leads to the condition 2.1.

(IV) Finally we show the existence of

simple coverings $\mathcal{U}_n^i(S_{\lambda\mu_1}^i, \dots, S_{\lambda\mu_k}^i)$ and sets of $\mathcal{C}^\infty$ functions $\mathcal{U}_{\lambda_1, \dots, \lambda_k}^{i,n}$

subordinate to $\mathcal{U}_n^i(S_{\lambda\mu_1}^i, \dots, S_{\lambda\mu_k}^i)$. We start with suitable

neighbourhoods $I_{\lambda\mu}^i(S_{\lambda\mu}^i)$ as well as $T_{\lambda\mu}^i(S_{\lambda\mu}^i)$.

Take a series $S_{\lambda\mu_1}^i \leftarrow \dots \leftarrow S_{\lambda\mu_k}^i$. The series $S_{\lambda\mu_1}^i \leftarrow \dots \leftarrow$

$\leftarrow S_{\lambda\mu_k}^i$ means the series of projections of $S_{\lambda\mu_1}^i, \dots, S_{\lambda\mu_k}^i$.

Recall that $T_{\lambda\mu}^i(S_{\lambda\mu_1}^i, \dots, S_{\lambda\mu_k}^i)$ is a fiber space

with base $T_n^{i-1}(S_{\lambda\mu_1}^{i-1}, \dots, S_{\lambda\mu_k}^{i-1})$ and fiber an open

segment (C.f. explicit expressions (2.1.36)₁, (2.1.36)₂, (2.1.36)₃).

For each element $A_{n, \lambda_1, \dots, \lambda_k, \sigma}^{i-1} \in \mathcal{U}(S_{\lambda_1}^{i-1}, \dots, S_{\lambda_k}^{i-1})$, define an

open set $A_{n, \lambda_1, \dots, \lambda_k, \sigma}^i$ by $A_{n, \lambda_1, \dots, \lambda_k, \sigma}^i = \pi_{i,i}^{-1}(A_{n, \lambda_1, \dots, \lambda_k, \sigma}^{i-1}) \cap P_{\sigma, \lambda}^i(S_{\lambda_1}^i, \dots, S_{\lambda_k}^i)$.

It is clear that $\mathcal{U} = \{A_{n, \lambda_1, \dots, \lambda_k, \sigma}^i\}$ is a simple covering of $T_{n, \sigma, c}^i(S_{\lambda_1}^i, \dots, S_{\lambda_k}^i)$.

For each $A_{n, \lambda_1, \dots, \lambda_k, \sigma}^i$ define a c^∞ function $u_{n, \lambda_1, \dots, \lambda_k, \sigma}^i$ by $\pi_{i,i}^*(u_{n, \lambda_1, \dots, \lambda_k, \sigma}^{i-1})$.

Then from the induction hypothesis, we know

that the function $u_{n, \lambda_1, \dots, \lambda_k, \sigma}^i$ has an asymptotic behavior

w.r.t. $\mathcal{F}_{j-1, i}^{-1}(\text{Fron}(S_{\lambda_1}^{i-1}))$. But from (2.1) and

from that the function $u_{n, \lambda_1, \dots, \lambda_k, \sigma}^i$ is considered in $T_{n, \sigma, c}^i(S_{\lambda_1}^i, \dots, S_{\lambda_k}^i)$,

the function $u_{n, \lambda_1, \dots, \lambda_k, \sigma}^i$ has asymptotic behavior w.r.t. $\text{Fron}(S_{\lambda_1}^i)$.

Thus the remained problem is to verify the

condition (2.1). This will be done by

dividing the cases: A common method in every

cases is to integrate a given form in the direction

of the fiber at first and reduce the problem

to the case of $j-1$. (i) In the case

in which the strata in series are composed of

first type only. In this case, for a given

The case in which the series contains at least t

two strata of second kind . $(i \ i)_2$ In the

case in which the strata of second kind appear

in the series . In the case of $(i \ i)_2$ the form $\psi_{\delta}^{*2/}$

and $\psi_{\delta}^{*2/} = \psi_{\delta}^{*2/} d\psi_{\delta}^{*2/}$ have asymptotic behavior w.r.t. $\pi_{\delta-1,\delta}^{-1}(\bar{F}_{\text{ron}}(S_{\lambda,\mu}^{*i-1}))$

Thus the problem is reduced to the case $(i-1)$.

In the second case, we divide our consideration in

$T_{\sigma_{\lambda,\mu}^{*i-1}, \sigma_{\lambda,\mu}^{*i-1}}(S_{\lambda,\mu}^{*i-1})$ and outside of $T_{\sigma_{\lambda,\mu}^{*i-1}, \sigma_{\lambda,\mu}^{*i-1}}(S_{\lambda,\mu}^{*i-1})$. For a

point $Q_{\lambda,\mu}^{*i-1} \in T_{\sigma_{\lambda,\mu}^{*i-1}, \sigma_{\lambda,\mu}^{*i-1}}(S_{\lambda,\mu}^{*i-1})$ the form $\psi_{\delta}^{*2/}$ has asymptotic

behavior w.r.t. $\pi_{\delta-1,\delta}^{-1}(\bar{F}_{\text{ron}}(S_{\lambda,\mu}^{*i-1}))$. On the otherhand, in

$T_{\sigma_{\lambda,\mu}^{*i-1}, \sigma_{\lambda,\mu}^{*i-1}}(S_{\lambda,\mu}^{*i-1})$ the matter is as follows: Recall that

that for a point $Q_{\lambda,\mu}^{*i-1} \in T_{\sigma_{\lambda,\mu}^{*i-1}, \sigma_{\lambda,\mu}^{*i-1}}(S_{\lambda,\mu}^{*i-1})$, $d(Q_{\lambda,\mu}^{*i-1}, \bar{F}_{\text{ron}}(S_{\lambda,\mu}^{*i-1}))$

$= d(Q_{\lambda,\mu}^{*i-1}, \bar{S}_{\lambda,\mu}^{*i-1})$ holds. Moreover for a point

$Q_{\lambda,\mu}^{*i-1}$ in $T_{\sigma_{\lambda,\mu}^{*i-1}, \sigma_{\lambda,\mu}^{*i-1}}(S_{\lambda,\mu}^{*i-1})$, there exists

a unique points $Q_{\lambda,\mu}^{*i-1}$, $Q_{\lambda,\mu}^{*i-1}$ so that the relations

$$(2.1.40) \quad \pi_{\delta-1,\delta}^{-1}(Q_{\lambda,\mu}^{*i-1}) = \pi_{\delta-1,\delta}^{-1}(\bar{Q}_{\lambda,\mu}^{*i-1}) = \pi_{\delta-1,\delta}^{-1}(\bar{Q}_{\lambda,\mu}^{*i-1}).$$

$$(2.1.40)_2 \quad Q_{\lambda,\mu}^{*i-1} \in \bar{T}_{n, \sigma_{\lambda,\mu}^{*i-1}}(S_{\lambda,\mu}^{*i-1}) - T_{n, \sigma_{\lambda,\mu}^{*i-1}}(S_{\lambda,\mu}^{*i-1})$$

hold .

Note that the comparison relation : $d(\theta_{\lambda', \mu'}^{*j}, \frac{1}{S_{\lambda', \mu'}^{*j}}) \sim$

$d(Q_{\lambda', \mu'}^{*j}, Q_{\lambda', \mu'}^{*j})$ is valid . Put $x_i^{j \pm} = x_i(Q_{\lambda', \mu'}^{*j})$

$x_i(Q_{\lambda', \mu'}^{*j})$. After these preparatory considerations

consider the integration $\int_{x_i} y_i^q$. In this case we integrate

the form explicitly as follows .

$$(2.1.41) \quad y_i^{q+1} = \int_{x_i(Q_{\lambda', \mu'}^{*j})} y_i^q : x_i(Q_{\lambda', \mu'}^{*j+1}) = x_i(Q_{\lambda', \mu'}^{*j}) + \frac{1}{2} \{x_i(Q_{\lambda', \mu'}^{*j+1}) - x_i(Q_{\lambda', \mu'}^{*j})\} ,$$

then it is clear that the form y_i^{q+1} has asymptotic

behavior w.r.t. $\pi_{\lambda', \mu'}^{-1}(\text{Fon}(S_{\lambda', \mu'}^{*j}))$ outside $T_{\lambda', \mu'}^j(S_{\lambda', \mu'}^{*j})$. On the

other hand , in the set $T_{\lambda', \mu'}^j(S_{\lambda', \mu'}^{*j})$, the

integration y_i^{q+1} has asymptotic behavior w.r.t. $\bar{S}_{\lambda', \mu'}^{*j+1}$

(Here we speak about the case where $S_{\lambda', \mu'}^{*j}$ has two walls . Other cases are treatable in the same manner in the whole the form y_i^{q+1} and the difference $y_i^q = y_i^{q+1} - dy_i^{q+1}$

has asymptotic behavior w.r.t. $\text{Fon}(S_{\lambda', \mu'}^{*j})$. by the same reason

as in () , the problem is reduced to (i-1) .

() , now the arguments in () $\sim$ ()

leads to the desired fact : Indeed the conditions

(2.1.36) for the suitability shows that we can

choose a directed set $T(\mathcal{R}^*)$ of $\mathcal{R}^*$ so

§ 2.2. Retractions with quantity

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§ 2.2 n.1. Our position here is to give

precise meanings of properties of retraction

maps which we call simply 'quantitative

properties of retraction maps.'

Let $(U, V, P$

be a triple composed of a neighbourhood U ,

a variety V in U and an 'origin' P of V .

In this and later arguments, when we speak of

a retraction map τ of a pair (U, V)

to a point P we are always concerned

with not only with the couple (U, V)

but also with the couple (U, V) endowed with

a pre-stratification $\mathcal{S}^*$ of V (and so a natural stratifi-

cation of U induced from $\mathcal{S}_0^*$) Our problem

will be formulated in terms of a given pre-stratification

of $\mathcal{S}_0^*$ rather than the couple (U, V) itself.

Let $I = [0, 1]$ be an interval in $\mathbb{R}$. Variables in I

will be denoted by ℓ . A continuous map $\tau :$

$\Gamma \times (U, V) \rightarrow (U, V)$ is said to be a retraction of $\mathcal{S}_0^*$ if the following conditions are satisfied.

(2.2.1)₁ $\tau(0, U) = P$, $\tau(1, Q) = Q$, for any $Q \in U$.

(2.2.1)₂ For each stratum $S^* \in \mathcal{S}^*$, an inclusion relation

$$\tau: (I - \epsilon) \times S^* \hookrightarrow S^*$$

is true.

Given a subclosed set W of V and triples (β) of positive numbers, a retraction map τ of $\mathcal{S}^*$

will be called to satisfy $(\beta) - (\beta')$

distance preserving property (w.r.t. W) if the

following inequality is valid.

$$(2.2.2)_1 \quad \beta_1 d(Q, W)^{\beta_2 \beta_3} \leq d(Q_\epsilon, W) \leq \beta'_1 d(Q, W)^{\beta'_2 \beta'_3}, \text{ for any point}$$

$Q \in U$ and $\epsilon \in (0, 1]$.

in the above $Q_\epsilon = \tau(\epsilon, Q)$. In the sequel the

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following abbreviation of the inequality will be used.

$$(2.2.2)' \quad (\beta) d(Q, W) \leq d(Q, W) \leq (\beta) d(Q, W)$$

Given a pair (W_1, W_2) of closed sets of V so

that $W_1 \supsetneq W_2$ holds. we define a notion of

inclusion property in the following manner:

Let $(\sigma), (\sigma')$ be couples. A retraction map τ

of $\mathcal{S}^*$ has $(\sigma) - (\sigma')$ -inclusion property

(w.r.t. (W_1, W_2)) $(\sigma) - (\sigma')$ -i.p. w.r.t. (W_1, W_2)

if the following condition is valid.

$$(2.2.2)_2 \quad \tau : [0, 1] \times N_\sigma(W_1, W_2) \hookrightarrow N_{\sigma'}(W_1, W_2)$$

On the other hand, for a given pair $(\sigma'), (\sigma'')$ of couples,

a map τ is said to have $(\sigma') - (\sigma'')$ -exclusion

property w.r.t. (W_1, W_2) $(\sigma') - (\sigma'')$ -ex.p. w.r.t. (W_1, W_2)

if the following condition is valid.

$$(2.2.2)'_2 \quad \tau : [0, 1] \times \{v \in N_{\sigma'}(W_1, W_2)\} \rightarrow v \in N_{\sigma''}(W_1, W_2)$$

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Examine certain immediate consequence

of the above definitions : Let W_λ ($\lambda = 1, \dots, \lambda_0$)
be a finite set of sub closed sets of V .

We assume the following d.p. properties

$$(\beta^\lambda) \cdot d(Q, W_\lambda) \leq d(Q_e, W_\lambda) \leq (\beta'^\lambda) \cdot d(Q, W_\lambda) \quad (\lambda = 1, \dots, \lambda_0).$$

, for each subset W_λ with suitable couples.

Then a simple observation leads to the following

$$\beta_1 \cdot d(Q, \bigcup_\lambda W_\lambda) \cdot \beta_3 \leq d(Q_e, \bigcup_\lambda W_\lambda) \leq \beta'_1 \cdot d(Q, \bigcup_\lambda W_\lambda) \cdot \beta'_3,$$

where $(\tilde{\beta})$, $(\tilde{\beta}')$ are triples defined to be

$$(\tilde{\beta}) = (\min_\lambda \beta_1^\lambda, \max_\lambda \beta_2^\lambda, \max_\lambda \beta_3^\lambda) \quad (\tilde{\beta}') = (\max_\lambda \beta_1'^\lambda, \min_\lambda \beta_2'^\lambda, \min_\lambda \beta_3'^\lambda).$$

Next we examine elementary relations between the notions

of d.p. and (i.p., ex.p.). We speak relations in terms

of neighbourhoods $\mathbb{E}'_\lambda$. Let τ be a retraction

map of $\mathcal{S}_0^*$, and let (W_1, W_2) be a pair

of subvarieties of V . We assume that τ has

$$(\sigma) \rightarrow (\sigma^1) \quad (\text{resp. } (\sigma) \rightarrow (\sigma'^2)) \quad \text{d.p.}$$

w.r.t. W_1 (resp. W_2). Take a point $Q \in U$

which satisfies the inequality

$$(2.2.3)' \quad d(Q, W_1) \leq h_1 d(Q, W_2)^{h_2}$$

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Moreover, we assume that the map τ has $(\beta) -$

$(\beta') -$ d.p. w.r.t. W_1 (and $(\hat{\beta}_1) -$

$(\hat{\beta}_2) -$ d.p. w.r.t. W_2). Then a simple

computation leads to the following

$$(2.2.3)' \quad d(Q_e, W_1) \leq \beta'_1 \cdot (\hat{\beta}_1)^{-\frac{\beta'_2}{\beta_2} h_2} \cdot h'_1 \cdot d(Q_e, W) \cdot (\frac{\beta'_2}{\beta_2}) h_2 \cdot \rho(\frac{\beta'_2}{\beta_2} \beta_2 \cdot h_2)$$

Thus we know the following fact .

(2.2.3), If the inequalities

$$\frac{\beta'_3}{\beta'_2} > (\frac{\hat{\beta}_3}{\hat{\beta}_2}) \cdot \delta_2, \quad \delta_1^{\beta'_2} \cdot \rho(\frac{\beta'_2}{\beta_2} \beta_2 \cdot h_2) \cdot (\hat{\beta}_1)^{-\frac{\beta'_2}{\beta_2} \delta_2} \leq \frac{1}{2}$$

is valid, then the inclusion relation

$$\tau : (0, 1] \times N_s(W_1, W_2) \hookrightarrow N_{s'}(W_1, W_2)$$

with $s' = (s'_1, s'_2) : s'_1 = \delta_1^{\beta'_2} \cdot \rho(\frac{\beta'_2}{\beta_2} \beta_2 \cdot h_2)$, $s'_2 = (\frac{\hat{\beta}_3}{\hat{\beta}_2}) \cdot \delta_2$ holds .

On the other hand, assume the following

$$(2.2.3)'_2 \quad d(Q, W_2) \leq h'_1 \cdot d(Q, W_1)^{h'_2}$$

Then we know the following

$$(2.2.3)''_2 \quad d(Q_e, W_2) \leq \hat{\beta}'_1 \cdot (\beta_1)^{-\frac{\beta'_2}{\beta_2} h'_2} \cdot h'_1 \cdot d(Q_e, W) \cdot (\frac{\beta'_2}{\beta_2}) h'_2 \cdot \rho(\frac{\beta'_2}{\beta_2} \beta_2 \cdot h_2)$$

The above inequality

Then combining (2.2.) and (1.) we know

(2.2.3)' inequalities $\hat{s}_2'(\hat{\beta}_3'/\hat{\beta}_2') \geq \beta_3'/\beta_2'$, $\hat{\beta}_1' \cdot \hat{\beta}_1' \cdot \hat{s}_4' \leq (\beta_1'/\beta_2')$

implies the following exclusion relation

(2.2.3)' $\tau: I \times \mathcal{P}_\sigma(W_1, W_2) \hookrightarrow \mathcal{P}_{\sigma^2}(W_1, W_2)$
 with $(\sigma^2) = (\sigma_1^2, \sigma_2^2)$ $\sigma_1^2 = (\hat{\beta}_1') \cdot \hat{\beta}_1' \cdot \hat{s}_4'$, $\sigma_2^2 = (\hat{\beta}_2'/\hat{\beta}_1') \cdot \hat{s}_2'$

In connection with the above observations, we shall

remark the following. Concerning the exclusion

property, (2.2)' tells us that, by taking

(σ) sufficiently small, the

exclusion relation is assured (2.2.1). In this sense we

say that the d.p. implies the ex. p. On

the otherhand, if we consider the i.p., the situation

becomes a little subtle: The relation (2.2.) shows

that, if we take (σ) sufficiently large

then the inequality relation is formally valid.

But we point out that the couple (σ) should be

suitably small in order that the set $N_\sigma(S)$ reflects

the property of S . In this sense we regard

that i.p. is not an immediate consequence

of d.p. Further relations among the above notion

will be discussed in later .

Now we shall be concerned with our formulation of our problems : Let U be a

domain in $\mathbb{R}^N$, and let V be an

analytic variety in U . Moreover, let P be

a point P in V . Moreover, let $\mathcal{R}$

be a regular series and let $\mathcal{F}$ be a normalizing

datum of $\mathcal{R}$. Of course we assume that

$(\mathcal{R}, \mathcal{F})$ is attached to V . We assume the

condition (R, C) . Our problem will

be formulated in terms of the series $\mathcal{R}^*$ (c.f. §2.1)

We note that we formulate the i.p. and ex. p.

in terms of the T - neighbourhoods rather than

N - neighbourhoods. A series $\mathcal{T}(\mathcal{R}^*) = \{\tau^i\}_{i=1, \dots, N}$ will

be called a $\mathcal{C}$ -retraction of the series $\mathcal{R}^*$

if the following conditions are valid.

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160 (2.2.4)₀ $\pi_{i,i} \cdot \tau^{i-1} = \tau^i \cdot \pi_{i,i-1}$
 (2.2.4)₁ Maps τ^i are C^∞ differentiable

on each set $(0,1] \times Q^i$; $Q^i \in U^i$

(2.2.4)₂ Maps τ^i are C^∞ differentiable in
 $(0,1] \times U - \mathbb{R}^i$.

(Remark) Because of the existence of

ramification loci, it is not possible to sharpen

differentiability condition (2.2)₂ concerning the ramifications
 in general .

Now our concerns about quantitative properties

of retraction maps come from sources : (i)

d.p with respect to closures of strata S^* .

(i i) Sizes of neighbourhoods $\mathcal{P}_{\sigma}^i$ where

the i.p. and the ex. p. are valid.

(i i i) Quantitative properties of C^∞ differentiable

properties of τ^i . These problems will be

formulated in the following manner.

(i) Let us assume that a series of assignments

($[\beta^i]$) , ($[\beta'^i]$) are given : $[\beta^i] : S_{\lambda\mu}^i \rightarrow \mathcal{P}_{\lambda\mu}^i(S_{\lambda\mu}^i)$,

$[\beta'^i] : S_{\lambda\mu}^i \rightarrow \mathcal{P}_{\lambda\mu}^i(S_{\lambda\mu}^i)$, where (β^i) , (β'^i) are triples

of positive numbers . The collection $\{\beta^i\}_{i=1, \dots, N}, \{\beta'^i\}_{i=1, \dots, N}$ will

be denoted by $[\beta], [\beta']$. A retraction τ of

$\mathcal{R}$ will be called $\frac{([\beta]) - ([\beta']) -$

distance preserving if the map τ^i is $(\beta^i) - (\beta'^i) -$

d.p. for each pre-stratification $\mathcal{S}^i ([\beta]) - ([\beta']) - \text{d.p.}$.

(i i) given a series of couples $\{(\sigma^i)_{i=1, \dots, N}\}, \{(\sigma'^i)_{i=1, \dots, N}\}$

$(\sigma^i), (\sigma'^i)$, where $(\sigma^i), (\sigma'^i), (\sigma^i)$ and (σ'^i)

are couples of positive numbers . the collections

$\{(\sigma^i)_{i=1, \dots, N}\}, \{(\sigma'^i)_{i=1, \dots, N}\}$ will be denoted by $[\sigma], \dots, [\sigma']^2$.

A map τ is $\frac{([\sigma]) - ([\sigma']^1)}{([\sigma]) - ([\sigma']^2)}$ i.p.

or $\frac{([\sigma]) - ([\sigma']^2)}{([\sigma]) - ([\sigma']^1)}$ ex.p. if the map τ

has the properties : τ^i is $(\sigma) - (\sigma')$

i.p. (or $(\sigma) - (\sigma'^2) - \text{ex.p.}$) for each pre-stratification

ation $\mathcal{S}^i$: $(\sigma), (\sigma')$.

(i i i) To state quantitative properties of

$\mathcal{C}^\infty$ -differentiability, we introduce the following

notations . Let $(\mathbb{R}, \mathbb{K}^i)$ be a set of

non negative integers $\in \mathbb{Z} \times \mathbb{Z}^i (i=1, \dots, N)$. Let τ be

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a map of $\mathcal{R}^*$ and let $(\quad)$ be the collection.

will be denoted by $[\gamma]$. A map τ will be called to have $([\gamma])$ differentiable

property if the following is valid.

(2.2.5), For each $D_{K, \alpha}^{j, k}$ and for a point Q

(τ, τ_{α}^j) the following estimation is valid.

$$(2.2.5)_2 \quad |D_{K, \alpha}^{j, k} \tau_{\alpha}^j| \leq \gamma_{K, \alpha}^j \cdot d(Q, D) \cdot e^{-\gamma_{K, \alpha}^j} \cdot e^{-\gamma_{K, \alpha}^j}$$

Using the above notations, our problem will be formulated in the following fashion.

Lemma 2.2.1. (Retractions with quantities)

For a given series $\mathcal{R}^*$ and normalizing

functions τ , there exists a set of assignments $[\beta], [\beta']$,

$[\sigma], [\sigma'], [\sigma''], [\sigma''']$ as well as a series of maps $P = \{P_i, P_j\}$ in

such a manner that the following are true.

(2.2.6), For any series of positive

numbers (τ) of type (cf §1.), there exists a

retraction map $\tau = \tau(\mathcal{R}^*)$ of $\mathcal{R}^*$, so that the map τ

has the following properties.

(2.2.6)₁, τ is of $(\beta) - (\beta')$

d.p.

(2.2.6)₂, τ is of $(\sigma) -$

$(\sigma^1) -$ i.p. and $(\sigma') - (\sigma'^1) -$ ex.p.

(2.2.6)₃, τ is of $(\gamma) - ()$

Remark . Our arguments will be done in such a manner that explicit computations of data $(\beta), \dots, (\gamma)$ are computable in an inductive way, though we do not enter into tedious computations. Key points which enables us to make computations of $(\beta), \dots$ are as follows:
The above lemma is spoken in terms of a fixed point P_0 .

Next we will be concerned with the variance of

the retraction with quantities'

Remakk . In the connection with the basic distance preserving properties (c.f. (2.2.)), we shall add the following : Let $S_{\lambda, \mu}^{*j'}$ be a stratum $\in \mathcal{S}^{*j}$, then , for a retraction map τ^j with properties (2.2.) , it may happen that the d.p. is sharpened in a small neighbourhood of $S_{\lambda, \mu}^{*j'}$:

By this reason , we introduce one another notion or local distance preserving property as follows.

As in (2.2.) , a map τ^j is said to have (β_{loc}) - (β'_{loc}) -local distance property around S^* (or more strictly in $T^j_r(S^*)$) if the inequalities

$$(\beta_{loc}^{(j)}) \cdot d(Q, \bar{S}^*) \leq d(Q, \bar{S}^*) \leq (\beta_{loc}^{(j')}) \cdot d(Q, \bar{S}^*)$$

are valid for a point P in $T^j_r(S^*, T_{loc}(S^*))$.

This notion will be used for a map τ defined locally in a neighbourhood of S^{*j} only :

Our arguments will be done *in a way* by fixing

quantities $(\beta_{loc}^{(j)})$, $(\beta_{loc}^{(j')})$ at first in local situations and after that we make use of the obtained results to yield a ' global ' result .

is assured for j . Take a stratum S^* of the dimension $(i+1)$. If a point Q^* is in $N_i(S^*, \mathcal{F}_m(S^*))$, then we have the following

$$(2.2.7)_1 \quad (\beta_{loc}(S^*)) \circ d(Q, \overline{S}^*) \leq d(Q, \overline{S}^*) \leq (\beta'_{loc}(S^*)) \circ d(Q, \overline{S}^*).$$

Assume that a point Q is not in $N_i(S^*, \mathcal{F}_m(S^*))$. From the induction hypothesis we know the following

$$(2.2.7)_2 \quad \hat{\beta}(S^*) \circ d(Q, \mathcal{F}_m(S^*)) \leq d(Q, \mathcal{F}_m(S^*)) \leq \hat{\beta}'(S^*) \circ d(Q, \mathcal{F}_m(S^*))$$

with suitable triples $\hat{\beta}(S^*)$, $\hat{\beta}'(S^*)$. Combining this inequality with (2.2.) we have

the following

$$(2.2.7)$$

with

This finishes our verification of (1.7) for j

The above observation enables us to concentrate our attention to local distance preserving property in each neighbourhood of S and exclusive property

(as in (), ()).

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In connection with the above remark we shall make the following observation : Let τ be a retraction of $\mathcal{S}$. Moreover, we assume the following properties of the map τ .

(i) For each stratum $S^* \in \mathcal{S}^*$, there exist a couple (δ) , triples $(\beta_{loc}(S^*))$, $(\beta'_{loc}(S^*))$ in such a manner that the map τ has

$(\beta_{loc}) - (\beta'_{loc})$ - d.p. property in $N_S(S^*, \tau_{loc}(S^*))$.

(ii) For each $S^* \in \mathcal{S}^*$, there exist couples (δ') $(\leq (\delta))$ and (δ'^1) in such a way that the map τ has the (δ') - (δ'^1) - ex.p.

Then we know that (d.p.) the map τ has $(\beta) - (\beta')$ - d.p. (in $\mathcal{U}$) for each S with suitable $(\beta)(\beta')$'s.

This is shown quickly by the induction on the dimension of S^* . If $\dim(S) = 0$, then the assumption (i) gives an answer for (d.p.). Assume that (d.p.)

Before we enter into details, we shall make preparations

What we will be concerned here are notational properties :

As neighbourhoods, we take those neighbourhoods $T(S_{\lambda,\mu}^{*i}, \text{Fron}(S_{\lambda,\mu}^{*i}))$

rather than neighbourhoods $T(S_{\lambda,\mu}^{*i}, \text{Fron}(S_{\lambda,\mu}^{*i}))$. This use of neighbourhood^T is for purpose of discussing inductive steps. The symbols

$(\hat{\beta}_{\lambda,\mu}^{*i})$, $(\hat{\beta}_{\lambda,\mu}^{*i})$ are used to express distance relations to

$\text{Fron}(S_{\lambda,\mu}^{*i})$. Then $(\hat{\beta}_{\lambda,\mu}^{*i})$'s and $(\hat{\beta}_{\lambda,\mu}^{*i})$'s are compared in the following manner.

$$(\hat{\beta}_{\lambda,\mu}^{*i}) \quad (\hat{\beta}_{\lambda,\mu}^{*i}) = \left(\hat{\beta}_1^{*i-1} \cdot \ell_1^{*i-1} \cdot \frac{\hat{\beta}_2^{*i-1}}{\ell_2^{*i-1}}, \hat{\beta}_2^{*i-1} / \ell_2^{*i-1}, \hat{\beta}_3^{*i-1} / \ell_3^{*i-1} \right),$$

$$(\hat{\beta}_{\lambda,\mu}^{*i}) = \left(\hat{\beta}_1^{*i-1} \cdot \ell_1^{*i-1} \cdot \hat{\beta}_2^{*i-1} \cdot \ell_2^{*i-1}, \hat{\beta}_2^{*i-1} \cdot \ell_2^{*i-1}, \hat{\beta}_3^{*i-1} \cdot \ell_3^{*i-1} \right).$$

Also in a small neighbourhood of $S_{\lambda,\mu}^{*i}$, the distance to $S_{\lambda,\mu}^{*i}$ will be replaced by $\hat{d}$. This is also for purpose of discussing inductive procedures. Finally to express natures of points in a neighbourhood of $S_{\lambda,\mu}^{*i}$, rules () will be preserved.

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n. 2, Remaining parts of the present section will be devoted to verifications of the lemmas stated in n. 1. Before we enter into details, we shall outline our arguments. Principally our argument will be done by the induction on j . Namely we ask possibilities of lifting a given retraction map τ^j with desired quantities $(\delta), \dots, (\gamma)$. Here we note that, if we ask quantitative conditions then a problem of a liftability becomes subtle that a case where we do not consider quantitative conditions. It seems reasonable, to the author's investigation, to impose certain additional quantitative properties for τ^j in order that the given map τ^j is liftable to a map τ^{j+1} with suitable quantities. Therefore our argument will be done in such a manner that cares are taken of about conditions on $\{(\delta), \dots, (\gamma)\}$ so that τ^j is liftable to a map τ^{j+1} with quantities.

Remaining parts of this section will be devoted

to verifications of assertions in n. 1°.

Our arguments will be done inductively on j .

n. 2.1. We start with local situations

around a stratum $S_{\lambda\mu}^{*j}$. An underlying

datum is a series of retraction maps $\tau^{i'} (i=1, \dots, j-1)$

with quantitative properties: $(\sigma^{i'}) - (\sigma^{i'+1}) -$

inclusion, $(\beta^{i'}) - (\beta^{i'+1}) -$ distance property, and

$(\beta_{loc}^{i'}) - (\beta_{loc}^{i'+1}) -$ local distance property ($i=1, \dots, j-1$).

We assume also $(\gamma^{i'}) -$ differentiable property ($i=1, \dots, j-1$).

Let us consider a map τ^{*j} defined in a set $T_{\hat{\sigma}_{\lambda\mu}^{*j}, \hat{\sigma}_{\lambda\mu}^{*j}}(S_{\lambda\mu}^{*j})$.

$\bigcap_{\substack{\lambda, \mu \\ \hat{\sigma}_{\lambda\mu}^{*j}, \hat{\sigma}_{\lambda\mu}^{*j}}} \{ \bigcap_{\substack{\lambda, \mu \\ \hat{\sigma}_{\lambda\mu}^{*j}, \hat{\sigma}_{\lambda\mu}^{*j}}} T_{\hat{\sigma}_{\lambda\mu}^{*j}, \hat{\sigma}_{\lambda\mu}^{*j}}(S_{\lambda\mu}^{*j}) \} : S_{\lambda\mu}^{*j} \in \mathcal{S}^j(S_{\lambda\mu}^{*j})$. We consider following

conditions for such a map τ^{*j} :

(2.2.)₁ $\pi_{i+1, i} \cdot \tau^{*i} = \tau^{*i+1} \cdot \pi_{i+1, i}$,

(2.2.)₂ For each $S_{\lambda\mu}^{*j}$, τ^{*j} has $(\hat{\sigma}_{\lambda\mu}^{*j}, \hat{\sigma}_{\lambda\mu}^{*j}) -$
 $(\hat{\sigma}_{\lambda\mu}^{*j}, \hat{\sigma}_{\lambda\mu}^{*j+1}) -$ inclusion property.

(2.2.)₃ τ^{*j} has $(\beta_{loc, \hat{\sigma}_{\lambda\mu}^{*j}}^{*j}) - (\beta_{loc, \hat{\sigma}_{\lambda\mu}^{*j+1}}^{*j+1}) -$ local

distance properties with quantities (β_{loc}^{*j}) , (β_{loc}^{*j+1}) .

in each $T_{\hat{\sigma}_{\lambda\mu}^{*j}, \hat{\sigma}_{\lambda\mu}^{*j}}(S_{\lambda\mu}^{*j}, T_{\hat{\sigma}_{\lambda\mu}^{*j}, \hat{\sigma}_{\lambda\mu}^{*j}}(S_{\lambda\mu}^{*j}))$.
 (2.2.)₄ τ^{*j} is C^∞ -differentiable outside $(\partial) \cdot \mathcal{D}^j$

and is estimated in the manner

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$$(2.2) \quad |D_{\mathbf{K}^i}(\alpha^i)| \leq \gamma_{\mathbf{K}^i} d(Q, \mathcal{D})^{-\gamma_{\mathbf{K}^i, 2}} e^{-\gamma_{\mathbf{K}^i, 3}}$$

Moreover, we consider the following

$$(2.2) \quad \text{The } \mathbf{x}_{\bar{\theta}} - \text{coordinate } \alpha^{\bar{\theta}}(\tau^i(e, Q)) \text{ is } C^\infty \text{ and differentiable in } \mathcal{D} \text{ the variable } \mathbf{x}_{\bar{\theta}}.$$

Furthermore, if a point Q is not in $\mathcal{D}$ then

$$(2.2) \quad \frac{d\tilde{\alpha}^i(e, \mathbf{x})}{d\mathbf{x}^i} \geq \gamma_i \cdot d(Q, \mathcal{D}^i)^{\gamma_i^2} e^{\gamma_i^3} \text{ if } Q \notin \mathcal{D}$$

and if Q is in $\mathcal{D}$ then

$$(2.2) \quad \frac{d\tilde{\alpha}^i(e, \mathbf{x})}{d\mathbf{x}^i} \geq \gamma_i \cdot d(Q, \mathcal{D}^i)^{\gamma_i^2} e^{\gamma_i^3}$$

where S is a (uniquely determined) stratum

containing the point Q .

The above conditions correspond to ()

Finally, as a technical condition, we consider

the following

$$(2.2) \quad \text{for each } S_{\lambda\mu}^{**} \in \mathcal{S}(S_{\lambda\mu}^*), \text{ there exists}$$

a couple $(\bar{\sigma}_{\lambda\mu}^*)$ so that, for a point $Q \in \mathcal{P}_{\bar{\sigma}_{\lambda\mu}^*, \bar{\sigma}_{\lambda\mu}^*}$,

the relation

$$\left| \frac{d\tilde{\alpha}_i}{d\mathbf{x}_i} \right| \leq K$$

is valid with a constant K .

If a map , defined in $\bigcup_{\lambda, \mu} T(S_{\lambda, \mu}^{*j}, T_{\text{con}}(S_{\lambda, \mu}^{*j}))$, satisfies

the above cited conditions (2.2) $\sim$ (2.2'),

we say that a map is adequate .

Let Σ be a subset of $T(S_{\lambda, \mu}^{*j}, T_{\text{con}}(S_{\lambda, \mu}^{*j}))$. A map defined τ

in Σ , will be called an extension of τ (in

Σ) if it coincides with τ in

a set of the form $\bigcup_{\lambda, \mu} T(S_{\lambda, \mu}^{*j}, T_{\text{con}}(S_{\lambda, \mu}^{*j})) \cap \Sigma$. Here we

note that $T_{\sigma, \alpha} \subsetneq T_{\sigma, \beta}$ holds in general .

Our problem will be concerned with an extension

of a given map τ . In doing our arguments ,

quantitative problems which we will be concerned a

are of the following two features .

(2.2) differential properties of τ :

(2.) Local distance preserving properties of $\tau_{\lambda\mu}^{*i} \frac{s_{\lambda\mu}^{*i}}{s_{\lambda\mu}^{*i}}$

Our discussion will be divided into two

parts : (1) arguments in a set of

the form $U_{\sigma_{\lambda\mu}, c_{\lambda\mu}}(S_{\lambda,\mu}^{*i})$ and (i i)

arguments in a set of the form $T_{\sigma_{\lambda\mu}, \hat{\sigma}_{\lambda\mu}} - U_{\sigma_{\lambda\mu}, c_{\lambda\mu}}$.

Differentiable properties will be cared in

the first situation while distance relations

will be examined in the second part . Arguments

in both cases are quite elementary . However

discussions will be done to certain details .

N. 2. First we show discuss a problem of

extending the given map τ to a set $U_{\sigma, c}$.

Arguments in this part will be done as follows :

Take a stratum $S_{\lambda,\mu}^{*i} \in \mathcal{S}(S_{\lambda,\mu}^{*i})$ and let $S_{\lambda,\mu}^{*i-1}$ be its projection

We first construct an extension of a map τ

in the set of the form $\Pi_{\sigma_{\lambda,\mu}}^{-1}(T_{\sigma_{\lambda,\mu}}(S_{\lambda,\mu}^{*i-1}, \text{Fron}(S_{\lambda,\mu}^{*i}))$:

First we fix certain notations used here .

Take a point $Q_{\tilde{\lambda}, \tilde{\mu}}^{*i-1}$ in $T_{\tilde{\sigma}_{\tilde{\lambda}, \tilde{\mu}}^{*i-1}}(S_{\tilde{\lambda}, \tilde{\mu}}^{*i-1}, \text{Fron}(S_{\tilde{\lambda}, \tilde{\mu}}^{*i-1}))$.

Then points $Q_{\tilde{\lambda}, \tilde{\mu}}^{*i}$'s lying over $Q_{\tilde{\lambda}, \tilde{\mu}}^{*i-1}$ (in $\bigcup T_{\tilde{\sigma}_{\tilde{\lambda}, \tilde{\mu}}^{*i}}(S_{\tilde{\lambda}, \tilde{\mu}}^{*i}, \text{Fron}(S_{\tilde{\lambda}, \tilde{\mu}}^{*i}))$)

satisfies an inequality of the following form

$$(2.2) \quad |X_g(Q_{\tilde{\lambda}, \tilde{\mu}}^{*i-1}) - X_g(Q_{\tilde{\lambda}, \tilde{\mu}}^{*i-1}, p)| \leq (\tilde{a}_{\tilde{\lambda}, \tilde{\mu}}^{*i-1}) \cdot d(Q_{\tilde{\lambda}, \tilde{\mu}}^{*i-1}, \text{Fron}(S_{\tilde{\lambda}, \tilde{\mu}}^{*i-1}))$$

with suitable positive numbers $(\tilde{a}_{\tilde{\lambda}, \tilde{\mu}}^{*i-1}) = (\tilde{a}_{\tilde{\lambda}, \tilde{\mu}_1}^{*i-1}, \tilde{a}_{\tilde{\lambda}, \tilde{\mu}_2}^{*i-1})$.

from (2.2) we obtain an inequality of the form

$$(2.2) \quad |X_g(Q_{\tilde{\lambda}, \tilde{\mu}}^{*i}, p) - X_g(Q_{\tilde{\lambda}, \tilde{\mu}}^{*i}, p)| \geq \tilde{a}_{\tilde{\lambda}, \tilde{\mu}}^{*i-1} \cdot (\tilde{a}_{\tilde{\lambda}, \tilde{\mu}_1}^{*i-1}) \cdot d(Q_{\tilde{\lambda}, \tilde{\mu}}^{*i-1}, \text{Fron}(S_{\tilde{\lambda}, \tilde{\mu}}^{*i-1}))$$

for a couple $(\tilde{a}_{\tilde{\lambda}, \tilde{\mu}}^{*i-1})$, define a point $Q_{\tilde{\lambda}, \tilde{\mu}}^{*i}(a)$ by $(\tilde{a}_{\tilde{\lambda}, \tilde{\mu}_1}^{*i-1}, \tilde{a}_{\tilde{\lambda}, \tilde{\mu}_2}^{*i-1})$.

$$\pi_{g-1, g}(\tilde{Q}_{\tilde{\lambda}, \tilde{\mu}}^{*i}(a)) = Q_{\tilde{\lambda}, \tilde{\mu}}^{*i-1}, \quad X_g(Q_{\tilde{\lambda}, \tilde{\mu}}^{*i}(a)) = X_g(Q_{\tilde{\lambda}, \tilde{\mu}}^{*i-1}) \pm (a) \cdot h(Q_{\tilde{\lambda}, \tilde{\mu}}^{*i-1}).$$

Here $Q_{\tilde{\lambda}, \tilde{\mu}}^{*i}$'s are all the points (lying over

$Q_{\tilde{\lambda}, \tilde{\mu}}^{*i-1}$) in $\bigcup T_{\tilde{\sigma}_{\tilde{\lambda}, \tilde{\mu}}^{*i}}(S_{\tilde{\lambda}, \tilde{\mu}}^{*i}, \text{Fron}(S_{\tilde{\lambda}, \tilde{\mu}}^{*i}))$ ordered by : $X_g(Q_{\tilde{\lambda}, \tilde{\mu}}^{*i}) > X_g(Q_{\tilde{\lambda}, \tilde{\mu}}^{*i-1})$.

Now we show the existence of functions $\hat{x}_x^i(t, x)$

$t=1, \dots, t_0$, defined in a set Σ_x so that the conditions

(2.2) are valid. Moreover we assume that $\hat{x}_x^i(t, x)$

coincides with the previously given x^0 in the set $\bigcup_{x=1}^t \tilde{\Sigma}_x$. This is shown inductively on t .

Assume that our assertion was shown for $t-1$ (≤ 1),

using functions define $\tilde{v}^t$ by

$$(2) \quad \tilde{v}^t = \frac{d\tilde{x}}{dt} \mathcal{L}(a, \hat{x}) + (1 - \mathcal{L}(a, \hat{x})) \tilde{v}_t$$

$$: \mathcal{L}(a, \hat{x}) \leq \mathcal{L}(a, \hat{x}) + \frac{\mathcal{L}(a, \hat{x}) - \mathcal{L}(a, \hat{x})}{2}$$

$$= \frac{d\tilde{x}}{dt} (\mathcal{L}(a, \hat{x}) + \mathcal{L}(a, \hat{x})) \cdot \tilde{v}_t$$

we regard that this function is defined in $\pi_{i-1}^{-1}(\pi_{i-1}^{-1}(T_{\sigma}^{i-1}(S_{\sigma}^{i-1})))$

Then quantitative properties of , which we make use

are as follows .

(2.2) $\tilde{v}^t$ is C^∞ -~~diff~~ function of x_0 .

(2.2) $\tilde{v}^t$ satisfies estimation (), () with suitable (r) in $\pi_{i-1}^{-1}(\pi_{i-1}^{-1}(T_{\sigma}^{i-1}(S_{\sigma}^{i-1})))$

(2.2) $\tilde{v}^t$ satisfies ()

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is estimated in a manner ~~as~~ with suitable

Define a function

$\hat{x}_t^i(\rho, x)$ by

$$(2.2)_1, \quad \hat{x}_t^i(\rho, x) = x_{t-1}^i \quad (x^i \leq (Q_i(a)) + (a))$$

$$(2.2)_2 \quad \hat{x}_t^i(\rho, x) = \int_{x_0^i(a)}^{x_t^i} v_t^i \cdot dx_t^i$$

then the difference $\hat{x}_t^i - x_t^i$ is independent of x_0^i and

is estimated as $|\hat{x}_t^i - x_t^i| \leq \int_{Q(a)}^{Q(a)+d(a)} |v_t^i - \frac{dx_t^i}{dt}| dx_t^i + \int_{Q(a)}^{Q(a)-} |v_t^i - \frac{dx_t^i}{dt}| dx_t^i$

when taking (a') sufficiently small one can assume

in view of (2.1)

an inequality of the following form

$$(2.2)_3 \quad |\hat{x}_t^i - x_t^i| \leq a_1 \cdot d(Q, F_m(S))^{a_2} \cdot \rho^{a_3}$$

with arbitrary given (a)

Define a function $\hat{x}_t^i$ by $\hat{x}_t^i = \hat{x}_{t-1}^i + (1 - \rho) \hat{x}_t^i$. This function

satisfies conditions (2.2)

because of () :

To be sure it is quite clear that

we check required condition

coincides with $\hat{x}_t^i$ in a set on the

and so satisfies (2.2).

otherhand it is also quite clear that the

derivatives are estimated from above)

a form What is remained is to see that

Note that the above checked facts are valid for any choice of (a')

is estimated from below as in (2.2); $2 - 2 - 2/$

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Because k_t^i is independent of x_i , we have

$$\frac{d\hat{x}_t^i}{dx_i} = \frac{d\hat{x}_t^i}{dx} + \left(\frac{d(1-f_{t,t})}{dx}\right) k_t^i$$

Note that $\frac{d\hat{x}_t^i}{dx_i} = \tilde{v}_t$ is estimated as in (),

so that quantities $(\tilde{v})$'s are chosen independently

from $(\tilde{q}')$. On the otherhand $x_{t,t}$ is obviously

independent of $(\tilde{q}')$. Because $|k_t^i|$ can be

assumed arbitrarily small (w.e.) by taking

$(\tilde{q}')$ sufficiently small, the estimation

(from below) for x is assured.

ext extend a map to the set $\mathcal{L}_{\mathcal{H}_i}^{-1}(T^i(S_{\lambda\mu}^{*ij})) \cup$

this will be done in an entirely same way as above.

for a point Q_0^{*ij} define $Q_{0,e}^{*ij}$ by $\mathcal{X}_i(Q_{0,e}^{*ij}) = c_i \hat{d}(Q_0^{*ij})^{\hat{Q}_i}$.

Because the given map satisfies $(\) - '(\) -$

inclusion property, we know that inequalities

$$\mathcal{X}_i(Q_{0,e}^{*ij}) > \mathcal{X}_i(Q_{0,t}^{*ij}) > \dots \quad \mathcal{X}_i(Q_{0,\mu}^{*ij}) > \mathcal{X}_i(Q_{0,e}^{*ij})$$

and estimation

$$|\mathcal{X}_i(Q_{0,e}^{*ij}) - \mathcal{X}_i(Q_{0,t}^{*ij})|, \quad |\mathcal{X}_i(Q_{0,\mu}^{*ij}) - \mathcal{X}_i(Q_{0,e}^{*ij})| > c_i d(Q_{0,\mu}^{*ij}) \cdot \rho^{(Q_i)}$$

is valid with suitable

clearly $x(Q)$'s are C^∞ differentiable outside

and are estimated in a manner $(\)$

with suitable $(\)$'s.

Define V^{+0} by

$$(2.2). \quad V_0^+ = \frac{\pm \mathcal{X}_i(Q_{0,e}^{*ij}) \mp \mathcal{X}_i(Q_{0,t}^{*ij})}{\pm \mathcal{X}_i(Q_{0,e}^{*ij}) \mp \mathcal{X}_i(Q_{0,t}^{*ij})}$$

Moreover define $\tilde{V}_0^+$ by $\tilde{V}_0^+ = \mathcal{X}_0(Q_0(a), (d)) \frac{d\hat{Q}_i}{d\mathcal{X}_i} + (1 - \mathcal{X}_0(Q_0(a), (d))) \cdot V_0^+$

and $\mathcal{X}_0(Q_0(a), (d)) \frac{d\hat{Q}_i}{d\mathcal{X}_i} + (1 - \mathcal{X}_0(Q_0(a), (d))) \cdot V_0^-$

Then define a function $\mathcal{X}_0^+$ by $\mathcal{X}_0^+ = \int \tilde{V}_0^+ d\mathcal{X}_i$. This function

coincides with $\mathcal{X}_0^+$ in S_t and $\mathcal{X}_0^- = \int \tilde{V}_0^- d\mathcal{X}_i$ and $\mathcal{X}_0^- = \hat{\mathcal{X}}_i$

is estimated as $\int_Q |\tilde{V}_0^+| d\mathcal{X}_i$. As in $(\)$ we obtain a func

tion x so that the conditions $(\)$ 2-2-2

Now it is easy to derive an extension of in
a set U (S) so that conditions are satisfied

This is done inductively on the length of S :

Namely we show the following facts inductively on

(2.2) $_l$ For a set $\bigcup_{\substack{\sigma \in S \\ \sigma \neq \tau}} T_{\sigma, \hat{\sigma}}(S_{\sigma, \hat{\sigma}}^{*+}) \cup \tau$ there exists an τ
extension of τ^i in $\pi_{\sigma, \hat{\sigma}}^{-1}(U_{\sigma, \hat{\sigma}}) \cap U_{\sigma, \hat{\sigma}, c}$ so that $(S_{\sigma, \hat{\sigma}}^{*+}) \leq l$

and () are valid .

(2.2) $\mathcal{X}_\sigma^+(\theta_{\sigma, \hat{\sigma}}^{*+}) = c_1 \hat{d}(\theta_{\sigma, \hat{\sigma}}^{*+})^{c_2}$
we show the above inductively on : If l

= 1, the assertion is simple : Let S be

a stratum of length l . Choose a function $\mathcal{X}$.

by $\mathcal{X} = \mathcal{X}\left(\frac{g(Q; \sigma)}{h(Q; \sigma)}\right)$. Define a function $\hat{\mathcal{X}}$ as

in the step () . To see the induction step

it is sufficient to put $\mathcal{X} = x \hat{\mathcal{X}} + (1-x) \hat{\mathcal{X}}^{i-1}$

Now we summarize the above ~~computations~~ observations :

(2.2.) For an arbitrarily given adequate map τ^i ,
there exists an extension τ in $U_{\sigma, \hat{\sigma}}(S_{\sigma, \hat{\sigma}}^{*+})$
so that (i) τ is adequate and, furthermore,

$$(2.) \quad \mathcal{X}_\sigma(\theta_{\sigma, \hat{\sigma}}) = c_1 \cdot \mathcal{X}_\sigma^{c_2}$$

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No2 we shall continue arguments in a neighbourhood of S .

Our arguments here are also quite elementary : We start with the following situation : Given a series of maps so that these series satisfy quantitative conditions

() ,....., (), assume that a map τ^* in $\mathcal{U}_{\bar{\sigma}, \bar{c}}(S_{\lambda, \mu}^{*j})$

is already given : Namely we assume the existence of a map τ^* , defined in $\mathcal{U}_{\bar{\sigma}, \bar{c}}(S_{\lambda, \mu}^{*j})$, so that τ^* is adequate

and moreover τ^* satisfies the condition

$$() \quad \mathcal{X}_j(\mathcal{Q}_e^{*j}) = c_1 \mathcal{X}_e(\mathcal{Q}_e^{*j})^{c_2}$$

Now we want to extend this map τ in a set $\mathcal{T}_{\bar{\sigma}, \bar{c}}(S_{\lambda, \mu}^{*j})$ so that ' main part ' of τ will be $\mathcal{P}^n$ outside $\mathcal{U}_{\bar{\sigma}, \bar{c}}$.

Here we assume the inequality : $(\bar{c}) > (2c_1, c_2)$.

For a given $\eta > 0$ we define a function $\mathcal{V}(\eta)$ in $\mathcal{T}_{\bar{\sigma}, \bar{c}}(S_{\lambda, \mu}^{*j})$ from (S^{*j}) by

$$\mathcal{V}(\eta) = \mathcal{P}^n \mathcal{X}(\mathcal{Q}_e^{*j}) + \mathcal{V}_t \mathcal{X}$$

This function $\mathcal{V}(\eta)$ has similar properties as were considered

in the previous numero : (i) is $\bar{c}$ -function outside $\mathcal{D}$

and whose derivatives are estimated as $(2, 2)$ with

*
$$\mathcal{X} = \mathcal{X}_0 \left(\frac{\mathcal{X}_j - \mathcal{X}_j(\theta^*)}{a_1 \hat{\mathcal{A}}(\mathcal{Q}^*)^{c_2}} \right) \quad 2-2-25$$

a suitable η (ii) τ is estimated (from below) as ()

with a suitable η . Moreover we note that $N(\tau)$ is

estimated as : $|N(\tau)| \leq K$ in a set ~~of the~~

~~for a~~ (with a suitable σ^1) .

To investigate quite simple properties of such a map we shall make the following observations . Consider

inequalities

$$(2.2.) \quad t \leq \hat{\sigma}_1 d(Q, T_{\text{con}}(S))^{\hat{\sigma}_2}$$

$$(2.2.) \quad t \leq c'_1 d(Q, T_{\text{con}}(S))^{c'_2}$$

Then a simple calculation leads to the following

(2.2.) an inequality

~~implies an inclusion~~
implies the relation $\hat{\sigma}_1^{j_1} \geq \hat{\sigma}_1^j \cdot (c \cdot \hat{\beta}_1^j)^{-\frac{\hat{\sigma}_2^j}{\beta_2^j}}$, $\hat{\sigma}_2^{j_2} \geq \hat{\sigma}_2^{j_1}$, $\eta \geq \hat{\sigma}_2^j \cdot (\frac{\beta_1^j}{\beta_2^j})$

$$t_p = \tau^\eta t \leq \hat{\sigma}_1^j \cdot d(Q_p, T_{\text{con}}(S))^{\hat{\sigma}_2^j}$$

(2.2.)

Also

$$\hat{\sigma}_1^{j_1} \geq c_1^j \cdot (c \cdot \hat{\beta}_1^j)^{-\frac{c_2^j}{\beta_2^j}}$$

Moreover a simple computation shows the following fact .

(2.2.)

an inequality

$$\hat{\sigma}_1^j \geq c_1^j \cdot (\hat{\sigma}_1^j \cdot \hat{\beta}_1^j)^{-\frac{c_2^j}{\beta_2^j}} \cdot \hat{\sigma}_2^{j_1} \cdot c_2$$

$$\hat{\sigma}_2^j \leq \hat{\sigma}_2^{j_1} \cdot c_2^{j_1}$$

implies an inclusion relation

$$\mathcal{U}_{\hat{\sigma}_1^j, c} \subset \mathcal{U}_{\hat{\sigma}_1^j, \sigma^1}$$

Examine simple properties of a map $\tau(\cdot)$.

(2.2.2) From the definition of τ and from ()

we know that an inequality

$$() \quad \sigma^2 \geq c_1^2 (\bar{\sigma}_1^2)^{c_1^2} (c_2^2 \bar{\sigma}_2^2)^{c_2^2} \frac{\bar{\sigma}_2^{c_2^2}}{\bar{\sigma}_1^{c_1^2}}, \quad \frac{\bar{\sigma}_2^{c_2^2}}{\bar{\sigma}_1^{c_1^2}} = \frac{c_2^2}{\bar{\sigma}_1^{c_1^2}} \frac{\bar{\sigma}_2^{c_2^2}}{2}$$

implies the following inclusion relation of $\tau(\cdot)$.

$$(2.2.) \quad \tau(\cdot): \mathcal{P}_{\bar{\sigma}, \hat{\sigma}}(S) \hookrightarrow \mathcal{P}_{\bar{\sigma}', \hat{\sigma}'},$$

with $\frac{\bar{\sigma}^2}{\hat{\sigma}^2} = \left(\frac{c_1^2 (\bar{\sigma}_1^2)^{c_1^2}}{c_2^2 (\bar{\sigma}_2^2)^{c_2^2}} \right)^{\frac{c_1^2}{c_2^2}}$

Assume the inequality (2.2.) and so the inclusion

relation (2.2.) . Then we obtain the following distance

relations to S .

$$(2.2.)_1 \quad \text{For } R \in \mathcal{P}_{\bar{\sigma}, \hat{\sigma}}(S),$$

$$\chi_i(R_e) \leq \chi_i(R) e^{\frac{1}{2}} + c_1 \chi_p^{c_2}$$

$$(2.2.)_{2,1} \quad \text{For } R \notin \mathcal{P}_{\bar{\sigma}, \hat{\sigma}}(S) - \mathcal{U}_{\bar{\sigma}, \hat{\sigma}}(S),$$

$$\chi_i(R_e) \geq \frac{1}{2} \cdot e^{\frac{1}{2}}$$

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$$(2.2.)_{2,1} \quad \text{For } R \in \mathcal{U}_{\bar{\sigma}, \hat{\sigma}}(S), \text{ we use the following obvious relation}$$

$$\chi_i(R_e) + \chi_i(R_e) \geq \chi_i(R_e)$$

From the aboves we know easily the following distance

preserving properties

$$(2.2) \quad \bar{P}_1(\bar{c}_1) \times d \cdot e^{\bar{P}_2/\bar{c}_2} \leq \hat{d}(P, S) \leq 2 \bar{c}_1 \bar{c}_2 \cdot \hat{d}(P, S) \cdot e^{\bar{P}_2/\bar{c}_2} \cdot \bar{P}_3 \bar{c}_2 \gamma$$

From the above elementary observations we remark that

$$(\bar{\sigma}_1^i | \bar{c}(\hat{\beta}_1^i) | \bar{\sigma}_2^i / \bar{\beta}_2^i) \text{ defined by is small enough only}$$

if (there exist certain relations between , so

far as we use . Consider following expressions ;

Then substituting (), () we obtain

recursive relations

$$\left(\bar{\sigma}_1^i | \bar{c}(\hat{\beta}_1^i) | \bar{\sigma}_2^i / \bar{\beta}_2^i \right) = c_1^i \cdot (\bar{\sigma}_1^i) \cdot (\bar{c}(\hat{\beta}_1^i)) \cdot \bar{\sigma}_2^i / \bar{\beta}_2^i, \quad \bar{\sigma}_2^i / \bar{\beta}_2^i = \bar{c}_2^i \cdot \bar{\beta}_2^i \cdot \bar{\sigma}_2^i / (\bar{\beta}_2^i)^2$$

$$\times (b_1^{-1/2}) \cdot \bar{\sigma}_2^i \cdot c_1^i \cdot \bar{\beta}_2^i$$

Being suggested from the above relation , we say that

is satisfies $(\bar{\sigma}_1^i, \bar{\sigma}_2^i, \bar{\beta}_2^i) - (\mathcal{E}_1)$, $(\bar{\sigma}_3^i) - (\mathcal{E}_2)$ condition

if the following additional relations are valid .

$$\left(\bar{\sigma}_1^i | \bar{c}(\hat{\beta}_1^i) | \bar{\sigma}_2^i / \bar{\beta}_2^i \right) < \mathcal{E}_2, \quad \bar{\sigma}_2^i \equiv (\hat{\beta}_2^i)^{m_3} \cdot \mathcal{E}_2^{-1}$$

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80 far as $(c_1, c_2) > (c^0)$

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From (2.), (2.) we derive statements below easily

() For an arbitrarily given $(\hat{G}^{i_1}), (m_1^i, m_2^i, n_1^i)$ choosing

$(m_1, m_2, \varepsilon_1), (m_3, \varepsilon_2)$ sufficiently small enables us to

assure the existence of a map τ^i so that the inclusion

relation holds with suitable (σ) .

Moreover one can assume that satisfies $(n_1 - n_2) - (\varepsilon)$

~~condition~~

$(m_3) - (\varepsilon_2) - \text{condition}$

n. From the above observations our assertions in n.1

follow quite easily : Actually we show the lemma 2.2.1

in the following manner (: We consider additional conditions

for series in question from technical reasons .

() A series of maps is said to satisfy

() - () - conditions if inequalities () - ()

are valid for any stratum of $\mathcal{S}^*$ besides conditions given in

n^o. 1 .

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We show the above assertion inductively on .

Because , for $\ell = 1$, no new conditions imposed by means of

()- () conditions we assume that () was shown for

() . We first show the ex in a neighbourhood of .

This part will be sh argued in the following way . We show the following

() For each l and a any () , () there exists a map , defined in a set , so that is adequate and (ii) satisfies ()- () condition . Moreover one can assume a further condition ()

This is easily checked to be true : If $\ell = 1$, it suffices to define a map by : $\chi_{\sigma} = \ell^q \cdot \chi^j$, where (σ_1, σ_2) is small enough . On the otherhand , for $=$, statements (2.2) *and ℓ is large enough* (2.2) suffices for our purpose .

In order to extend a map in a neighbourhood of to a whole space , there is no difficulty in arguments if we follow procedure in n. *2-2-3* . this finishes our *good* inductive step .

§ 9.1. Cohomology with algebraic growth condition. 187

3.1 n.1. Let $\mathbb{C}^N$ be an N - dimensional complex

Euclidean space with coordinates $(x_1, \dots, x_N)$

and let U be a bounded domain in $\mathbb{C}^N$. coordinates

of $\mathbb{C}^N$ will be denoted by $(w_1, \dots, w_n)$. The

product $\mathbb{C}^N \times U$ will be denoted by $\Sigma_{n,N}$. In

the sequel we simplify, without essential loss of generality, U to be a product of rectangles.

$U = \prod_{i=1}^k [a_i, b_i] \times \prod_{j=1}^n [c_j, d_j]$. We assume that $n_{i,j}(\bar{a}_i, \bar{c}_j, \bar{b}_i, \bar{d}_j)$ is smaller than

1. we will be at first concerned with the

structure sheaf $\mathcal{O}$ over $\mathbb{C}^N \times \mathbb{C}^n$. We note that when

we speak of a neighbourhood of a point Q in $\Sigma_{n,N}$,

we are concerned with a set in $\mathbb{C}^N \times \mathbb{C}^n$ rather than

its restriction to $\Sigma_{n,N}$. For a point $Q \in \Sigma_{n,N}$

we mean by $\mathcal{U}^s(\Sigma_{n,N})$ the open set $\{Q' : |x_i(Q') - x_i(Q)| < s \cdot (s_i(Q) + 1)^{-s_i}$,

For a fixed couple $(s) = (s_1, \dots, s_n)$ we mean by

$\hat{\mathcal{U}}^s(\Sigma_{n,N})$ the open covering of $\Sigma_{n,N}$ defined to be $\{\hat{\mathcal{U}}^s(Q) : Q \in \Sigma_{n,N}\}$

This open covering is not locally finite. But it is

found to be commodiate to formulate our problem in

terms of such covering for our application.

A q - cochain will be said to be of

algebraic growth $(d) = (\alpha_1, \alpha_2)$ if the following estimation is valid .

$$(A, G) \quad |y_{\alpha_1, \dots, \alpha_k}^{(z, w)}| \leq \alpha_1 (|z| + 1)^{\alpha_2}$$

Our first concern will be to discuss a type of vanishing theorems for such cochains :

Namely we show the following lemma, which we

call a vanishing theorem with algebraic growth condition (We simplify to call v.a.) .

Lemma 3.1.1. (V. A)

There exists a datum (P_1, P_2, L)

with which the following facts are valid .

$(V, A, I)_1$ The datum (d, c, P, P_k) is depending on (N, n, q) only .

$(V, A, I)_2$ For a given cocycle $\psi^i \in C^i(N(\mathcal{X}(\bar{z}_n)))$ there exists a $(q-1)$ -cochain so that the equation

$$\tau^* \psi(\bigwedge_{v=1}^q \alpha_{I_v K_v}^{s'}) = \delta_{\text{cech}}(\psi^{i-1})(\bigwedge_{v=1}^q \alpha_{I_v J_v K_v}^{s'})$$

is valid with a suitable refinement map τ^* so far as $\alpha_{I_v J_v K_v}^{s'} \in \mathcal{C}^n \cup \emptyset$ ($v=1, \dots, q$).

$(V, A, I)_3$ Quantities $(\mathcal{Q}'_1, \mathcal{Q}'_2)$ are given

by

$$\mathcal{Q}'_1 = \min_v (\tilde{a}_v, \tilde{b}_v)^{-c} P_1(a, \delta_1) e^{P_2(a_2, \delta_2)}, \quad \mathcal{Q}'_2 = L(a_2 + \delta_2)$$

Remark . In the above statement , the equality

$(V, A, I)_2$ is understood so that the right hand side is understood to be a restriction of to by taking a suitable refinement . We note that we can assume (δ') is smaller than (δ) . In the sequel we understand the meaning of coboundary relations as is understood here .

Before we enter into details of proof, we shall

fix certain notations. Let $\mathbb{C}^l$ be a complex

Eukclidean space. We say that a geometric figure $\square$

is of elementary type if $\square$ is a product of

rectangulars : $\square = \prod_{j=1}^l \square_{a_j, b_j}$. Unless we do not

mention otherwise, we assume that $m(\bar{a}_j, \bar{b}_j)$ is smaller

than 1. Consider a figure $\sum_{i=1}^N \square_i$, where $\square_i$

is a figure of elementary type. We shall introduce

certain notations used here : Henceforth we

assume that (δ_1, δ_2) is always a couple of

positive integers in this numero : n. 1. By P_I ,

we mean the point in $\mathbb{C}^N$ with coordinates $(i_1^1, i_1^2, \dots,$

$i_N^1, i_N^2)$. On the otherhand we mean by P_{IJ}^s

the point in $\mathbb{C}$ with coordinate $(x_{P_I}) + \frac{J}{\delta_1(I+1)\delta_2}$,

$(\delta_1 J \leq \delta_1(I+1)\delta_2)$. Furthermore we mean by P_{IJK}^s the point

in $\mathbb{C}^N \times \mathbb{C}^n$ with coordinates $(x(P_{IJ}^s)) \times \left\{ \left(\frac{x_{P_I} \times \delta_1^J}{\delta_1(I+1)\delta_2}, \frac{\tilde{x}_{P_I} \times \delta_1^J}{\delta_1(I+1)\delta_2} \right) : 0 \leq K \leq \delta_1(I+1)\delta_2 \right\}$.

By $\square^s(P_{IJK}^s)$ we mean the neighbourhood of P_{IJK}^s

defined to be $\{Q : |x_{P_I}(Q) - x_{P_I}(P_{IJK}^s)| < \frac{1}{\delta_1(I+1)\delta_2} (J=1, \dots, N), |w_{P_I}(Q) - w_{P_I}(P_{IJK}^s)| < \frac{\tilde{x}_{P_I}}{\delta_1(I+1)\delta_2}\}$.

Moreover we mean by $\tilde{\mathcal{U}}^s(\bar{z}_{n,N})$ the open covering of

defined to be $\tilde{\mathcal{U}}^s(\bar{z}_{n,N}) = \{ \square^s(P_{IJK}^s), I \in \mathbb{Z}^{2N}, 0 \leq J, K \leq \delta_1(I+1)\delta_2 \}$.

Here x_{P_I} 's are real and imaginary parts of x .

Our formulation of lemma 3.1.1. is found to be suitable for our later discussions (cf. § 4. Our formulation of lemma 3.1.1. is found to be quite suitable to discuss theories of differentiable forms). However, in our practical argument, it is more suitable to work with $\tilde{\partial}^s$ than $\hat{\partial}^s$.

A cochain $\tilde{\gamma}^i \in N(\tilde{\mathcal{K}}_{n,n}^i)$ is, in a similar way to (A.G) of algebraic growth (∂_1, ∂_2) if $\tilde{\mathcal{K}}^q$ is estimated in the manner (A.G). Then we show the following lemma 3.1.1' which is completely similar to lemma 3.1.1.

Lemma 3.1.1'. There exists a datum

$\{\mathcal{L}, \epsilon, \mathcal{P}_1, \mathcal{P}_2, \mathcal{L}\}$ depending on (N, n, i) only so that

the following conditions are valid.

(V.A.I) For a given cocycle

of algebraic growth (∂_1, ∂_2) , there exists

a $(q-1)$ -cochain $\gamma \in \mathcal{C}_{n,n}^{q-1}$ so that

the equation

$$(\quad) \quad \gamma^x \gamma_{\substack{\partial_1, \partial_2 \\ \partial_1, \partial_2}}^{\partial_1, \partial_2} = \text{Scch}(\gamma^{\partial_1, \partial_2}) \left(\prod_{v=1}^{\partial_1, \partial_2} \mathcal{P}_v \right)$$

as well as relations $(\partial_1', \partial_2') = \left(\min(\partial_1, \partial_2) - \epsilon, \mathcal{P}_1(\partial_1, \partial_2) e^{\mathcal{P}_2(\partial_1, \partial_2)}, \mathcal{L}(\partial_1, \partial_2) \right)$

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it is easy to see that lemma 3.1.1. implies lemma 3.1.1 .

Before we enter into detailed arguments of the above lemma, we shall give an outline of our proof.

Outline of proof.

As in the case of the standard vanishing theorem on Stein variety, our discussion, where quantitative properties enter into, will be done in two steps: (I) One dimensional case where examinations of Cousin integrals and a method of an approximation play central rolls. (II) General cases where inductive devices from lower dimensional cases to higher dimensional cases play key rolls. In both cases quantitative arguments are important, though arguments themselves are completely elementary. We shall discuss both cases in details for the sake of completeness and also for the purpose of such a basic fact as the vanishing theorem on Stein variety through our problem.

(I) Let D be a domain in $\mathbb{C}^n$. We do not assume the boundedness of D . By $\mathcal{O}^{\delta, \varepsilon}$, we mean the open covering

of $D \times \mathbb{C}^{N(N-1)}$ composed of all the open sets of the form $D \times \square_{i_1, i_2}^{\varepsilon}$

$$\square_{i_1, i_2}^{\varepsilon} = \{x : |x_1 - x_1(i_1)|, |x_2 - x_2(i_2)| < \frac{1}{2} + \varepsilon\} : (i_1, i_2) \in \mathbb{Z} \times \mathbb{Z}.$$

Given a 1-cocycle $\gamma \in C^1(N(\mathcal{A}), \mathbb{C})$ estimated as

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$$(3.1.1) \quad |\gamma'_{i_1, i_2}(z, w)| \leq \alpha_1 \cdot \{ |z|^{\alpha_2} + |w|^{\alpha_2} + 1 \}$$

We show the existence of absolute independent constants*

$c_0, c_1, \dots, c_6$, with which the following statements are valid.

(3.1.2) There exists a $p-1$ cochain $\{\gamma^0_{i_1, i_2}\} \in C^0(N(\mathcal{A}), \mathbb{C})$ so that the equation

$$(3.1.2)_1 \quad \hat{s}_{\text{cch}}(\gamma^0_{i_1, i_2}) = \gamma'$$

and an estimation of the following forms

$$(3.1.2)_2 \quad |\gamma^0_{i_1, i_2}(w, z)| \leq \varepsilon^{-c_1} \cdot c_2^{\alpha_2 + c_4} \cdot \alpha_1 \cdot \{ |z|^{\alpha_2 + c_4} + |w|^{\alpha_2 + c_4} + 1 \}$$

are valid.

Our proof is of a completely elementary nature and will be given in two steps: (i) First we show the existence of constants $c'_0, \dots, c'_6$ with which the following assertions, which is weaker than (3.1.2),

are valid:

(3.1.2)' There exist absolutely independent constants $c'_0, \dots, c'_6$ in such a way that the following (3.1.2)' holds.

* Here we mean by 'absolutely independent' the fact

that datum is depending on the dimension of $\mathcal{C} = 1$

(3.1.2)' There exists a 0-cochain $\{\gamma_{i_1 i_2}^0\} \in \mathcal{L}^0(N(\pi^{i_0}), \theta)$ with

which the relation

$$(3.1.2)'_1 \quad \gamma^1 = \sum_{\text{cell}} (\gamma_{i_1 i_2}^0) \quad \text{is zero on}$$

the intersections $\square_{i_1 i_2}^1$ of the form $\square_{i_1 i_2}^1 = \square_{i_1 i_2} \cap \square_{i_1 i_2}$.

as well as the estimations of the following forms

$$(3.1.2)'_2 \quad |\gamma_{i_1 i_2}^0| \leq \varepsilon^{-\delta'} c_1^{i_1 d_1 + \delta'} \cdot d_1' \cdot (|z|^{c_2 d_2 + \delta'} + |w|^{c_3 d_3 + \delta'} + 1)$$

are valid.

Let P_0 be the point with coordinates $(0, i_2, w)$

We let $\Psi_{i_1 i_2}^2$ be the Cousin integrals defined by

$$(3.1.3) \quad \Psi_{i_1 i_2}^1 = \left(\frac{1}{2\pi i}\right) \int_{\gamma_{i_1 i_2}^1} \gamma_{i_1 i_2}^1 (z-z)^{-1} d\zeta, \quad \Psi_{i_1 i_2}^2 = \left(\frac{1}{2\pi i}\right) \int_{\gamma_{i_1 i_2}^2} \gamma_{i_1 i_2}^2 (z-z)^{-1} d\zeta$$

, where $\gamma_{i_1 i_2}^j$ ($j=1,2$) are characterized by the requirements $\bigcup_{j=1}^2 \gamma_{i_1 i_2}^j = \partial(\square_{i_1 i_2}^E)$, and $z \in \gamma_{i_1 i_2}^1$ or $z \in \gamma_{i_1 i_2}^2$ according to whether $\operatorname{Re}(z) \leq i_1 + 1/2$ or $\operatorname{Re}(z) \geq i_1 + 1/2$. For the value $i \geq 0$, we expand P in the following form: $\Psi_{i_1 i_2}^1 = \sum_{j=0}^{\infty} \Psi_{i_1 i_2, j}^1(w) z^j$ where

$$\Psi_{i_1 i_2, j}^1(w) = (-1)^j \left(\frac{1}{2\pi i}\right) \int_{\gamma_{i_1 i_2}^1} \gamma_{i_1 i_2}^1 \zeta^{-(j+1)} d\zeta$$

is holomorphic in w .

Define a polynomial

$\Psi_{i_1, i_2}(P)$ in z by

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$$(3.1.3)_1 \quad \Psi_{i_1, i_2}(P) = \sum_{j=0}^{d_2+2} \Psi_{i_1, i_2, j}(w) \cdot (z - i_2)^j$$

Also define a holomorphic function $\Psi_{i_1, i_2}(\infty) =$ infinite part of Ψ_{i_1, i_2}

by

$$(3.1.3)_2 \quad \Psi_{i_1, i_2}(\infty) = \Psi_{i_1, i_2} - \Psi_{i_1, i_2}(P).$$

Note that the above function $\Psi_{i_1, i_2}(\infty)$ is considered in the set

$$\mathcal{D}' \times \{z : \operatorname{Re}(z) \leq i_2 + \frac{1}{2} + \varepsilon, i_2 - \frac{1}{2} - \varepsilon < \operatorname{Im}(z) < i_2 + \frac{1}{2} + \varepsilon\}$$

Also note that the function $\Psi_{i_1, i_2}(\infty)$ coincides with the

following power series

$$(3.1.3)_2' \quad \sum_{j=d_2+3}^{\infty} \Psi_{i_1, i_2, j}(w) \cdot (z - i_2)^j$$

in the set $\mathcal{D}' \times \{z : \operatorname{Re}(z) < i_2, i_2 - \frac{1}{2} - \varepsilon < \operatorname{Im}(z) < i_2 + \frac{1}{2} + \varepsilon\}$.

From the explicit expressions of $\Psi_{i_1, i_2}(w)$, $\Psi_{i_1, i_2}(P)$ and $\Psi_{i_1, i_2}(\infty)$

we know easily that following estimations are valid for functions

$$\Psi_{i_1, i_2, j}(w), \Psi_{i_1, i_2}, \Psi_{i_1, i_2}(P) \text{ and } \Psi_{i_1, i_2}(\infty).$$

$$(3.1.4)_1 \quad |\Psi_{i_1, i_2, j}(w)| \leq 4 \cdot 4^{d_2+1} \cdot d_1 \cdot \{ |w|^{d_2} + |i_1|^{d_2} + |i_2|^{d_2} + 1 \} \cdot (i_1 \neq \frac{1}{2})^{-j-1}$$

$$(i_1 \geq 0)$$

$$(3.1.4)_2 \quad |\Psi_{i_1, i_2}^{\pm}(w, z)| \leq 4 \cdot 4^{d_2+1} \cdot d_1 \cdot \{ |w|^{d_2} + |i_1|^{d_2} + |i_2|^{d_2} + 1 \},$$

In the second inequality (3.1.4.) , we consider

the function $\Psi_{i_1, i_2}^1(w, z)$ in the set where z satisfies the

condition : $\operatorname{Re} z \leq i_1 + \frac{1}{2} + \varepsilon, \quad |\operatorname{Im} z - k_{i_1}| \leq \frac{1}{2} + \varepsilon.$

Define functions $\sigma_{i_1, i_2}^0(i_1 \geq 0)$ by the following equations

$$(3.1.5) \quad \sigma_{i_1, i_2}^0 = \sum_{0 \leq i_1' \leq i_1 - 1} \Psi_{i_1', i_2}^2 - \sum_{0 \leq i_1' \leq i_1 - 1} \Psi_{i_1', i_2}^1(R) + \sum_{i_1 \leq i_1'} \Psi_{i_1', i_2}^1(\infty) \quad (i_2 \geq 0)$$

We note that these functions are considered in a set

where z satisfies the condition: $\{ z : |\operatorname{Re} z - i_1|, |\operatorname{Im} z - i_2| < \frac{1}{2} + \varepsilon \}$.

Examine properties of these functions σ_{i_1, i_2}^0 's .

At first it is clear that σ ' coboundary condition '

$$(3.1.6)_1 \quad \sigma_{i_1+1, i_2}^0 - \sigma_{i_1, i_2}^0 = \sigma_{(i_1, i_2), (i_1+1, i_2)}^1$$

holds .

On the otherhand , we obtain the following

estimations of σ_{i_1, i_2}^0 ' s in view of (3.1.4)_{1,2} .

$$(3.1.6)_2 \quad | \sigma_{i_1, i_2}^0 | \leq \varepsilon^{-\frac{1}{2}} c_1^{i_1 d_1 + l_1'} c_2^{i_2 d_2 + l_2'} \{ |w|^{l_3 d_3 + l_3'} + |i_2|^{l_4 d_4 + l_4'} + |i_1|^{l_5 d_5 + l_5'} + 1 \}$$

with absolutely independent constants $(c_1', \dots, c_5')$.

Repeat an entirely same procedure for $i_1 \leq 0$. Then we obtain

functions σ_{i_1, i_2}^{0-} for $i_1 \leq 0$, in such a manner that the equations

$$(3.1.6)'_1 \quad \sigma_{i_1, i_2}^{0-} - \sigma_{i_1-1, i_2}^{0-} = \sigma_{i_1, i_1-1, i_2}^{0-} \quad (i_1 \leq 0)$$

as well as the estimations of the forms

$$(3.1.6)'_2 \quad |\sigma_{i_1, i_2}^{0-}| \leq \mu_1^{c_{i_1}^{0-} + c_{i_2}^{0-}} \cdot (2)^{c_{i_1}^{0-}} \cdot |w|^{c_{i_1}^{0-} + c_{i_2}^{0-}} + \mu_1^{c_{i_1}^{0-} + c_{i_2}^{0-}} \cdot (2)^{c_{i_1}^{0-}} \cdot |w|^{c_{i_1}^{0-} + c_{i_2}^{0-} + 1}$$

are valid. (4)

We let $\sigma_{i_2}^0$ be the function defined to be $\sigma_{0, i_2}^+ - \sigma_{0, i_2}^-$

in $D \times \{z: -\frac{1}{2} - \frac{\varepsilon}{4} \leq \operatorname{Re} z \leq \frac{1}{2} + \frac{\varepsilon}{4}\}$. Define functions $\sigma_{i_1, i_2}^+ (i_1 \geq 0)$ and $\sigma_{i_1, i_2}^- (i_1 \leq 0)$

$$i_2 - \frac{1}{4} - \frac{\varepsilon}{4} \leq \operatorname{Im} z \leq i_2 + \frac{1}{4} + \frac{\varepsilon}{4}$$

in $D \times \{z: -\frac{1}{2} - \frac{\varepsilon}{4} - i_1 \leq \operatorname{Re} z \leq \frac{1}{2} + \frac{\varepsilon}{4} + i_1, -\frac{1}{2} - \frac{\varepsilon}{4} - i_2 \leq \operatorname{Im} z \leq \frac{1}{2} + \frac{\varepsilon}{4} + i_2\}$ respectively by

$$\text{Cousin integrals } \int_{\gamma_0^+} \sigma_{i_1, i_2}^0 (z-z')^{-1} dz' \quad \text{and} \quad \int_{\gamma_0^-} \sigma_{i_1, i_2}^0 (z-z')^{-1} dz' : \gamma_0^\pm = \partial(\square_{i_1, i_2}^\pm) \cap \{z: \operatorname{Re} z = 0\}$$

Define finally functions σ_{i_1, i_2} by $\sigma_{i_1, i_2} = \sigma_{i_1, i_2}^+ + \sigma_{i_1, i_2}^-$

$(i_1 = 0, \pm 1, \dots)$ in $D \times \{z: -\frac{1}{2} - \frac{\varepsilon}{4} - i_1 \leq \operatorname{Re} z \leq \frac{1}{2} + \frac{\varepsilon}{4} + i_1, -\frac{1}{2} - \frac{\varepsilon}{4} - i_2 \leq \operatorname{Im} z \leq \frac{1}{2} + \frac{\varepsilon}{4} + i_2\}$
Then, from (3.1.5), (3.1.6), (3.1.6)', (3.1.6)'₁, (3.1.6)'₂ and from an estimation

of σ_{0, i_2}^+ obtained from the above, our assertions (3.1.2)₁, (3.1.2)₂

are assured.

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(ii) Next let us consider the following situation. We let $\square_{i_2, \varepsilon'} =$

$\{z: i_2 - (1/2) - \varepsilon' \leq \operatorname{Im} z \leq i_2 + (1/2) + \varepsilon'\}$. Let $\tilde{\mathcal{U}}_\varepsilon$ be the covering of $D'_n \times \mathbb{C}$ composed of sets $\{D'_n \times \square_{i_2, \varepsilon'} : i_2 \in \mathbb{Z}\}$. Take a cocycle $\tilde{\varphi}^1 \in C^1(N(\tilde{\mathcal{U}}_\varepsilon), \mathcal{O})$

which is estimated as follows..

$$(3.37)_1 \quad |\tilde{\varphi}^1| \leq \tilde{\alpha}_1'' \cdot (|z| + |w| + 1)$$

Now we show, in an analogous way to $(\mathbb{I})_1$ the existence of $\tilde{\omega}^0 \in C^0(N(\tilde{\mathcal{U}}_\varepsilon), \mathcal{O})$ satisfying the following conditions.

$$(3.38)_1 \quad \hat{\partial} \tilde{\omega}^0 = \tilde{\gamma}^1 \text{ and } \varphi^{10} \text{ is defined in } \square_{i_2, \varepsilon/2}$$

$$(3.38)_2 \quad |\tilde{\gamma}^1| \leq c_1'' \cdot c_2'' \cdot c_3'' \cdot c_4'' \cdot \tilde{\alpha}_1'' \cdot (|z| + |w| + 1) \quad \text{with absolutely independent constants } c_1'', c_2'', c_3'', c_4'' \text{ and } c_5''.$$

We verify the above assertions $(3.38)_{1,2}$ as follows: We let S_{pi_2} be the geometric figure defined by $S_{pi_2} = \{z: -1/2 - p - \varepsilon' \leq \operatorname{Re} z \leq -1/2 + p + \varepsilon', -\varepsilon' - 1/2 + i_2 \leq \operatorname{Im} z \leq \varepsilon' + (1/2) + i_2\}$, and let S'_{pi_2} be $D'_n \times S_{pi_2}$. We

define the figures $\Gamma^1_{pi_2}$ and $\Gamma^2_{pi_2}$ by the requirements

$$(i) \bigcup_{i=1}^2 \Gamma^i_{pi_2} = \partial(S'_{pi_2} \cap S'_{pi_2+1}) \text{ and } (ii) \operatorname{Im} z \geq i_2 + (1/2) \text{ or}$$

$\operatorname{Im} z \leq i_2 + (1/2)$ according to $z \in \Gamma^1_{pi_2}$ or $z \in \Gamma^2_{pi_2}$. We expand the Cousin

$$\text{integral } \Psi^1_{pi_2} = \int_{\Gamma^1_{pi_2}} \gamma_{i_2} \cdot (3-z)^{-1} dz \text{ at the origin } 0 \text{ in the form}$$

$$\Psi'_{p, \lambda_2} = \sum_{j=0}^{\infty} \Psi_{p, \lambda_2, j}$$

In this case the following estimation for $\Psi'_{p, \lambda_2, j}$ is

valid.

$$(3.1.9)_1 \quad |\Psi_{p, \lambda_2, j}| \leq 4^{d_2} \alpha_1 \cdot (p^{d_2} + |\lambda_2|^{d_2} + |w|^{d_2}) \cdot (2p+2) \cdot (\lambda_2 + \frac{1}{2})^{-j-1} \quad (\lambda_2 \geq 0)$$

It is clear that Cousin integral Ψ'_{p, λ_2} itself is estimated in the

manner

$$(3.1.9)_2 \quad |\Psi_{p, \lambda_2}^{\tau}| \leq 4^{d_2} (\varrho')^{-1} \alpha_1 \cdot (p^{d_2} + |\lambda_2|^{d_2} + |w|^{d_2} + 1) (2p+2) \quad (\lambda_2 \geq 0)$$

($\tau=1, 2$) in the set $\mathcal{D} \times \{z : \frac{I_m(z)}{I_m(z)} < \lambda_2 + \frac{1}{2} + \frac{\varepsilon}{2}, -p - \frac{1}{2} - \frac{\varepsilon}{2} < \operatorname{Re}(z) < p + \frac{1}{2} + \frac{\varepsilon}{2}\}$

In an analogous way to (), define functions $\Psi_{p, \lambda}^+(\mathbb{P})$

and $\Psi_{p, \lambda}^+(\infty)$ by

$$(3.1.9)_3 \quad \Psi_{p, \lambda}^+(\mathbb{P}) = \sum_{j=0}^{d_2+2} \Psi_{p, \lambda, j}^+ z^j, \quad \Psi_{p, \lambda}^+(\infty) = \Psi_{p, \lambda}^+(\infty) - \Psi_{p, \lambda}^+(\mathbb{P})$$

Using the estimation (3.1.9)_{1,2} the above functions $\Psi_{p, \lambda}^+(\mathbb{P})$ and

$\Psi_{p, \lambda}^+(\infty)$ are estimated respectively in the following ways.

$$(3.1.10)_1 \quad |\Psi_{p, \lambda}^+(\mathbb{P})| \leq 4^{d_2+2} \alpha_1 \cdot (p^{d_2} + |\lambda_2|^{d_2} + |w|^{d_2} + 1) (2p+2) \left(\sum_{j=0}^{d_2+2} |\lambda_2|^j \right)$$

$$(3.1.10)_2 \quad |\Psi_{p, \lambda}^+(\infty)| \leq 4^{d_2+2} \alpha_1 \cdot (p^{d_2} + |\lambda_2|^{d_2} + |w|^{d_2} + 1) (2p+2) |\lambda_2|^{d_2+3} \cdot (\lambda_2 + \frac{1}{2})^{-d_2-2}$$

in $\mathcal{D} \times \{z : -p - \frac{1}{2} - \frac{\varepsilon}{2} \leq \operatorname{Re} z \leq p + \frac{1}{2} + \frac{\varepsilon}{2}, \operatorname{Im} z \leq \lambda_2\}$.

Next define a function σ_{p,i_2} in the set $\mathcal{D} \times \{z: |\operatorname{Re} z| < p + \frac{1}{2} + \frac{\varepsilon}{2}, \operatorname{Im} z <$

$\operatorname{Im} z < i_2 + \frac{1}{2} + \frac{\varepsilon}{2}\}$

by

$$(3.1.11) \quad \sigma_{p,i_2}^{0,+} = \sum_{i_2 > i_2^0} \Psi_{p,i_2}^2 + \sum_{i_2 > i_2^0} \Psi_{p,i_2}^+(\mathbb{P}) - \sum_{i_2 \geq i_2} \Psi_{p,i_2}^+(\infty)$$

Then, from the definition of σ_{p,i_2} itself, the relation

$$(3.1.11)_1 \quad \sigma_{p,i_2+1}^{0,+} - \sigma_{p,i_2}^{0,+} = \sigma_{i_2}$$

is valid. in $\mathcal{D} \times \{z: |\operatorname{Re} z| < p + \frac{1}{2} + \frac{\varepsilon}{2}, i_2 + \frac{1}{2} - \frac{\varepsilon}{2} < \operatorname{Im} z < i_2 + \frac{1}{2} + \frac{\varepsilon}{2}\}$

On the other hand, from estimations (), (), and ()

we obtain estimations of the functions

$$\sigma_{p,i_2}^{0,+} \text{ in}$$

the following ways.

$$(3.1.11)_2 \quad |\sigma_{p,i_2}^{0,+}| \leq 4^{5i_2+2} (\varepsilon)^{-1} \alpha_1 \cdot \left\{ p^{2i_2+6} + i_2^{2i_2+6} + |w|^{3i_2+6} + 1 \right\}.$$

$$\cdot (2p+2),$$

in $\mathcal{D} \times \{z: |\operatorname{Re} z| < p + \frac{1}{2} + \frac{\varepsilon}{2}, i_2 + \frac{1}{2} - \frac{\varepsilon}{2} < \operatorname{Im} z < i_2 + \frac{1}{2} + \frac{\varepsilon}{2}\}$

($j = 0, 1, \dots$).

Repeat an entirely same arguments as above for $i \leq 0$.

Also repeat an entirely same argument as in the part (I),

(c.f. the end of I) . Then we obtain

functions $\sigma_{p,i}^0$ ($i = 0, \pm 1, \dots$) in $\mathcal{D} \times \{z: |\operatorname{Re} z - p|, |\operatorname{Im} z - \lambda_2|$

$< \frac{1}{2} + \frac{\xi}{4}\}$ so that the equation

$$(2.1.12)_1 \quad \sigma_{p,i+1}^0 - \sigma_{p,i}^0 = \sigma_{i,i+1}^1 \quad (i = 0, \pm 1, \dots)$$

as well as an estimation

$$(3.1.2)_0 \quad |\sigma_{p,i}^0| \leq 4 \cdot \frac{\tilde{\ell}_2 + \tilde{\ell}_1}{\tilde{\ell}_1} \cdot \frac{1}{\epsilon(E)} \times \alpha_1 \times \left\{ p^{\tilde{\ell}_2 + \tilde{\ell}_1} + |\mu_2| + |\lambda_2| + 1 \right\}$$

are valid ,

let $p' > p$ be two positive integers . Then

functions $\sigma_{p'}^0 = \sigma_{p,i_2}^0 - \sigma_{p,i_2}^0$'s are independent of i and defined

a global function in a neighbourhood of the set $\mathcal{D} \times$

$\{z: |\operatorname{Re} z - p + \frac{1}{2}|\}$. Especially we put $\sigma_p = \sigma_{p+1,p}$. The follow

ing estimation is valid .

$$(3.1.3) \quad |\sigma_p(\omega, z)| \leq \epsilon^{-\tilde{\ell}_1} \cdot \frac{\tilde{\ell}_2 + \tilde{\ell}_1}{\tilde{\ell}_1} \cdot \alpha_1 \times \left\{ p^{\tilde{\ell}_2 + \tilde{\ell}_1} + |\mu_2| + |\lambda_2| + 1 \right\}$$

so far as the inequality $|\operatorname{Im} z| \leq \tilde{\ell}_2$ is valid.

Here constants $(\tilde{\ell}_1, \dots, \tilde{\ell}_4)$ are also absolutely

independent.

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Now we adjust functions $\sigma_{i_2, p, \lambda}^0$ to obtain desired functions

$\sigma_{i_2, \lambda}^0$. For this purpose we expand functions σ_p' at the origin 0 in the following form.

$$(3.1.13)_1 \quad \sigma_p = \sum_{i=0}^{\infty} \sigma_{p,i}(w) \cdot z^i$$

Then the coefficients $\sigma_{p,i}(w)$'s are estimated in the manner

$$(3.1.13)_2 \quad |\sigma_{p,i}(w)| \leq \varepsilon^{-\ell_1''} \cdot \ell_1''^{\ell_2'' \omega_2 + \ell_3''} \cdot d_1 \cdot \left\{ p^{\ell_4'' \omega_2 + \ell_5''} + |w|^{\ell_4'' \omega_2 + \ell_5''} + 1 \right\} \cdot (p + \frac{1}{2})^{-i-1} \cdot \ell_1'' \cdots \ell_5''.$$

As before we define functions $\sigma_p(\mathbb{R})$ and $\sigma_p(\infty)$

by

$$(3.1.14) \quad \sigma_p(\mathbb{R}) = \sum_{i=0}^{\ell_4'' \omega_2 + \ell_5'' + 2} \sigma_{p,i}(w) \cdot z^i, \quad \sigma_p(\infty) = \sigma_p - \sigma_p(\mathbb{R}).$$

We consider the second function $\sigma_p(\infty)$ in the set $\mathcal{D} \times \{z : 2|Re z|/L_2 \leq |P|\}$.

Finally we define functions $\sigma_{i_2}(\lambda_2=0, 1, \dots)$ in the set $\mathcal{D} \times \{z :$

$$i_2 - \frac{1}{2} - \frac{\varepsilon}{4} \leq Im z \leq i_2 + \frac{1}{2} + \frac{\varepsilon}{4} \} \quad \text{by}$$

$$(3.1.15) \quad \sigma_{i_2}^0 = \sigma_{i_2, p}^0 + \sum_{\lambda \in P} \sigma_{\lambda}(\infty) - \sum_{\lambda \in P} \sigma_{\lambda}(\mathbb{R}), \quad \text{for the points}$$

in the set $\mathcal{D} \times \{z : 2|Re z| < |P|, |Im z - i| \leq \frac{1}{4} + \frac{\varepsilon}{4}, (\frac{|P|}{2} \leq \omega_2 + 1)\}$.

It is found easily that these functions σ_{i_2}' 's are

defined in the set $\mathcal{D} \times \{z : |Im z - i| \leq \frac{1}{4} + \frac{\varepsilon}{4}\}$ globally. (i.e. independent of P)

On the otherhand it is clear that the equation

$$(3.1.15)_1 \quad \sigma_{i_2+1}^0 - \sigma_{i_2}^0 = \sigma_{i_2+1, i_2}^1 \quad (i_2 = 0, \pm 1, \dots)$$

is valid .

Estimations of these functions are easily obtained .

To be sure we give estimations of these functions

in the following form :

$$\begin{aligned} (3.1.15)_{2,1} \quad |\sigma_p^0(p)| &\leq \varepsilon^{-\tilde{c}_0''} \cdot c_1'' \cdot \tilde{c}_2'' \tilde{c}_3'' \cdot \{ p^{\tilde{c}_4'' \tilde{c}_2 + \tilde{c}_5''} + |w| + 1 \} \cdot \left\{ \sum_{i=0}^{\tilde{c}_4'' \tilde{c}_2 + \tilde{c}_5'' + 2} |z^i| \right\}, \\ (3.1.15)_{2,2} \quad |\sigma_p^0(\infty)| &\leq \varepsilon^{-\tilde{c}_0''} \cdot c_1'' \cdot \tilde{c}_2'' \tilde{c}_3'' \cdot \{ p^{\tilde{c}_4'' \tilde{c}_2 + \tilde{c}_5''} + |w| + 1 \} \cdot |z| \cdot \frac{c_4'' \tilde{c}_2 + c_5'' + 3 - c_4''}{(p/2)}, \\ &\quad (p \geq 1, \quad \operatorname{Re} z, \operatorname{Im} z < \sqrt{p}/2), \\ (3.1.15)_{2,3} \quad |\sigma_{i_2}^0| &\leq |\sigma_{i_2, p}^0| + \varepsilon^{-\tilde{c}_0''} \cdot c_1'' \cdot \tilde{c}_2'' \tilde{c}_3'' \cdot \{ p^{\tilde{c}_4'' \tilde{c}_2 + \tilde{c}_5''} + |w| + 1 \} \\ &\quad \times \left\{ \sum_{i=0}^{\tilde{c}_4'' \tilde{c}_2 + \tilde{c}_5'' + 3} |z|^i \right\} \quad (p \geq 1, \quad |\operatorname{Re} z|, |\operatorname{Im} z| < \sqrt{p}/2). \end{aligned}$$

In the above $\tilde{c}_0'', \dots, \tilde{c}_5''$ are absolutely independent.
From the above estimations we know easily the desired

estimation $(3.1.18)_{1,2}$.

Now it is obvious that $(3.1.)$ and $(3.1.18)_{1,2}$ suffice for our purpose .

(II) Now we shall discuss simple quantities

, which is also used in the later .

3-1-17

Discussions here are also quite elementary. We shall enter into details for completeness. Our arguments will be done for two types of geometric figures. Differences of two situations considered here are quite small. However, to discuss both cases in one time causes technical confusions

(to the author). (A) Our first situation is

explained in the following manner: Let $\mathbb{C}^{l_1+l_2}$ be a complex

Euclidean space with coordinates $(x_1, \dots, x_{l_1}, x_{l_1+1}, \dots, x_{l_1+l_2})$.

Assume that a geometric figure $\square = \prod_{v=1}^{l_1+l_2} \square_{a_v, b_v}$ of elementary type

in $\mathbb{C}^{l_1+l_2}$ is given. For a given series of positive

integers $\{m_v^+, n_v^+\}_{v=1, \dots, l_1+l_2, i=1, 2}$, we write indices $\{j_v^+, t_v\}_{v=1, \dots, l_1+l_2, i=1, 2}$ as

(S, T) . Define a neighbourhood $\square(P_{ST})$ of P_{ST}

$$P_{ST} = \{(x): x_{v_1}(P_{ST}) = a_{v_1} + \frac{j_{v_1}}{m_{v_1}}, x_{v_2}(P_{ST}) = b_{v_2} + \frac{t_{v_2}}{n_{v_2}}\}_{v=1, \dots, l_1+l_2} \text{ by}$$

$$\square(P_{ST}) = \{(x) = (x_1, \dots, x_{l_1+l_2}): |Re x_i - Re x_i(P_{ST})| \leq \frac{j_i}{m_i}, \frac{t_i}{n_i},$$

$$|Im x_i - Im x_i(P_{ST})| \leq \frac{j_i}{m_i}, \frac{t_i}{n_i} \{v=1, \dots, l_1+l_2\}.$$

Furthermore, assume that another figure $\square' \subset \mathbb{C}^{l_1+l_2}$ of

elementary type is given. By $\mathcal{U}(\square, \square')$, we mean

the subset of $\{\square(P_{ST})\}$ composed of all the figures

satisfying the following condition.

$$(3.1.16) \quad \square(P_{ST}) \cap \square' \neq \emptyset$$

3-1-18

Obviously the open set $|\mathcal{N}(\square, \square')| = \bigcup \square(P_{ST}) \cdot \square(P_{ST})$ is also

a geometric figure of elementary type. We assume that a

domain $\mathcal{D}$ in $\mathbb{C}^{\ell_2}$ is given. By $\tilde{\square}(P_{ST})$, ~~$\square(P_{ST})$~~ and $\tilde{\mathcal{N}}(\square, \square', \mathcal{D})$

By $(\square(P_{ST}))$, we mean the projection $P_{ST} \square(P_{ST})$.
 , we mean $\square(P_{ST}) \times \mathcal{D}$ and $\{\square(P_{ST}) \cdot \square(P_{ST})\}$ respectively.

Take a set $\tilde{\mathcal{F}} = \{\square(P_{ST})\}_{\mu=1, \dots, s}$ of figures in $\mathbb{C}^{\ell_2} \times \mathbb{C}^{\ell_3}$ * composed of
 s elements. Henceforth such a set $\{\square(P_{ST})\}_{\mu=1, \dots, s}$ will be denoted by

$\mathcal{F}_s^{\ell_2}$. We mean by $\tilde{\mathcal{N}}(\square, \square', \mathcal{D}, \mathcal{F}_s^{\ell_2})$ the

set of all the figures $\square(P_{ST}) \in \tilde{\mathcal{N}}(\square, \square', \mathcal{D})$ satisfying the following
 two conditions.

$$(3.1.17) \quad \square(P_{ST}) \in \mathcal{F}_s^{\ell_2} : \quad P_{ST}(\square(P_{ST})) = \text{pr}_{\mathcal{F}_s^{\ell_2}}(\square(P_{ST}))$$

It is clear that the set $\tilde{\mathcal{N}}(\square, \square', \mathcal{D}, \mathcal{F}_s^{\ell_2})$ is an open covering of $|\tilde{\mathcal{N}}(\square, \square', \mathcal{D})|$.

Our first assertion, which is quite elementary, is spoken

for such a set. A q -cochain $\varphi^i \in C^q(N(\tilde{\mathcal{N}}(\square, \square', \mathcal{D}, \mathcal{F}_s^{\ell_2})), \mathbb{C})$

is said to be defined already in $\tilde{\mathcal{N}}^M(\square, \square', \mathcal{D}, \mathcal{F}_s^{\ell_2})$, if, for

each set $\{\square_{ST_r}\}_{r=0, \dots, s} : \bigcap_{r=0}^s \square_{ST_r} \neq \emptyset$, $\varphi^i(\bigwedge_{r=0}^s \square_{ST_r})$ is

already defined in $\bigwedge_{r=0}^s \square_{ST_r}^M \times \mathcal{D}$. (Note that the above does not

mean that φ^i is a restriction of a cochain $\tilde{\varphi}^i$

in $C^q(N(\tilde{\mathcal{N}}(\square, \square', \mathcal{D}, \mathcal{F}_s^{\ell_2})))$. A q -cochain φ^i is of

cotype s' , if $\varphi^i(\bigwedge_{r=0}^s \square_{ST_r}) = 0$, unless $\{\text{pr}_{\mathcal{F}_s^{\ell_2}}(\square_{ST_r})\}$ is

composed of (at least) s' elements. Let $\varphi^i \in$

* $\square(P_{ST}) = P_{ST}(\square(P_{ST})) : \mathbb{C}^{\ell_2} \times \mathbb{C}^{\ell_3} \rightarrow \mathbb{C}^{\ell_2} \times \mathbb{C}^{\ell_3}$.

a q - cochain of cotype Δ . In this case, a further definition will be made use of. For this purpose fix an order $\{\square_{T_v}\}_{v=1, \dots, d}$ of $\{\square_{T_v}\}_{v=1, \dots, d}$ once and for all. With this fixed order, a q - cochain y^q of cotype Δ is further said to be of cotype (s, k) if the following further conditions are valid,

$$(3.1.17)_2 \quad y^q(\bigwedge_{v=0}^q \square_{S_v T_v}) = 0, \quad \text{unless the}$$

set $\{\square_{S_v T_v}\}_{v=0, \dots, q}$ has the property: The cardinal number of the $\{\text{pr} \square_{S_v T_v}\}_{v=0, \dots, q}$ is (at least) k .

Using the above notations, what we make use of will be stated in the following fashion.

(A') $_{\Delta, k}^{l_1, l_2}$ Assume that $2 \leq \Delta \leq q$ and that a q - cocycle $y^q \in C(\mathcal{N}(\pi(\mathcal{M}), \mathcal{Q}))$ is given. Furthermore, we assume that

$\|y^q\| = \sup_{(x, x', v)} \|y^q(x, x', v)\|$ is finite.

If y^q is of cotype

(s, k) , then there exists a $(q-1)$ - cochain $y^{q-1} \in C(\mathcal{N}(\pi(\mathcal{M}), \mathcal{Q}))$

such that the following conditions are true.

(A') $_{\Delta, k, 1}^{l_1, l_2}$ y^{q-1} is defined in $\mathcal{M}(\pi(\mathcal{M}), \mathcal{Q}, \mathcal{L}_2)$; $M' = e \cdot \mathcal{M}$, and moreover,

$$\|y^{q-1}\| \leq c_1 \cdot M^{l_2} \cdot \left\{ \prod_{v=1}^{q+l_2} \max(\pi_v, \tau_v) \right\} \cdot \prod_{v=1}^{l_2} \max(\tilde{\pi}_v, \tilde{\tau}_v)^{-l_1} \cdot \|y^q\|.$$

(A') $_{\Delta, k, 2}^{l_1, l_2}$ $\delta(y^{q-1}) + y^q$ is of cotype $(s, k+1)$ if

$$k < p+2-\Delta$$

is zero if $k = q+2-\Delta$,

(A') $_{\Delta, k}^{l_1, l_2}$ y^{q-1} is of cotype (s, k) if $k \geq q+2-\Delta$,
 $(s, k-1)$ if $k = q+2-\Delta$.

In the above, quantities $e = e_{s,k}^{l,l_2}$, $c_k = c_{s,k}^{l,l_2}$ are depending on (l_1, l_2, q, s, k) only.

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(12) Our second assertion, which is easily deduced from $(A)_{s,1}^{l,l_2}$,

is stated in the following manner.

(A')_s^{l,l_2} Assume that $1 \leq s \leq q$. Also assume that a q -cocycle

γ^q of cotype s is given, ~~where~~ γ^q is already defined in

$\hat{N}(\square, \square, \square, \mathbb{S}_s^k)$. Then we find a $(q-1)$ -cochain γ^{q-1}

of cotype s so that the equation

$$(\hat{A}')_{s,1}^{l,l_2} - \delta_{\text{cch}}(\gamma^{q-1}) + \gamma^q = 0; \quad \gamma^{q-1} \text{ is of}$$

as well as the estimation

$$(\hat{A}')_{s,2}^{l,l_2} \|\gamma^{q-1}\| \leq c' M' \left\{ \prod_{v=1}^{l+l_2} \max(\pi_v, \pi_v) \right\}^{\frac{1}{2}} \times \left\{ \prod_{v=1}^{l+l_2} \min(\tilde{\pi}_v, \tilde{\pi}_v) \right\}^{\frac{1}{2}} \|\gamma^q\|$$

hold, so far as $M' \leq c' M > 1$.

In the above quantities $(c', l', l'_2, c'_s, c'_k, c')$ are

depending on (l, l_2, q, s) only.

For the sake of completeness, we shall formulate one

another problem. Let $l_2 = 0$, and let $l = l_1$. Then we

know the following.

(A'')^l There exists a set of positive numbers

$\{e'', c''_{k=1,2,3,4}\}$ determined by (l, l_2) with which the statement below is valid.

(A'')^l For a pair $\{\square, \square'\}$ and a

q -cocycle $\gamma^q \in \mathbb{Z}^q(N(\mathcal{N}(\square, \square'), \mathcal{O}))$, defined in

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in $\mathcal{C}^M(\square, \square')$, there exists a $(q-1)$ -cochain γ^{q-1} defined in

$\mathcal{C}^M(\square, \square')$ which satisfy the equation

$$(A'')^L_1 - \delta_{\text{cech}}(\gamma^{q-1}) + \gamma^q = 0$$

as well as the estimation

$$(A'')^L_2 \|\gamma^{q-1}\| \leq c' \cdot M^{L_2} \prod_{i=1}^{L_2+L_1} \{ \max(\tilde{a}_i, \tilde{b}_i) \}^{L_2} \cdot \left\{ \prod_{i=1}^{L_2+L_1} (\tilde{a}_i, \tilde{b}_i) \right\}^{-L_1} \|\gamma^p\|$$

are valid, where $M' = c' M$ is assumed to be : $M' \geq 1$.

(remark) In the above statement, nature of the figure

$\square'$ does not appear. We are formulating our prob

problem including the figure $\square'$, for purposes of applications.

(B) Our second concern is figures in $\Sigma_{n,N}$.

our assertions here are parallel to statements $(A'')^{L_1, L_2}_{A, B, 1, 2}$

and $(A)^{L_1, L_2}_{A, B, 1, 2}$. Though arguments are parallel to (A),

a care is necessary for quantitative arguments. (A care

, we take of, is quite simple. (C.f.)).

In this situation we assume the estimation of γ^q of the

following form ;

$$\|\gamma^q(z, w)\| \leq d_1 \cdot \{ |z|^{2_1} + |w|^{2_2} + 1 \}$$

Then, our assertions corresponding to $(A)^{L_1, L_2}_{A, B, 1, 2}$ and $(A)^{L_1, L_2}_A$

are stated in the following manner: First we shall

need to add some notations which are similar to ones in (A).

Let us return to the figure $\Sigma_{n,N}$. We mean

by pr_z and pr_w the natural projections: $h_z(z,w) = (z)$ and

$h_w(z,w) = (w)$. Assume that a figure $\Xi \in \mathbb{C}^* \mathbb{C}^N$ of an elementary

type is given. Also assume that an ordered set $\tilde{\mathcal{B}}_s$ of figures $\{\tilde{\Xi}_{i,j}^s\}_{i,j=1,\dots,n}$

in $\mathbb{C}^N$ of elementary types are given. We let $\mathcal{N}(\Xi, \tilde{\mathcal{B}}_s)$ be

the set of all the sets $\tilde{\Xi}_{i,j}^s$ which satisfy the following condition

$$(3.1.18) \quad \tilde{\Xi}_{i,j}^s \cap \tilde{\Xi}' \neq \emptyset, \text{ and } h_z(\tilde{\Xi}_{i,j}^s) \in \tilde{\mathcal{B}}_s$$

Notations used in (A) are used here under the following modifications. A set of open sets $\mathcal{N} = \{\tilde{\Xi}_{i,j}^s\}_{i,j=1,\dots,n}$ of type s if

the set $\{pr(\tilde{\Xi}_{i,j}^s)\}_{i,j=1,\dots,n}$ is composed of s elements. For a

a set $\mathcal{N}'$ of type s , we say that $\mathcal{N}'$ is of type (s, k)

if the number of elements in $\mathcal{N}'$, whose projection is $\tilde{\Xi}_{i,j}^s$,

is exactly k . A cochain $\psi \in C^p(\mathcal{N}(\Xi, \tilde{\mathcal{B}}_s), \mathbb{C})$ is said to be

defined in $\mathcal{N}(\Xi, \tilde{\mathcal{B}}_s)$ if $\psi(\tilde{\Xi}_{i,j}^s)$ is defined in $\tilde{\Xi}_{i,j}^s$ so far as $\tilde{\Xi}_{i,j}^s$

$\neq \emptyset$. Now assume that a p -cocycle ψ , defined in $\mathcal{N}(\Xi, \tilde{\mathcal{B}}_s)$

is given. In a similar way to (A), we show

the following two assertions respectively.

(B) $\mathcal{B}_{s,k}^{n,N}$ Assume the inequality $2 \leq s \leq q$:

Moreover, we assume that a q -cocycle $\psi \in G(N, \mathcal{B}_{s,k}^{n,N})$ of cotype (s, k) , defined already in $\mathcal{N}^{M_0}(\square, \mathcal{B}_{s,k}^{n,N})$, is given. The following estimation of ψ^i is assumed.

$$(B)_1 \quad |\psi^i(z, w)| \leq \alpha_1 (|z|^{2_1} + |w|^{2_2}) \quad z = 1$$

Then, corresponding to $(A)_{s,k}^{2_1, 2_2}$, we know the follow

(B) $\mathcal{B}_{s,k}^{n,N}$ there exists a $(q-1)$ -cochain $\psi^{i-1} \in G(N, \mathcal{B}_{s,k}^{n,N})$

*) with which the following facts are valid. (Moreover, ψ^{i-1} is assumed to be defined already in $\mathcal{N}^{M_0}(\square, \mathcal{B}_{s,k}^{n,N})$).

(B) $\mathcal{B}_{s,k}^{n,N}$ according to $k < q + 2 - s$ or $k = q + 2 - s$,

the relations

$$(B)_{s,k}^{n,N} \quad \psi^i - \delta_{\text{cch}}(\psi^{i-1}) = \text{of cotype } (s, k-1)$$

and ψ^{i-1} is of cotype (s, k) ;

if $k < q + 2 - s$.

$$(B)_{s,k}^{n,N} \quad \psi^i - \delta_{\text{cch}}(\psi^{i-1}) = 0,$$

and ψ^{i-1} is of cotype $(s, k-1)$

if $k = q + 2 - s$.

(B) $\mathcal{B}_{s,k}^{n,N}$ In both cases of $k < (q+2-s)$ or $k = q + 2 - s$,

the following estimation of ψ^{i-1} is valid,

$$(B)_{s,k}^{n,N} \quad |\psi^{i-1}| \leq \alpha'_1 (|z|^{2'_1} + |w|^{2'_2})$$

In the above, quantities $(M', \alpha'_1, \alpha'_2)$ are

$$g = 1 - 2/3$$

determined by the following equations

$$(B_{1,5}) \quad \tilde{M}' = e^{-\tilde{\alpha}_2 \delta_2} \tilde{M}, \quad \alpha'_1 = P_1(\tilde{\alpha}_1) P_2(\tilde{\alpha}_2) \tilde{\alpha}_2 P_3(\tilde{\alpha}_1, \tilde{\alpha}_2) \cdot \min(\tilde{\alpha}_1, \tilde{\alpha}_2)^{-\tilde{\alpha}_2}$$

$$\alpha'_2 = L(\alpha_2 + \delta_2)$$

, where $\tilde{\alpha}_2 = e^{\tilde{\alpha}_2 \delta_2} \in \mathbb{R}^+$, $\tilde{\alpha}_1 = e^{\tilde{\alpha}_1 \delta_1} \in \mathbb{R}^+$, P_i, L are depending on (N, n, q, s, κ) only.

Corresponding to the assertion $(B_{1,5})$, we know the following facts.

$(B'_{1,1})_{\Delta, k}^{n, N}$ Assume the inequality: $1 \leq s \leq q$

Also assume that a q -cocycle $\psi^q \in \mathbb{Z}^q(N(\tilde{\alpha}, \tilde{\alpha}_2), \mathbb{C})$ of cotype s , defined in $\mathcal{U}(\tilde{\alpha}, \tilde{\alpha}_2)$ and estimated as

$$(B'_{1,1})_{\Delta, k}^{n, N} |\psi^q| \leq \tilde{\alpha}_1 \cdot (|\tilde{\alpha}_2| + 1)$$

is given. Then we obtain a $(q-1)$ cochain

ψ^{q-1} , defined in $\mathcal{U}(\tilde{\alpha}, \tilde{\alpha}_2)$, which satisfies the

following conditions.

$$(B'_{1,1})_{\Delta, k}^{n, N} \psi^{q-1} - \delta_{\text{cch}}(\psi^{q-1}) = 0,$$

$$(B'_{1,2})_{\Delta, k}^{n, N} |\psi^{q-1}| \leq \tilde{\alpha}_1 \cdot (|\tilde{\alpha}_2| + 1)$$

Here quantities $(\tilde{M}', \tilde{\alpha}_1, \tilde{\alpha}_2)$ are determ-

because the steps (I) involve considerable details we shall outline each step taken in the second step.

In the first place (I), we settle a set of neighbourhood to each stratum $S_{\lambda_i \mu_j}^{*j}$. This is done by dividing the strata into first and second type, and this part is of essential (but of quite simple and elementary) nature in the remaining parts of this section. Once we fix a set of neighbourhoods to each stratum $S_{\lambda_i \mu_j}^{*j}$, we investigate intersection relations of neighbourhoods for a series $S_{\lambda_i \mu_j}^{*j} \rightarrow S_{\lambda_i \mu_j}^{*j}$. This part will be divided into three cases : (i) The case in which the series is composed of strata of first type only . (i i) The case where the series is composed of strata of second type only and finally (i i i) the mixed case where the series contains strata of both types. If the intersections takes a simple form which is commodiate for our purpose, we say that the associated

neighbourhoods are suitable,

we ask quantitative conditions

in order that the associated neighbourhoods are

suitable . This will be done by dividing the cases

into three cases (i) , (ii) , and (iii) , and is

also of quite elementary nature. In (iii)

the condition (2.1.1) is quickly shown for

the neighbourhoods which are suitable. Finally

in (iv) we show the condition ().

Our arguments in the last two problems will be

done easily on basis of explicit expressions of

intersections of neighbourhoods.

Finally we note that, in the case of $\beta = 1$

U^* is expressed as a disjoint finite union of

points and one dimensional open segments . Therefore

to verify all the desired conditions in proposition 2.5 is

an obvious matter, and we do not enter into.

Actually, we first see that assertions $(Imp)_1 \sim (Imp)_5$ lead to the assertions $(A)_{d,k}^{l,l_2}$, $(A)_d^{l,l_2}$ and $(A)^l$.

For this purpose, note that the following series of implications are valid.

$$\begin{aligned} & (\widetilde{Imp})_1^1 \quad (A)^{l'} \quad (l' = 1, \dots, l-1) \Rightarrow (A)_1^{l-1,1} \Rightarrow \\ & (A)_k^{l-1,2} \quad (k = 1, \dots, q_{l-1}) \Rightarrow (A)_q^{l-1,2} \Rightarrow (A)_{l,k}^{l-1,2} \\ & \dots \Rightarrow (A)_{l,k}^{l-1,1} \Rightarrow (A)_i^{l-1,1} \Rightarrow (A)_{q+1}^{l-1,1} \Rightarrow (A)^l. \end{aligned}$$

In the above sequence, the first assertion follows from $(Imp)_3$ and the last assertion follows from $(Imp)_5$ on the otherhand, intermediate sequences are because of $(Imp)_2$.

thus we know that the assertions $(Imp)_1 \sim (Imp)_5$ implies the validity of the assertion $(A)^l$ ($l=1, \dots$).

then it is clear that the assertions $(Imp)_1$, $(Imp)_2$ and $(Imp)_3$ lead to the assertions $(A)_{d,k}^{l,l_2}$, $(A)_d^{l,l_2}$.

In a similar reason as above, it is clear that the assertions $(Imp)_1'$, $(Imp)_2'$ and $(Imp)_3'$ imply the assertions $(B)_{d,k}^{n,N}$ and $(B)_d^{n,N}$. Now we show the assertions

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$(\text{Imp})_{1 \sim 5}$ (resp. $(\text{Imp})_{1 \sim 3}$) in the following

manner. At first note that it is clear that the

repeated uses of $(A)_{\Delta, k}^{l_1, l_2}$ (resp. $(B)_{\Delta, k}^{l_1, N}$) lead -

easily to assertions $(A)_{\Delta}^{l_1, l_2}$ (resp. $(B)_{\Delta}^{l_1, N}$). on the

other hand, $(\text{Imp})_{\frac{1}{3}(\text{Imp})_3}$ is shown quickly as follows :

Take an element $\square_{\circ}^{l_2} \in C^{l_2}$ (resp. $\square_{r, \circ}^s \in C^N$) respectively . we mean by $\mathbb{A}_1^{l_2}$ (and $\mathbb{A}_1^N$) the set

composed of one element $\mathbb{A}_{\circ}^{l_2}(\square_{r, \circ}^s)$ only. Write the

given figure $\square^{l_1, l_2}$ in C^{l_1, l_2} (resp. $\square^{n, N}$ in $C^{n, N}$) as

$$\square^{l_1, l_2} = \square^{l_1} \times \square^{l_2}, \quad \square^{n, N} = \square^n \times \square^N$$

Then $\mathcal{U}(\square_{\circ}^{l_1, l_2}, \mathbb{A}_{\circ}^{l_2}, \mathcal{D})$ (resp. $\mathcal{U}(\square^{n, N}, \mathbb{A}_1^N)$) is composed of figures

$$\square_{\mu}^{l_1} \times \square_{\circ}^{l_2} \times \mathcal{D} \quad (\text{resp. } \square_{\mu, K_{\mu}}^s \times \square_{r, \circ}^s \times \mathcal{D}) \quad \text{which satisfy}$$

$$(3.1.19) \quad \square_{\mu}^{l_1} \cap \square^{l_1} (\neq \emptyset) \quad (\square_{\mu, K_{\mu}}^s \cap \square^s \neq \emptyset).$$

Denote by $\mathcal{U}(\square_{\circ}^{l_1, l_2}, \mathbb{A}_{\circ}^{l_2}, \mathcal{D})$ ($\mathcal{U}(\square^{n, N}, \mathbb{A}_1^N)$) the set

of figures $\{ \square_{\mu}^{l_1} \times \mathcal{D} \} \quad (\{ \square_{\mu, K_{\mu}}^s \})$ so that the relation

(3.1.19) holds. Define a map $i(i')$ by

$$3-1-28$$

$$(3.1.20) \quad i: G^*(N(\pi(\square_{\mu}^{l_2}, \emptyset), \emptyset) \ni \psi \rightarrow G^*(N(\pi(\square_{\mu}^{l_1}, \emptyset), \emptyset) \ni i(\psi)$$

$$(3.1.20)' \quad : \quad \psi(\square_{\mu}^{l_1}, \emptyset) = \psi(\square_{\mu}^{l_2}, \emptyset)$$

$$i': G^*(N(\pi(\square_{\mu}^{l_1}, \emptyset), \emptyset) \ni \psi' \rightarrow G^*(N(\pi(\square_{\mu}^{l_2}, \emptyset), \emptyset) \ni i'(\psi')$$

$$: i'(\psi')(\square_{\mu}^{l_1}, \emptyset) = i(\psi)(\square_{\mu}^{l_2}, \emptyset)$$

The above bijective map $i(i')$ commutes with the Čech coboundary operator $\delta_{\text{Čech}}$. Then it is clear that $(A)^{l'}$

$((A)^{l'})$ implies the assertion $(A)^{l_1, l_2} \implies (B)^{n, n'}$ therefore

the remaining problem is to show assertions $(\text{Imp})_2$

$((\text{Imp})'_4)$, $(\text{Imp})_5$ (and $(\text{Imp})'_2$). We consider the assertion $(\text{Imp})_2$ $((\text{Imp})'_2)$.

Verifications of the assertion $(\text{Imp})_2$ $((\text{Imp})'_2)$.

Arguments will be done by dividing the cases: $k \neq q+2-s$

and $k = q+2-s$. First we remark the following simple

facts. Let M, M' be two positive numbers ($M, M' \geq 1$),

Moreover, let $\square_1, \square_2$ (resp. $\square_{I,J,K_1}^s, \square_{I,J,K_2}^s$) be figures so

that $\square_1 \cap \square_2 \neq \emptyset$ (resp. $\square_{I,J,K_1}^s \cap \square_{I,J,K_2}^s \neq \emptyset$) holds.

Then there exist positive numbers $e = e_{s,l_1}$ (resp. $e_{\mu} = e_{\mu, l_1, l_2}$)

with which the following are valid.

$$(3.1.20) \quad \text{If } M \geq e \cdot M', \text{ then } \square_1^M \supset \square_2^{M'}$$

$$(3.1.20)' \quad \text{If } M \geq e_{\mu} \cdot M', \text{ then } \square_1^{Ms} \supset \square_2^{M's}$$

We show assertions $(\text{Imp})_2$ $((\text{Imp})'_2)$ as follows.

3-1-29

(Imp &)_k $k > q + 2 - s$: Start with a datum $(\square, \square', \mathcal{B}_s^{\ell_2})$

(resp. $(\tilde{\square}, \tilde{\mathcal{B}}_s^N)$) as in assertions $(A)_{s, \ell_2}^{\ell_1, \ell_2} ((B)_{s, \ell_2}^{N, \ell_2})$

We express the ordered set $\mathcal{B}_s^{\ell_2}$ (resp. $\tilde{\mathcal{B}}_s^N$) in the

following way. $\mathcal{B}_s^{\ell_2} = \{ \square_{I_\mu J_\mu} \}_{\mu=1, \dots, s}$, $\tilde{\mathcal{B}}_s^N = \{ \tilde{\square}_{I_\mu J_\mu K_\mu}^s \}_{\mu=1, \dots, s}$. Take elements

$(\square_{I_\nu J_\nu}) : \bigcap_{\mu=1, \dots, k} \square_{I_\mu J_\mu} \neq \emptyset$ (resp. $(\tilde{\square}_{I_\nu J_\nu K_\nu}^s) : \bigcap_{\mu=1, \dots, k} \tilde{\square}_{I_\mu J_\mu K_\mu}^s \neq \emptyset$) ($\nu = 1, \dots, k$)

Instead of the given figure $\square \in \mathcal{C}^{\ell_2}$ (resp. $\square \in \mathcal{C}^{N\ell_2}$) we

consider the figure $\square_{\ell_2}^* = \square_{\ell_2}(\bigcap_{\mu=1}^{\ell_2} \square_{I_\mu J_\mu})$ (resp. $\tilde{\square}_{\ell_2}^* = \tilde{\square}_{\ell_2}(\bigcap_{\mu=1}^{\ell_2} \tilde{\square}_{I_\mu J_\mu K_\mu}^s)$). Also

we start with the ordered set $\tilde{\mathcal{B}}_{s-1}^{\ell_2} = \{ \square_{I_\mu J_\mu} \}_{\mu=1, \dots, s-1}$ (resp. $\tilde{\mathcal{B}}_{s-1}^N = \{ \tilde{\square}_{I_\mu J_\mu K_\mu}^s \}_{\mu=1, \dots, s-1}$)

instead of $\mathcal{B}_s^{\ell_2}$ (resp. $\tilde{\mathcal{B}}_s^N$). In a little while

our argument will be done with the data $(\square, \square', \mathcal{B}_{s-1}^{\ell_2})$

(resp. $(\tilde{\square}, \tilde{\square}', \tilde{\mathcal{B}}_{s-1}^N)$). For a given cocycle γ^{ℓ_2}

$\in C^{\ell_2}(N(\mathcal{U}(\square, \square', \mathcal{B}_s^{\ell_2}), \mathcal{O}))$ (resp. $\tilde{\gamma}^{\ell_2} \in C^{\ell_2}(N(\mathcal{U}(\tilde{\square}, \tilde{\square}', \tilde{\mathcal{B}}_s^N), \mathcal{O}))$)

we obtain a $(q - k)$ -cocycle $\gamma_*^{\ell_2 - k} \in C^{\ell_2 - k}(N(\mathcal{U}(\square, \square', \mathcal{B}_{s-1}^{\ell_2}), \mathcal{O}))$

($\tilde{\gamma}_*^{\ell_2 - k} \in C^{\ell_2 - k}(N(\mathcal{U}(\tilde{\square}, \tilde{\square}', \tilde{\mathcal{B}}_{s-1}^N), \mathcal{O}))$) by the following equation.

$$(3.1.21) \quad \gamma_*^{\ell_2 - k} \left(\bigcap_{\mu=1}^{\ell_2 - k + 1} \square_{I_\mu J_\mu} \right) = \gamma^{\ell_2} \left(\bigcap_{\mu=1}^{\ell_2 - k + 1} \square_{I_\mu J_\mu} \bigcap_{\mu=\ell_2 - k + 1}^{\ell_2} \square_{I_\mu J_\mu} \right),$$

if $\text{pr}_{x^*} \{ \square_{I_\mu J_\mu} \}_{\mu=1, \dots, \ell_2 - k + 1}$ coincide with $\mathcal{B}_{s-1}^{\ell_2}$.

$$(3.1.21)' \quad \tilde{\gamma}_*^{\ell_2 - k} \left(\bigcap_{\mu=1}^{\ell_2 - k + 1} \tilde{\square}_{I_\mu J_\mu K_\mu}^s \right) = \tilde{\gamma}^{\ell_2} \left(\bigcap_{\mu=1}^{\ell_2 - k + 1} \tilde{\square}_{I_\mu J_\mu K_\mu}^s \bigcap_{\mu=\ell_2 - k + 1}^{\ell_2} \tilde{\square}_{I_\mu J_\mu K_\mu}^s \right) \text{ if } \mathcal{B}_{s-1}^{\ell_2} = \mathcal{B}_{s-1}^N,$$

$$3-1-30 \quad = 0, \text{ otherwise}$$

moreover, we assume that the cocycle γ^2 (resp. $\tilde{\gamma}^2$)

is already defined in $\mathcal{N}^{\mathcal{M}}(\square, \square', \mathcal{L}_{\Delta}^{l_2})$ (resp.

$\mathcal{N}^{\tilde{\mathcal{M}}^s}(\square', \mathcal{L}_{\Delta}^{\mathcal{N}})$) . Then, in view of (),

we can assume that γ_{*}^{2-k} (resp. $\tilde{\gamma}_{*}^{2-k}$) is already

defined in $\mathcal{N}^{\mathcal{M}'}(\square, \square', \mathcal{L}_{\Delta}^{l_2})$ (resp. $\mathcal{N}^{\tilde{\mathcal{M}}^s}(\square', \mathcal{L}_{\Delta}^{\mathcal{N}})$), $M' = e^* M, \tilde{M}' = \tilde{e}^* \tilde{M}$

here constants e^* ($\tilde{e}_1^*, \tilde{e}_2^*$) are depending on (q, l_2)

(n, N) only . By taking M sufficiently large, we can assume

the inequality $M' \geq 1$. For the above

cocycle γ_{*}^{2-k} (resp. $\tilde{\gamma}_{*}^{2-k}$) , we apply the

assertion (A) $_{\Delta-1}^{l_1, l_2}$ (resp. (B) $_{\Delta-1}^{n, N}$) .

Then we obtain a ($q-k-1$) - cochain γ

$$\gamma_{*}^{2-k-1} \in C^{2-k-1}(\mathcal{N}^{\mathcal{M}}(\square, \square', \mathcal{L}_{\Delta-1}^{l_2}), \mathcal{O}) \quad (\text{ resp. } \tilde{\gamma}_{*}^{2-k-1} \in$$

$C^{2-k-1}(\mathcal{N}^{\tilde{\mathcal{M}}^s}(\square', \mathcal{L}_{\Delta-1}^{\mathcal{N}}), \mathcal{O})$) , already defined in $\mathcal{N}^{\mathcal{M}}(\square, \square', \mathcal{L}_{\Delta-1}^{l_2})$

(resp. in $\mathcal{N}^{\tilde{\mathcal{M}}^s}(\square', \mathcal{L}_{\Delta-1}^{\mathcal{N}})$) in such a manner that

the equation

$$(3.3.22) \quad \gamma_{*}^{2-k} - \delta_{\text{Cech}}(\gamma_{*}^{2-k-1}) = 0,$$

$$(\text{ resp. } \tilde{\gamma}_{*}^{2-k} - \delta_{\text{Cech}}(\tilde{\gamma}_{*}^{2-k-1}) = 0 ,)$$

holds.

3-1-31

The estimation of ψ_{*}^{q-k-1} (resp. $\tilde{\psi}_{*}^{q-k-1}$) is obtained from $(A)_{s-1}^{l_1, l_2}$ (resp. $(B)_{s-1}^{n, N}$) in the following fashion.

$$(3.1.23) \quad \|\psi_{*}^{q-k-1}\| \leq c_i^{*} M^{l_2^{*}} \{ \min(\tilde{a}_v, \tilde{b}_v) \}_{v=1}^{n_1^{*}} \{ \max(m_v, n_v) \}_{v=1}^{n_2^{*}} \|\psi_{*}^{q-k}\|$$

$$(3.1.23)' \quad \|\tilde{\psi}_{*}^{q-k-1}\| \leq d_i^{*} \cdot (1 + d_2^{*} + 1)$$

In the above, quantities $(c_1^{*}, \dots, c_s^{*})$

(d_1^{*}, d_2^{*}) are given by the $(A)_{s-1}^{l_1, l_2}$ (resp. $(B)_{s-1}^{n, N}$). Using the above cochain

ψ_{*}^{q-k-1} (resp. $\tilde{\psi}_{*}^{q-k-1}$), we define a

$(q-1)$ -cochain $\psi^{q-1} \in C^{q-1}(N(\mathcal{M}(\square, \square, \beta_2^q), \mathcal{O}))$

(resp. $\tilde{\psi}^{q-1} \in C^{q-1}(N(\mathcal{M}(\square, \square, \beta_A^N), \mathcal{O}))$) by the

following equation.

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(3.1.24) $\psi^{i-1}(\bigwedge_{v=1}^i \square_{I_v J_v}) = \psi_*^{i-k-1}(\bigwedge_{v=1}^{i-k-1} \square_{I_v J_v} \bigwedge_{v=i-k}^i \square_{I_v J_v})$, in the set

$\{\square_{I_v J_v}\}_{v=1, \dots, i-1}$ satisfies the following condition,

(3.1.24)' there exists exactly k figures $\{\square_{I_v J_v}\}_{v=1, \dots, k}$ whose projection $p_{I_v J_v}(\square_{I_v J_v})$ is $\square_{I_v J_v}$ and $\langle \square_{I_v J_v} \rangle_{v=1, \dots, i}$ is $\{\square_{I_v J_v}\}_{v=1, \dots, i-k} - \{\square_{I_v J_v}\}_{v=i-k+1, \dots, i}$, otherwise. $\square_{I_v J_v}$

In the case of $(B)_{A, k}^{2N}$ we define ψ^{i-1} by

(3.1.24)* $\psi^{i-1}(\bigwedge_{v=1}^i \square_{I_v J_v}) = \psi_*^{i-k-1}(\bigwedge_{v=1}^{i-k-1} \square_{I_v J_v})$ if the set $\{\square_{I_v J_v}\}_{v=1, \dots, i}$

satisfy the condition

(3.1.24)* there exists exactly k figures $\{\square_{I_v J_v K_v}\}_{v=1, \dots, k}$ whose projection $p_{I_v J_v K_v}(\square_{I_v J_v K_v})$ is $\square_{I_v J_v}$, and in $(\cdot)$, $\{\square_{I_v J_v K_v}\}_{v=1, \dots, i-k} - \{\square_{I_v J_v K_v}\}_{v=i-k+1, \dots, i}$ is $\{\square_{I_v J_v K_v}\}_{v=1, \dots, i-k}$, otherwise.

Moreover,, in view of the simple remarks (3.1.20)', we

can assume that this cochain ψ^{i-1} (resp. ψ^{i-1})

is already defined in $\mathcal{M}'(\square, \beta_s^{e_2})(\mathcal{M}'(\square, \beta_s^{e_2}))$, where

$\mathcal{M}'$ is defined by $\mathcal{M}' = e \cdot \mathcal{M}$ (resp. $\mathcal{M}' = e_1^{e_2} \cdot \mathcal{M}$)

with constants e (e_1, e_2) depending on

(l_1, l_2) (resp. (N, n)) only.

On the otherhand, estimations of cochains $\psi^{i-1}, \tilde{\psi}^{i-1}$,

are obtained from the induction hypothesis $(A)_{s-1}^{l_1, l_2} \{ (B)_{s-1}^{2N} \}$.

thus required quantitative properties required for

cochains φ^{i-1} ($\tilde{\varphi}^{i-1}$), are satisfied. On the

otherhand, it is clear from the equations (), ()

that the following relations are valid.

(3.1.25) $\varphi^k - s_{\text{def}} (\varphi^{k-1})$ is of cotype

(s, k+1). Finally it is also obvious, from (),

, that the cochain φ^{k-1} is of cotype (s, k+1).

This finishes our proof in this case.

(Imp 2) $k = q + 2 - s$; in this case our

method of reducing the problem to the lower case is not applicable as in the case (). we proceed

as follows: Take a set of figures $\{\square_{I_v J_v K_v}\}_{v=1, \dots, \delta-1}$

as follows: Take a set of figures $\{\square_{I_v J_v K_v}\}_{v=1, \dots, \delta-1}$

(resp. $\{\square_{I_v J_v K_v}\}_{v=1, \dots, \delta-1}^\delta$) so that the conditions

(3.1.26) $\bigcap_{v=1}^{\delta-1} \square_{I_v J_v K_v} \neq \emptyset$, (resp. $\bigcap_{v=1}^{\delta-1} \square_{I_v J_v K_v}^\delta \neq \emptyset$)

(3.1.26)' $\{B_{\alpha}(\square_{I_v J_v K_v})\}_{v=1, \dots, \delta-1} = \{\square_{J_v}\}_{v=1, \dots, \delta-1}$, $\{B_{\alpha}(\square_{I_v J_v K_v}^\delta)\}_{v=1, \dots, \delta-1} = \{\square_{I_v J_v K_v}^\delta\}_{v=1, \dots, \delta-1}$

are valid.

In this case we fix a set $\{\square_{I_v J_v K_v}\}_{v=1, \dots, \delta-1}$ (resp. $\{\square_{I_v J_v K_v}^\delta\}_{v=1, \dots, \delta-1}$)

at a moment . Define a figure $\square_{\delta-1}^*$ (resp. $\widetilde{\square}_{\delta-1}^*$)

by $\square_{\delta-1}^* = \square_{\delta-1}^* (\bigwedge_{v=1}^{\delta-1} \square_{I_v J_v})$ (resp. $\widetilde{\square}_{\delta-1}^* = \widetilde{\square}_{\delta-1}^* (\bigwedge_{v=1}^{\delta-1} \square_{I_v J_v})$) .

Moreover, define a set of figures $\mathcal{N}(\square, \square_{\delta-1}^*, \square_{\delta-1}^*)$ by $\mathcal{N}(\square, \square, \square_{\delta-1}^*) = \{ \square, \square_{\delta-1}^* \}$

$\Rightarrow \square_{J_\delta}, \square_{\delta-1}^* \} \quad (\text{resp. } \mathcal{N}(\square, \widetilde{\square}_{\delta-1}^*, \widetilde{\square}_{\delta-1}^*) = \{ \square_{\delta-1}^*, \square_{\delta-1}^* \})$. Then

it is clear that a $(q - s + 1) -$ cocycle $\gamma_*^{q-s+1} \in G(\mathcal{N}(\square, \square_{\delta-1}^*, \square_{\delta-1}^*), \mathcal{D})$ (resp. γ_*^{q-s+1}

$\in G(\mathcal{N}(\square, \widetilde{\square}_{\delta-1}^*, \widetilde{\square}_{\delta-1}^*), \mathcal{D})$) is obtained from a $q -$ cocycle $\gamma_*^q \in G(\mathcal{N}(\square, \square_{\delta-1}^*, \square_{\delta-1}^*), \mathcal{D})$ (resp. $\gamma_*^q \in G(\mathcal{N}(\square, \widetilde{\square}_{\delta-1}^*, \widetilde{\square}_{\delta-1}^*), \mathcal{D})$) derived

already in $\mathcal{N}(\square, \square_{\delta-1}^*, \square_{\delta-1}^*)$ (resp. $\mathcal{N}(\square, \widetilde{\square}_{\delta-1}^*, \widetilde{\square}_{\delta-1}^*)$) by the following formula.

$$(3.1.27) \quad \gamma_*^{q-s+1} (\bigwedge_{v=1}^{q-s+1} \square_{I_v J_v}) = \gamma_*^q (\bigwedge_{v=1}^{q-s+1} \square_{I_v J_v} \bigwedge_{v=1}^{\delta-1} \square_{I_v J_v})$$

$$(3.1.27') \quad \gamma_*^{q-s+1} (\bigwedge_{v=1}^{q-s+1} \square_{I_v J_v K_v}) = \gamma_*^q (\bigwedge_{v=1}^{q-s+1} \square_{I_v J_v K_v} \bigwedge_{v=1}^{\delta-1} \square_{I_v J_v K_v})$$

In the above, we can assume that γ_*^{q-s+1} (resp. γ_*^{q-s+1})

is defined in $\mathcal{N}^{M'}(\square, \square_{\delta-1}^*, \square_{\delta-1}^*)$ (resp. $\mathcal{N}^{M'}(\square, \widetilde{\square}_{\delta-1}^*, \widetilde{\square}_{\delta-1}^*)$), where

M' is determined by $M' = e_+ M$ (resp. $M' = \tilde{e}_1^* \tilde{e}_2^* M$)

with constants $e_+ (\tilde{e}_1^*, \tilde{e}_2^*)$ depending on $(\ell_1, \ell_2, \delta, s)$ (resp. $(M, \ell_1, \ell_2, \delta, s)$) only.

(The existence of constants $e_+ (\tilde{e}_1^*, \tilde{e}_2^*)$ are obvious

in view of (3.1.19) . Also our arguments will

be done by assuming that $M' \geq 1$. Here we

remark that γ_*^{q-s+1} (resp. γ_*^{q-s+1}) is independent

of $\{ \square_{I_v J_v} \}_{v=1, \dots, \delta-1}$ (resp. $\{ \square_{I_v J_v K_v} \}_{v=1, \dots, \delta-1}$) in the sense that

Apply the induction hypothesis $(\quad)_{l_1, l_2}^{l_1, l_2}$ (resp. $(\quad)_{l_1, l_2}^{n, N}$)

to the cocycle γ_*^{i-d+1} (resp. $\tilde{\gamma}_*^{i-d+1}$). Then we obtain a

$(q-1)$ cochain $\gamma_*^{i-d} \in C^i(N(\mathcal{M}(\square_{i,j}^s, \square_{i,j}^s)), \mathcal{O})$

(resp. $\tilde{\gamma}_*^{i-d} \in C^i(N(\mathcal{M}(\square_{i,j}^s, \square_{i,j}^s)), \mathcal{O})$) which satisfy the

following equation.

$$(3.1.28) \quad \delta_{\text{cch}}(\gamma_*^{i-d}) - \gamma_*^{i-d+1} = 0.$$

$$(\text{ resp. } (\quad)_{l_1, l_2}^{l_1, l_2} \delta_{\text{cch}}(\tilde{\gamma}_*^{i-d}) - \tilde{\gamma}_*^{i-d+1} = 0)$$

By $(A)_{l_1, l_2}^{l_1, l_2}$ (resp. $(B)_{l_1, l_2}^{n, N}$) we know

necessary quantitative properties : γ_*^{i-d} (resp. $\tilde{\gamma}_*^{i-d}$)

is defined in $\mathcal{M}^s(\square_{i,j}^s, \square_{i,j}^s)$ (resp. $\mathcal{M}^s(\square_{i,j}^s, \square_{i,j}^s)$) while

the estimation of γ_*^{i-d} (resp. $\tilde{\gamma}_*^{i-d}$) is given by

$(A)_{l_1, l_2}^{l_1, l_2}$ (resp. $(B)_{l_1, l_2}^{n, N}$). Using the above cochain γ_*^{i-d} , we

define a $(q-1)$ -cochain $\gamma_*^{i-d} \in C^{i-d}(N(\mathcal{M}(\square_{i,j}^s, \square_{i,j}^s)), \mathcal{O})$ by the following

equation.

$$\textcircled{1} \quad (3.1.29) \quad \gamma_*^{i-d}(\square_{i,j}^s, \square_{i,j}^s) = \gamma_*^{i-d+1}(\square_{i,j}^s, \square_{i,j}^s) \text{ if } \{\square_{i,j}^s\} \text{ is of type } (d, d-d)$$

$$(\text{ resp. } \gamma_*^{i-d} \in C^{i-d}(N(\mathcal{M}(\square_{i,j}^s, \square_{i,j}^s)), \mathcal{O}) \text{ by } \gamma_*^{i-d}(\square_{i,j}^s, \square_{i,j}^s) = 0, \text{ otherwise.}$$

In the above M (resp. M') is given

in the following form (in view of ()).

$$(3.1.29)' \quad M' = e^* M, \quad M' = e_1^* e_2^* \delta_2 M.$$

with constants $e^* (e_1^*, e_2^*)$ depending on (l_1, l_2)

(resp. (n, N)) only .

now it is easy to see that the following equation

$$(3.1.30) \quad \psi^i - \delta_{cck} (\psi^{i'}) = 0, \\ \text{resp.} \quad \tilde{\psi}^i - \delta_{cck} (\tilde{\psi}^{i'}) = 0,$$

holds:

first it is clear that $\psi^i - \delta_{cck} (\psi^{i'}) = 0$ ($\tilde{\psi}^i - \delta_{cck} (\tilde{\psi}^{i'}) = 0$),

except the cases where $\{\square_{i,j}\}$ satisfy the conditions

$\{\square_{i,j}\}_{i=1, \dots, n} (\{\square_{i,j,k}^s\}_{i=1, \dots, n})$ is of type s , and of type (s, k) or $(s, k+1)$.
Next, if $\{\square_{i,j}\} \{\square_{i,j,k}^s\}$ is of type (s, k) , then

, implies that $\psi^i - \delta_{cck} \psi^{i'} = 0$ (resp.

$\tilde{\psi}^i - \delta_{cck} (\tilde{\psi}^{i'}) = 0$), thirdly, for $(\square_{i,j,k}) (\square_{i,j,k}^d)$

of type $(s, k-1)$, $\psi^i - \delta_{cck} \psi^{i'} = 0$ ($\tilde{\psi}^i - \delta_{cck} (\tilde{\psi}^{i'}) = 0$)

because of the equation (3.1.29) .

this finishes our proof. q.e.d.

Remainin~ two assertions $(Imp)_4$ and $(Imp)_5$ are quite easy and are proven in the folloeing ways.

(A)^{4.12} Concerning $(Imp)_4$ we first note that the relation γ^i is of cotype $(q+1)$ implies

the existence of a q -cocycle $\gamma'^i \in C^i(N(\square_1^q, \square_2), \mathcal{O})$

$$(3.1.31) \quad \gamma(\square_1^q \times \square_2^q) = \gamma'(\square_1^q \times \square_2^q) \square_1^M = \square_1^M, \quad *$$

$$\square_2' = \square_2(\square_1') : \beta_{\square_1}(x_1, \dots, x_{q+1}) = (x_1, \dots, x_q), \beta_{\square_2}(x_1, \dots, x_{q+1}) = x_{q+1}.$$

Thus the problem is reduced to the one dimensional

case. (The existence of γ'^i is a consequence

of the relation : γ^i is of cotype $(q+1)$ and of

the existence of a chain remarked in (3.1.18)).

we find a $q-1$ cochain $\gamma'^{i-1} \in C^{i-1}(N(\square_1^q, \square_2), \mathcal{O})$

so that the equation $\gamma^i - \delta_{\text{eck}}(\gamma'^{i-1}) = 0$ holds. Moreover,

we define a $(q-1)$ -cochain γ'^{i-1} by

$$(3.1.32) \quad \gamma'^{i-1}(\square_1^q \times \square_2^q) = \gamma'^{i-1}(\square_1^q \times \square_2^q)$$

it is clear that $(Imp)_4$ is assured with this

cochain γ'^{i-1} .

(A)¹ The assertion is quite clear and we omit it

* $\square_{v_1}, \square_{v_2}$ are projections $\beta_{\square_{v_1}}, \beta_{\square_{v_2}}$ and the left side of (3.1.31) is the restriction to the set $(\bigwedge_{v=1}^q \square_{v_1}) \times (\bigwedge_{v=1}^q \square_{v_2}) \times \mathcal{O}$.

(III) After the above elementary preparations, we show quickly our assertion: Lemma 3.1. 1'. We start with the following.

(E . L)_K Eliminations of the indices K: There exists a datum $\{ \mathcal{L}, P_1, P_2, L, \tilde{c} \}$, depending on $(N, \mathcal{L})$ only, and a composition of a map $\mathcal{L}$ of e - L -type, $c \in \mathbb{Z}^+$, polynomials $P_i(i=1,2)$, and a linear functions L so that the following assertion is valid.

(B . L)_K¹ There exists a $(q-1)$ -cochain $\gamma^{q-1} \in \mathcal{Q}^{q-1}(N(\mathcal{N}^s(\bar{\Sigma}_{n,N}), \mathcal{O}))$ with algebraic growth (α'_1, α'_2) in such a manner that the relation

(B . L)_K² $\gamma^q - \delta_{\text{Coch}}(\gamma^{q-1})$ is of cotype $(q+1)$,

holds. (Here the right side γ^q is understood to

be an element in $\mathcal{Q}^q(N(\mathcal{N}^s(\bar{\Sigma}_{n,N}), \mathcal{O}))$ by means of a suitable refinement map.)

Moreover, quantities (δ') , (α'_1, α'_2) are determined

by

$$\delta' = \delta' = 2(\delta), \quad (\alpha'_1, \alpha'_2) = (\alpha_1, \alpha_2) = (40, 40) \\ 3 \rightarrow 1 - 40$$

$$\begin{aligned} (\delta'_1) &= \mathcal{L}(\delta), \quad \alpha_1 = \mathbb{P}_1(\alpha_1, \delta_1) e^{\mathbb{P}_1(\alpha_1, \delta_1)} \min(\alpha_1, \delta_1)^{-1}, \\ \alpha'_2 &= \mathcal{L}(\alpha_2 + \delta_2). \end{aligned}$$

In concluding this part we note the following :

For a given couple (δ) , we define an another type

of an open covering of $\mathbb{C}^n \times U$: Namely, by $\mathcal{U}^s(\mathbb{C}^n \times U)$

(Note that we are writing $\mathbb{C}^n \times U$ instead of)

the set of all the open sets $\{\square^s(\mathbb{P}_\alpha) \times U\}_{\alpha \in \mathbb{N}}$. Assume

that a cocycle $\mathcal{Y}^i \in \mathcal{K}^s(\Sigma_{n,K})$ of type $(q+1)$ is given.

Then there exists an uniquely defined cocycle $\tilde{\mathcal{Y}}^i \in \mathcal{K}^s(\mathbb{C}^n \times U)$

so that the equations

$$(2.1.31) \quad \tilde{\mathcal{Y}}^i(\cap \square^s_{I,J,K}) = \mathcal{Y}^i(\cap \square^s_{I,J,K}) \quad \text{in } \mathbb{C}^n \times U$$

for any indices (I, J) and K , hold.

This uniquely determined cocycle $\tilde{\mathcal{Y}}$ will be called

the restriction of $\mathcal{Y}^i$ to $\mathbb{C}^n \times U$ and will be denoted by

$$\mathcal{Y}^i|_{\mathbb{C}^n \times U}.$$

(E.L. J) Elimination of the index J : Let $\tilde{\mathcal{Y}}^i$ be

a cochain in $G^i(N(\mathcal{K}(\mathbb{C}^n \times U)))$. Such a cochain $\tilde{\mathcal{Y}}^i$ is of type

s if $\tilde{\mathcal{Y}}^i(\cap \square^s_{I,J,K}) = 0$ for any sets $\{I, J, K\}$ so that the

cardinal number of $\{I, J, K\}$ is at most s . A cochain $\tilde{\mathcal{Y}}^i$

is of algebraic growth (α_1, α_2) if the asymptotic

behavior () is true. Then we

know that there exists a datum (P', Q', L', c', d') depending on (N, n, q) only, so that the following are valid.

(E.L.J.) There exists a $(q-1)$ -cochain $\tilde{c}^q(N(\tilde{\mathcal{U}}^{\delta'}(\mathbb{C}^N \times \mathbb{U})), \emptyset)$ of type $(q+1)$ in such a manner that

(E.L.J.)₁ $\gamma^q \delta_{\text{cch}}(\gamma^{q-1})$ is of cotype $(q+1)$, as well as estimations

$$(E.L.J.)_2 \quad |\gamma^{q-1}| \leq \alpha'_1 (|\alpha'_1| + |\alpha'_2| + 1);$$

$$\alpha'_1 = P'_1(\alpha_1, \delta_1) \cdot e^{\frac{P_2(\alpha_2, \delta_2)}{\min(a, b)} - c}, \quad \alpha'_2 = L'(\alpha_2 + \delta_2), \quad \delta = d(\delta)$$

are true.

The above fact is shown by a similar and an easier manner than in (). Therefore we omit any details of them. We also note that if a cocycle

$\tilde{\gamma}^q$ is of cotype $(q+1)$, then we can regard $\tilde{\gamma}^q$ as

an element of $C^q(N(\mathcal{U}_{\frac{1}{2}}^0(\mathbb{C}^N \times \mathbb{U}), \emptyset)$. $\mathcal{U}_{\frac{1}{2}}^0(\mathbb{C}^N \times \mathbb{U})$ is an open covering of $\mathbb{C}^N \times \mathbb{U}$ composed of all the open sets $\{\square(P_r) \times \mathbb{U}\}$. More

precisely there exists a cocycle $\gamma^q(x) \in C^q(N(\mathcal{U}_{\frac{1}{2}}^0(\mathbb{C}^N \times \mathbb{U}), \emptyset)$ so that the following equations are valid for any indices

(I_p) and (J_v).

() =

Thirdly we consider the following

(E.L. I) Elimination of the indices I :

An element $\gamma_{(I)}^q$ in $C^q(N(\mathcal{V}_{\frac{1}{2}k}^0(\mathbb{C}^{N_0})), \mathcal{O})$ is of algebraic growth (∂_1, ∂_2) if an asymptotic behavior () is valid . Then we know the following fact from (), () and an elementary argument like ().

(E.L. I) There exists a datum (P_1, P_2, L, ϵ) , depending on (N, n, q) only, so that the following are true .

(3.1.32) For a cocycle γ^q of growth (∂_1, ∂_2) , there exists a cochain $\gamma_{(I)}^{q-1} \in C^{q-1}(N(\mathcal{V}_{\frac{1}{2}k}^0(\mathbb{C}^{N_0})), \mathcal{O})$ of growth $(\partial_1', \partial_2')$ so that

$$(3.1.33)_1 \quad \delta_{\text{Cech}}(\gamma_{(I)}^{q-1}) = \gamma_{(I)}^q,$$

as well as

$$(3.1.33)_2 \quad |\gamma_{(I)}^{q-1}| \leq \partial_1' \binom{\partial_1' + \partial_2' + 1}{k+1} : \quad \partial_1' = P_1(q) \cdot e^{P_2(\partial_2)}, \quad \partial_2' = L(\partial_2 + \delta_2).$$

hold.

() Now it is easy to deduce the original

lemma from the above arguments : This part is

quite obvious and might be excused . However we

discuss for sureness . Start with a cocycle $\gamma^q \in C^q(N(\mathcal{A}_n^1))$

, θ) of growth (α_1, α_2) . Then, from (E.L.K),

we know the existence of (δ^1) , (α^1) and a

refinement map $\gamma_K: \mathcal{U}(\mathcal{A}_n^1) \hookrightarrow \mathcal{U}(\mathcal{A}_n^1)$ so that the following

equation

$$(3.1.) \quad \gamma_K(\gamma^1) = \delta_{\text{cocl}} \gamma_1^{q-1} + \gamma_1^q$$

holds. Here $\gamma_1^{q-1}, \gamma_1^q$ are in $C^{q-1}(N(\mathcal{A}_n^1))$ and (δ^1) , (α^1) are

determined by (E, E, J) . Restrict γ_1^q to $C^N \times U$,

and apply (E.L.J.) . Then we obtain a cochain $\tilde{\gamma}^{q-1}$ in $C^{q-1}(N(\mathcal{A}_n^1))$

so that

$$(3.1.) \quad \tilde{\gamma}_J(\tilde{\gamma}_1^q) - \delta_{\text{cocl}}(\tilde{\gamma}_2^{q-1}) = \tilde{\gamma}_2^q$$

holds, where $\tilde{\gamma}_1^q$ is of cotype $(q+1)$ (w. r. t. J) ; $\tilde{\gamma}_2: \mathcal{U}(\mathcal{A}_n^1) \rightarrow \mathcal{U}(\mathcal{A}_n^1)$. Define a refinement map $\gamma_J: \mathcal{U}(\mathcal{A}_n^1) \hookrightarrow \mathcal{U}(\mathcal{A}_n^1)$ so

that the following conditions are valid.

$$(3.1.) \quad \gamma_J(\square_{IJK^2}) = \square_{IJK^1}, \quad \tilde{\gamma}_J(\square_{IJK^2 \times U}) = \square_{\tilde{I}\tilde{J}^1 \times U} \Rightarrow$$

$$(I^1, J^1) = (\tilde{I}^1, \tilde{J}^1)$$

For a couple (δ_1, δ_2) , $\mathcal{U}^{\delta_1}(\mathcal{A}_n^1)$ means the open

covering of $\tilde{\mathcal{Z}}_{n,N} = C^N \times U$ of open sets $\{\square_{IJK \cap U}\}_{IJK}$.

3-1-44.

() Now it is quite easy to deduce the lemma from

the above three steps : (E.L.I), (E.L.J) and (E.L.K).

We start with a cocycle $\psi^q \in C^q(N(\mathcal{U}(\mathbb{Z}_n), \mathcal{V}))$ of algebraic growth (α_1, α_2) .

Then from (E.L.K), we know the existence of (β') , (α'_1, α'_2)

and a refinement map $\tau_1: \mathcal{U}(\mathbb{Z}_n) \hookrightarrow \mathcal{U}(\mathbb{Z}_{n'})$ so that the following relation

$$(3.1.36) \quad \tau_1^*(\psi^q) = \delta_{\text{cch}}(\psi^{q-1}_{(1)}) = \psi^q_{(1)} \text{ is of type } (q+1),$$

is valid. In the above $\psi^{q-1}_{(1)}$ is a cochain in $C^{q-1}(N(\mathcal{U}(\mathbb{Z}_{n'}), \mathcal{V}))$ of

algebraic growth (α'_1, α'_2) . Of course (α'_1, α'_2) is determined by

(E.L.K). We write the uniquely determined cocycle $\psi^q_{(1)}|_{\mathcal{U} \times \mathcal{V}}$ in

$C^q(N(\mathcal{U}(\mathbb{Z}_{n'}), \mathcal{V}))$ by (3.1.32). Apply (E.L.J) to obtain a

a cochain $\psi^{q-1}_{(2)}|_{\mathcal{U} \times \mathcal{V}}$ in $C^{q-1}(N(\mathcal{U}(\mathbb{Z}_{n'}), \mathcal{V}))$ so that the relation

$$(3.1.37) \quad \tau_2^* \psi^q_{(1)}|_{\mathcal{U} \times \mathcal{V}} = \delta \psi^{q-1}_{(2)}|_{\mathcal{U} \times \mathcal{V}} = \psi^q_{(2)} \text{ is of type } (q+1)$$

is valid in the sense of (E.L.J), $\therefore \mathcal{U}(\mathbb{Z}_{n'}) \hookrightarrow \mathcal{U}(\mathbb{Z}_{n''})$, with a suitable refinement map τ_2 .

Then we define a refinement map $\tau: \mathcal{U}(\mathbb{Z}_n) \hookrightarrow \mathcal{U}(\mathbb{Z}_{n''})$ so that

the following condition is valid.

$$(3.1.38) \quad \text{If } \tau_2^*(\psi^{q-1}_{(2)}|_{\mathcal{U} \times \mathcal{V}}) = \psi^{q-1}_{(2)}|_{\tau^{-1}(\mathcal{U} \times \mathcal{V})}, \text{ then the}$$

relation

$$(\tau, \tau') = (\tau', \tau') : \tau_2^*(\psi^{q-1}_{(2)}|_{\tau^{-1}(\mathcal{U} \times \mathcal{V})}) \hookrightarrow \psi^{q-1}_{(2)}|_{\tau^{-1}(\mathcal{U} \times \mathcal{V})}$$

is valid.

We let $\square_{IJK}^{*,\delta} = \square_{IJK}^{\delta} \cap \mathcal{U}^N$. For a refinement map $r_j: \mathcal{U}^{\delta_2} \hookrightarrow \mathcal{U}^{\delta_1}$

, we define a map $\gamma_j(x): \mathcal{U}^{*,\delta_2}(\mathbb{C}^N \times \mathcal{U}) \rightarrow \mathcal{U}^{*,\delta_1}(\mathbb{C}^N \times \mathcal{U})$ by

$$(3.1) \quad \gamma_j(x)(\square_{IJK}^{*,\delta_2}) = \gamma_j(\square_{IJK}^{\delta_2})$$

Moreover define cochains $\psi_2^{*,i-1}, \psi_2^{*,i} \in C^{*,i}(N(\mathcal{U}^N), \mathcal{U})$ by the equations

$$(3.1) \quad \psi_2^{*,i}(\cap_j \square_{IJK}^{\delta_2} \cap \mathcal{U}) = \tilde{\psi}_2(\cap_j \square_{IJK}^{\delta_2} \times \mathcal{U})$$

Then we obtain the following equation

$$(3.1) \quad \gamma_j(x) \cdot \gamma_j^*(\psi_2^{*,i}) = \delta_{\text{cech}}(\gamma_j(x)(\psi_1^{*,i-1})) + \gamma_j(x) \psi_1^{*,i} \\ = \delta_{\text{cech}}(\gamma_j(x)(\psi_1^{*,i-1})) + \delta_{\text{cech}}(\psi_2^{*,i-1}) + \psi_2^{*,i}$$

Finally apply (E.L.I) to $\tilde{\psi}_2^{*,i}(\mathbb{C})$. Then we obtain

a cochain $\tilde{\psi}_2^{*,i}(\mathbb{C}) \in C^{*,i}(N(\mathcal{U}^N(\mathbb{C} \times \mathcal{U})), \mathbb{C})$ so that

$$(3.1) \quad \tilde{\psi}_2^{*,i}(\mathbb{C}) = \delta_{\text{cech}}(\tilde{\psi}_3^{*,i-1}(\mathbb{C}))$$

is valid. Quantitative properties of $\tilde{\psi}_3^{*,i}(\mathbb{C}) : (\mathbb{C}, \mathcal{U})$ are determi

ned by (E.L.I)

ned by (E.L.I). Finally we proceed as follows :

Define a covering $\mathcal{U}_\varepsilon^{*,\delta}(\mathbb{C}^N \times \mathcal{U})$ of $\mathbb{C}^N \times \mathcal{U}$ to be

$$\mathcal{U}_\varepsilon^{*,\delta}(\mathbb{C}^N \times \mathcal{U}) (\subset \mathcal{U}^{*,\delta}(\mathbb{C}^N \times \mathcal{U})) = \{ \square_{IJK}^{\delta_2,*}, P_{\varepsilon,x}(\square_{IJK}^{\delta_2}) \subset \square_{IJK}^{\delta_2}(P_I) \}$$

The map $i(x)$ stands for the injection map : $i(x): \mathcal{U}_\varepsilon^{*,\delta}(\mathbb{C}^N \times \mathcal{U})$

$\hookrightarrow \mathcal{U}^{*,\delta}(\mathbb{C}^N \times \mathcal{U})$. Then, from (), we obtain the

following equation

$$() \quad i(x)^* \cdot \gamma_j^*(x) \cdot \gamma_j^*(\psi_2^{*,i}) = \delta_{\text{cech}}(i(x)^*(\psi_1^{*,i-1})) + \delta_{\text{cech}}(i(x)^*(\psi_2^{*,i-1})) \\ + i(x)^*(\psi_2^{*,i})$$

Moreover define a cochain $\psi_3^{i-1} \in C^i(N(\pi_2^*(\mathcal{O}), \mathcal{O}))$ by $\psi_3^{i-1}(\pi_{1,2}^*(x)) = \psi_2^{i-1}(x)$.

Then we know the relation $\mathcal{L}(x) \cdot \psi_2^i(x) = \delta_{\text{cch}}(\psi_3^{i-1})$.

Then, finally we obtain an equality of the following form.

$$(3.1) \quad (\mathcal{L}(x) \cdot \pi_2^* \cdot \pi_k^*(\psi^i)) = \delta_{\text{cch}}(\psi_2^{i-1,*})$$

where $\psi_2^{i-1,*}$ is an element in $C^{i-1}(N(\pi_2^*(\mathcal{O}^n \times \mathcal{O}), \mathcal{O}))$

of algebraic growth $(\tilde{\alpha}_1, \tilde{\alpha}_2)$. Quantities

$(\tilde{\alpha}_1, \tilde{\alpha}_2)$ are clearly of the form $(\quad)$.

our assertion (of both lemmas 3.1 and 3.1.1' follow from

(3.1) easily. q.e.d.

n. . Next we consider a couple (U, D) of a bounded domain U in $\mathbb{C}^n$ and a divisor D in U defined by $h = 0$. A canonical lifting $\tilde{U-D}$ of

$U - D$ in $\sum_{n,i} = \mathbb{C}^n \times U$ is a divisor in $\mathbb{C} \times U$ defined by $S = 1 - w \cdot h(x)$. Fix

a point P in D and by P we mean a point near P . Now we show the following

Lemma 3.2.3. There exists a datum $(\quad)$ depending on (D, h) only in such a manner that the following are valid.

For a small positive number r , define an open

covering $\mathcal{U}^\delta(\Delta(P) - U)$ by $\mathcal{U}^\delta(\Delta(P) - U) = \{\Delta_\delta(q; b) \mid q \in \overline{\Delta(P) - U}\}$. An element $\gamma^\delta \in C^2(N(\mathcal{U}^\delta(\Delta(P) - U)))$ is of algebraic growth (α_1, α_2) if γ^δ is estimated as

$$(3.1) \quad |\gamma^\delta(q)| \leq \alpha_1 \cdot d(q; D)^{-\alpha_2}$$

Now we show the following

Lemma 3.1.3. There exists a datum (r_0, d, M_1, M_2, P, L)

, depending on (D, h) only so that the following
are true.

(3.1) For a given cocycle $\gamma^\delta \in C^2(N(\mathcal{U}^\delta(\Delta(P) - U)))$
of algebraic growth (α_1, α_2) there exists a
cochain $\gamma^{\delta-1} \in C^1(N(\mathcal{U}^{\delta'}(\Delta(P) - U)))$ of algebraic growth (α'_1, α'_2)

so that

$$(3.1)_1 \quad \gamma^\delta = \delta_{\text{Coch}}(\gamma^{\delta-1})$$

and

$$(3.1)_2 \quad \gamma' = M_1(\gamma), \quad \delta' = L(\delta), \quad \alpha'_1 = M_2(\alpha_1) \cdot P(\alpha_1, \delta) e^{P(\alpha_1, \delta)}, \quad \alpha'_2 = L(\alpha_2)$$

hold, so far as $r < r_0$ is valid.

(Remark) In the above our assertion of independence of data $(\delta', (\alpha'))$ on points P is important.

This will be called uniformity of v.a.

3-1-47

Proof. The above fact is easily shown by means

of lemma 3.1.1. and the inequality of Lojasiewicz .

(i) Note that the derivative $\frac{\partial w}{\partial x} : w = \frac{1}{h(x)}$ is estimated

as

(3.1) $\left| \frac{\partial w}{\partial x} \right| \leq a_1 d(\theta, D)^{-a_2}$: a_1, a_2 are depending on D only.

For a point $\tilde{Q}$ on $\widetilde{U - D}$, we consider a neighbourhood $\Delta_{r,r}(\tilde{Q}) = \Delta_r(R\theta) \times \{w : |w - w(\theta)| < r^2\}$ rather than $\Delta_x(\tilde{Q})$.

A neighbourhood $\Delta_{r,r}(\tilde{Q})$ is adequate if $\widetilde{U - D}$ is

parametrized on $\Delta_{r,r}(\tilde{Q})$ as $w = \frac{1}{h(\theta)}$. If $\Delta_{r,r}(\tilde{Q})$ is adequate then

there exist a unique projection $\tilde{\pi} : \Delta_{r,r}(\tilde{Q}) \rightarrow \Delta_{r,r}(\tilde{Q}) \cap \widetilde{U - D}$ so that

$\pi(\theta) = \pi \cdot \tilde{\pi}(\theta)$ holds. From (3.1) it is clear

that the following is valid.

(3.1) There exists a couple (δ_1^0, δ_2^0) , a monomial M in such a manner that the statement below is true.

(3.1) If $r < \tilde{\delta}_1^0 |\tilde{w}(\theta)|^{\tilde{\delta}_2^0}$ then $\Delta_{r,r}(\tilde{Q})$ is adequate : $\gamma = M(r)$. Here (δ_1^0, δ_2^0) , M are depending on only. Take a point P (near P_0) and consider a neighbourhood $U_r(P)$. Given a cocycle $\gamma^i \in N(U_r(P))$ of

algebraic growth (∂_1, ∂_2) , we associate a cochain $\tilde{\gamma}^i \in C^i(N(U_r(P)))$

4) as follows .

$$(3.1) \quad \tilde{y}(\tilde{\Delta}_i(\tilde{Q}_r)) = \pi^* y(\tilde{\Delta}_i(0,0)) ,$$

where $\tilde{Q}$'s are points on $\widetilde{U-D}$ while Q 's are projections of $\tilde{Q}$'s : $\tilde{Q} = \mathcal{L}(Q)$ with $\mathcal{L}$ depending on (D) only.

From the inequality of Lojasiewicz , we obtain estimations

$$(3.1) \quad |\tilde{y}^2| \leq d_1' |w|^{d_2'} : (\Delta_{1,3} d_2') \tilde{\Delta} = \mathcal{L}(Q, d_2)$$

with $(\mathcal{L})$ depending on (D, h) only .

Fix neighbourhoods $\mathbb{T}_{\delta, \delta}(\tilde{Q}) : Q \in \widetilde{U-D}$ and let $\mathbb{T}_{\delta, \delta}^*(\widetilde{U-D})$ be the union $\bigcup_{Q \in \widetilde{U-D}} \mathbb{T}_{\delta, \delta}(Q)$. Then it is easy to see the following

(3.1) There exist maps $\mathcal{L}^1, \mathcal{L}^2$ so that if Q is in $\mathbb{T}_{\delta, \delta}(\widetilde{U-D}) : \delta^1 \mathcal{L}(Q)$ then there exists a point $\tilde{Q} \in \widetilde{U-D}$ with a property : $\Delta_{\delta}(\tilde{Q}) \supset \Delta_{\delta}(Q) : \mathcal{L}^1(\tilde{Q}) = \delta^1$.

On the ptherhand it is also easy to see the following

(3.1) There exists a map $\hat{\mathcal{L}}$ with which we know the satement below .

() For a point $Q \in \mathbb{T}_{\delta}(\widetilde{U-D}), \mathbb{T}_{\delta}^*(\widetilde{U-D})$,

$T_{\tilde{s}}(Q) \cap T_{\tilde{s}'}(\widetilde{U-D}) = \phi : \tilde{s}' \xrightarrow{\hat{\phi}} \tilde{s}$ and moreover, at any point $Q' \in T_{\tilde{s}'}(Q)$, an estimation of the form : $\tilde{s}(Q) \geq \hat{a}_1 \hat{a}_2$

here $(\hat{a}_1, \hat{a}_2)$ is given in a form : $(\hat{a}_1, \hat{a}_2) = \hat{\phi}'(s'_1, s'_2)$.

In the above one can take $(\hat{\phi}, \hat{\phi}, \hat{\phi})$ to be absolutely independent while $\hat{\phi}$ is depending on $(0, 1)$ only

After the above observation, we proceed as follows :

(3.1) For a given cocycle of algebraic growth $(\tilde{s}, \tilde{s}')$, define a cochain in the following manner .

(3.1)₁ For points $\tilde{Q} \in \widetilde{U-D}$, fix a set of neighbourhoods $T_{\tilde{s}}(\tilde{Q}) : \tilde{s} = \hat{\phi}(s)$,

(3.1)₂ For a set $T_{\tilde{s}}(\widetilde{U-D}) - T_{\tilde{s}}$, define a map $\hat{\phi} : Q \in T_{\tilde{s}}(\widetilde{U-D}) \rightarrow \tilde{Q} \in T_{\tilde{s}}(\widetilde{U-D}) : \tilde{s}' = \hat{\phi}'(\tilde{s})$, so that

$T_{\tilde{s}'}(Q) \subset T_{\tilde{s}}(Q)$ holds : $\tilde{s}' = \hat{\phi}'(\tilde{s})$

(3.1)₃ For a point Q' in $\sum_{l,n} T_{\tilde{s}}(\widetilde{U-D})$, define a neighbourhood $T_{\tilde{s}'}(Q')$ so that (3.1)₁

is valid with $(\tilde{s}')$ in $T_{\tilde{s}}(\widetilde{U-D})$. Also we

can assume that $T_{\tilde{s}'}(Q'') \cap T_{\tilde{s}}(Q) = \phi$ if $Q'' \in \sum_{l,n} T_{\tilde{s}}(\widetilde{U-D})$, $Q \in T_{\tilde{s}}(\widetilde{U-D})$. 3-1-50

of course we can assume that $\mathcal{L}, \mathcal{L}^1, \mathcal{L}^2, \mathcal{L}^3$ are depending on (h, D) only and is particularly independent

of $\mathcal{P}$: Define a cochain $\hat{\varphi}^1 \in C^1(N(\mathcal{U}^1(\mathcal{Z}_{n,n}), \mathcal{O}))$

($\hat{\varphi}_+ = \min ((\hat{\varphi}_2), (\hat{\varphi}_3))$) in the following manner.

$$\begin{aligned} (3.1.1)_1 \quad \hat{\varphi}^1(q_0, \dots, q_i) &= i^* \hat{\varphi}^1(q_0, \dots, q_i) \text{ if } q_0, \dots, q_i \in \mathcal{P}_i(\widehat{U-D}), \\ (3.1.1)_2 \quad \hat{\varphi}^1(q_0, \dots, q_i) &= 0, \text{ otherwise.} \end{aligned}$$

Take the coboundary $\hat{\varphi}^{i+1} = \delta_{\text{cech}}(\hat{\varphi}^i)$ of $\hat{\varphi}^i$. The cocycle condition imposed on $\hat{\varphi}^i$ implies that $\hat{\varphi}^{i+1}$ is zero on $\widehat{U-D}$.

More precisely the following equations are valid.

$$\begin{aligned} (3.1.2)_1 \quad \hat{\varphi}^{i+1}(q_0, \dots, q_{i+1}) &= 0 \text{ if } q_0, \dots, q_{i+1} \text{ are in } \mathcal{P}_i(\widehat{U-D}), \\ (3.1.2)_2 \quad \hat{\varphi}^{i+1}(q_0, \dots, q_{i+1}) &= \hat{\varphi}^{i+1}(q_0, q_{i+1}) \text{ if a point } q_0 \text{ is in} \end{aligned}$$

but other points $q_1, \dots, q_i$ are outside $\widehat{U-D}$.

$$(3.1.3) \quad \hat{\varphi}^{i+1}(q_0, \dots, q_{i+1}) = 0 \text{ otherwise.}$$

In the above $\hat{\varphi}^{i+1}(q_0, \dots, q_{i+1})$ with $(\hat{\varphi}_1, \hat{\varphi}_2) = (\hat{\varphi}_1, \hat{\varphi}_2) \in \mathcal{L}(\mathcal{Z}_{n,n})$

It is clear that the collection $\{ \hat{\varphi}^{i+1}(q_0, q_{i+1}), 0 \}$ as

above defines a cocycle $\hat{\varphi}^{i+1} \in C^{i+1}(N(\mathcal{U}^1(\mathcal{Z}_{n,n}), \mathcal{O}))$. Apply lemma 3.1.1. to

decompose $\hat{\varphi}^{i+1}$ as

$$(3.1.4) \quad \hat{\varphi}^{i+1} = \delta_{\text{cech}} \hat{\varphi}^i$$

Then $\hat{\varphi}^i = \hat{\varphi}^i + \delta_{\text{cech}} \hat{\varphi}^{i-1}$ is again cocycle and we apply

lemma 3.1.31. once more. We obtain the equation

$$(3.1.5) \quad \hat{\varphi}^i = \delta_{\text{cech}}(\hat{\varphi}^{i-1}) \quad 3-1-5 /$$

Restricting the above equation to $\overline{U - D}$ and

using the biholomorphic property of $\pi: \widetilde{U-D} \rightarrow U-D$, our proof is finished.

Next we discuss asymptotic behaviors concerning coherent sheaves: We start with a domain U in $\mathbb{C}$ and a variety V in U . Assume that the dimension of V is $n-1$. Moreover we assume that a set of functions $(g_1, \dots, g_L)$ vanishing on V and a subvariety V' of V is given so that the following condition

$$(J) \quad \frac{\partial(g_1, \dots, g_L)}{\partial(x_{n+1}, x_n)} \neq 0$$

is valid at each point $P \in V - V'$.

In a small neighbourhood $T_\delta(V - V')$ of $V - V'$, a holomorphic function h in $T_\delta(V - V')$ is expanded in the form

$$(3.1) \quad h = \sum_J h_J g^J$$

where h_J 's are holomorphic functions on $V - V'$.

We mean by $V(h)$ the order of h a.r.t. V :

$V(h) = \min_{J \neq 0} h_J \cdot h_J \neq 0$. We say that h is of meromorphic type if

h_J 's are meromorphic on $V - V'$ (More strictly,

there exists a divisor D in U so that h_j 's are restrictions of meromorphic functions in $U - D$ on V)

Such a function h is of growth (α_1, α_2) if the following estimation

$$(3.1) \quad |h(\alpha)| \leq \alpha_1 d(\alpha, V)^{-\alpha_2}$$

is valid. If a function h is of meromorphic type and

if an expansion of the form (3.1) is valid with a suitable

(α_1, α_2) , we say that h is of type (α) .

Given a matrix K of type (α) (i.e. whose

coefficients are of type (α)), we examine a simple

property of ~~matrix~~ K .

An elementary observation

which we make here is as follows :

by $\mathcal{O}_P^*$ we

mean the sheaf over $V - V$ whose stalk at P is

characterized by $\mathcal{O}_P^* = \{ h_P^i = \text{the germ of a holomorphic} \}$

function in $T_i(V : V')$. The sheaf $\mathcal{K}$ is the

kernel sheaf over $V - V$ of the equation $K \cdot f = 0$.

The first simple assertion which we use later is as follows

(3.1) There exists a couple (α, β) a subvariety V' of V , polynomials

P_1, P_2 and a linear function L (all these

data are depending on K only) with which we know

the following

(3.1) There exists a finite set of functions

$\varphi_1, \dots, \varphi_n \in \Gamma(U, \mathcal{O}_U)$ of type (a) which generate the stalk at each point P in $V \rightarrow V'$.

(3.1) $\varphi_1, \dots, \varphi_n$ are estimated as

$$|\varphi_j(q)| \leq \alpha'_j d(q, V')^{\alpha'_j} : \alpha'_1 = R(\alpha_1, s_1) e^{\alpha_1}, \alpha'_2 = L(\alpha_2, s_2)$$

with

The above statement is concerned with a simple property of the sheaf $\mathcal{O}_U$. We ask an equation $K \varphi = \varphi$ with a given element φ . Concerning such a problem we use the following simple

(3.1) There exists a datum $(\mathcal{O}, V'', M_1, M_2, R, R_2, L)$ with which the following are valid.

(3.1) For each point $P \in V \rightarrow V''$ and an element $\varphi \in \mathcal{O}_P$, defined already in $\Delta(\Delta(\mathcal{O}) \mathcal{O})$ and satisfying

the relation $\varphi \in K(\mathcal{O}_P)$, we find an element $\varphi' \in \Gamma(\Delta(\mathcal{O}) \mathcal{O})$

with the following properties

$$(3.1) \quad K \varphi' = \varphi,$$

$$((3.1)) \quad |\varphi| \leq \alpha'_j d(q, V')^{\alpha'_j} |\varphi|$$

with

$$\alpha'_1 = M_1(\alpha_1), \quad \alpha'_2 = M_2(\alpha_2) e^{\alpha_2}, \quad \alpha'_3 = L(\alpha_3)$$

() $\varphi_1, \dots, \varphi_n$ are estimated as

3-1-54

The above two statements are easily shown along standard

lines to argue syzygy or coherentneses of holomorphic

function, namely let k be the number of the equation

of K ($=$ the length of K), n be the dimension of V

and i be the dimension associated to $T^i(\quad)$. It

is easy to see (3.1) and (3.1) inductively on these indices,

and we shall omit arguments. ~~Assume~~ Assume that a divisor D

in U is given and that a matrix K whose coefficients

are meromorphic in U (with the pole $D: h=0$). Further-

more assume that a variety $V (\supset D)$ is given.

Now we use the following elementary

Proposition . 3.1.2. There exists datum $(s^0, Q^0$

M) , depending on (D, K) , with which the following

are valid.

(() For a point Q and for an element $y^i \in \Gamma(\Delta_r(\theta); \mathcal{O}_Q^{\otimes r})$; $r < s^0 d(Q, V)^{s^0}$ satisfying the condition

$$y^i \in K \cdot \mathcal{O}_Q^{\otimes r}$$

there exists an element $\eta_a \in \Gamma(\Delta_Q, \mathcal{O}_Q^{\otimes r})$ so that

(3.1)

$$K \cdot \eta_a = y^i$$

and

()

$$|\eta_a| \leq \frac{M(r) \cdot d(Q, V)^{s^0}}{3 - 1 - 55} \cdot |y^i|$$

hold.

Proof

The above assertion follows immediately

from (), () and ().

Let us consider a matrix whose coefficients are meromorphic functions in U (with a pole D), and let

$$() \quad 0 \rightarrow \mathcal{O} \xrightarrow{\mathcal{I}_{q-1}} \dots \rightarrow \mathcal{O} \xrightarrow{\mathcal{I}_q} \mathcal{F} \rightarrow 0$$

be a resolution of $\mathcal{F}$. A cochain $\gamma^i \in C^i(N(\pi^*(\mathcal{A}_x(\mathcal{O})), \mathcal{F})$

is called of growth $(\tilde{\alpha}_1, \tilde{\alpha}_2)$ if has an expression

$$(3) \quad |\gamma^i(q_1, \dots, q_{q-1})| \leq \tilde{\alpha}_1 d(q_1, \dots, q_{q-1})^{\tilde{\alpha}_2},$$

where $\gamma^i(q_1, \dots, q_{q-1})$ is in $\Gamma(\pi^*(\mathcal{O}_{q_1, \dots, q_{q-1}}), \mathcal{O})$ so that, at each

point $q \in \bigcap_{\lambda=0}^i \mathcal{A}_\lambda(q_1, \dots, q_{q-1})$, the relation $\gamma^i(q_1, \dots, q_{q-1}) \in \mathcal{F}_q$ holds.

Then we show the following

Theorem 3.1. (v.a) There exists a datum

$(r, s, \mathcal{O}, P_1, P_2, L)$ depending on (more strictly on

a resolution of) only, so that the following

are valid.

(3.1) For a given cocycle $\gamma^q \in C^q(N(\pi^*(\mathcal{A}_x(\mathcal{O})), \mathcal{F})$

algebraic growth $(\tilde{\alpha}_1, \tilde{\alpha}_2)$, there exists a $(q-1)$ -

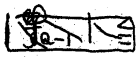
cochain $\gamma^{q-1} \in C^{q-1}(N(\pi^*(\mathcal{A}_x(\mathcal{O})), \mathcal{F})$ of algebraic growth $(\tilde{\alpha}_1', \tilde{\alpha}_2')$

so that the equality

$$(3.1) \quad \gamma^q = \delta_{\text{cch}}(\gamma^{q-1})$$

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and estimations

(3.1.)  $\tilde{\alpha}_1 = \max_{P \in P_0} P(\alpha_1) e^{P_0(\alpha_1)}$, $\tilde{\alpha}_2 = L(\alpha_1, \alpha_2)$
are valid, so far as $r < r_0$, $s < s_0$; P is a point near P_0 .

proof. this assertion is easily proven inductively

on the length of a resolution of . Keys are to
use proposition 3.1.2. in order to reduce the present

problem to lemma 3.1.1.. Because one reduction is done along
standard arguments we do not enter into details.

We derive an immediate corollary of the above theorem.

Let us start with a datum $(X, D, \mathcal{U}_X, (h))$ so

that the condition () is true. For a point P near P_1 ,

an element will be called of growth (α_1, α_2) so if

an estimation

$$() \quad |y^q(\alpha_1, \alpha_2)| \leq \alpha_1 \cdot d(\alpha_2) \alpha_2^{-\alpha_2}$$

is valid. The following is derived from theorem 3.

1.1.

Corollary 3.1.1. There exists a datum $(r, s, M, L, P, R,$

$\mathcal{U})$ with which assertions (), () for

the sheaf $\mathcal{U}$ is valid.

Proof. Here we reduce the assertions in this

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to theorem 3.1. This part is quite similar to procedures in lemma 3.1.2. We choose maps $\alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4$ depending on $(X, D, \mathcal{H})$ only with which the following are valid

(2.1.) If P is in $N_{\delta}(X)$ then there exists a point

$\tilde{P} \in X$ so that $N_{\delta}(\tilde{P}) \supset N_{\delta}(P)$ holds.

(2.1.) If a point P is not in $N_{\delta}(X)$ then $N_{\delta}(P)$ is outside $N_{\delta}(X \cup \partial X)$.

In the above, $(\delta, \delta^2, \delta^3, \delta^4, \delta^5)$ are determined

as $(\delta^k) = \alpha_k(\delta)$. Define a cochain $\psi^* \in C^q(N(X), \mathbb{R})$ as in :

(2.1.) If $Q_1, \dots, Q_q$ are in $N_{\delta}(X \cup \partial X)$,

$$\psi^*(Q_1, \dots, Q_q) = \psi^*(\tilde{Q}_1, \dots, \tilde{Q}_q).$$

() $\psi^{q+1} = 0$, otherwise.

Remark that ψ^{q+1} is zero on X .

Then, from proposition 3.1, we obtain an expression

$$() \quad \psi^*(Q_1, \dots, Q_q) = \sum_i \alpha_i \psi^*(\tilde{Q}_1, \dots, \tilde{Q}_q)$$

where α_i is in $\Delta_{\delta}(\partial X)$ of growth $(\tilde{\alpha}_1, \tilde{\alpha}_2)$.

Here $(\tilde{\alpha}_1, \tilde{\alpha}_2)$ are determined as $(\tilde{\alpha}_1) = \alpha'(\delta)$, $(\tilde{\alpha}_2) = \beta'(\alpha, \delta) \in \mathbb{R}_2^{(h, \delta)} / \mathbb{R}_2^{(h, \delta)}$

with α', β' depending on $(X, D, \mathcal{H})$ only.

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Applying the theorem 3.1. and following the procedure in the lemma 3.1.2, we know easily our assertion. *g.e.d*

We have discussed the cases lemma 3.1.2, corollary 3.1. because these two results are made use of in 3.2. we discuss quickly an another result, ~~which can~~

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3.2 . Cohomology with algebraic division and algebraic growth condition.

The arguments given in the section 3.1 provides a sufficient analytic basis when we are concerned with a theory of differential forms for a triple (X, V, D) where $V - D$ is smooth. On the otherhand, if we ask possible theories of differential forms for the triple (X, V, D) in which $V - D$ is (generally) not smooth, we should ask more materials on analytic sides. The purpose of this section is to propose a cohomology theory for this purpose and discuss our proposed subjects in details. A new essential feature of this section is that the powers of the ideal sheaf $\mathcal{O}_V(-D)$ enters into our problems besides the asymptotic behaviors w.r.t. the pole divisor D .

* In our actual arguments, the powers of ideals are discussed in a somewhat modified form.

$$\{ \varphi_{v_1 \dots v_i}^{m, i-1} \}_{1 \leq v_1 < \dots < v_i \leq l} \quad (\text{i.e. } \mathcal{E}_P^{m, i-1} = \mathcal{O}_P \{ \varphi_{v_1 \dots v_i}^{m, i-1} \}_{1 \leq v_1 < \dots < v_i \leq l}).$$

Especially the submodule $\mathcal{E}_P^{m, 0}$ of $\mathcal{O}_P$ stands for the one : $\mathcal{E}_P^{m, 0} = \mathcal{O}_P (g_1^m, \dots, g_l^m)$. Moreover,

we put $\mathcal{E}_P^{m, l} = \mathcal{O}_P \{ \varphi_{v_1 \dots v_l}^{m, l} \} \cong \mathcal{O}_P$. We define

$\mathcal{O}_P$ - homomorphisms $K(m, i)$ ($i=1, \dots, l$)

by the following formulas.

$$(3.2.1) \quad K(m, i) : \varphi_{v_1 \dots v_i}^{m, i} \rightarrow K(m, i)(\varphi_{v_1 \dots v_i}^{m, i}) = \sum_{1 \leq v_1 < \dots < v_i \leq l} f_{v_1 \dots v_i} \varphi_{v_1 \dots v_i}^{m, i-1}$$

It is quite easy to see that the equation $K(m, i)$,

$K(m, i) = 0$ holds. We will be concerned here with

submodules $\mathcal{E}_P^{m, i}$'s defined above. Let

$\varphi_{v_1 \dots v_i}^{m, i} = (f_{v_1 \dots v_i})_{1 \leq v_1 < \dots < v_i \leq l}$ be a vector in $\mathcal{O}_P^{(i)}$ which

satisfies the following equation.

$$(3.2.2)_1 \quad K(m, i) \cdot \varphi_{v_1 \dots v_i}^{m, i} = 0,$$

We express $f_{v_1 \dots v_i}$ in the following explicit form.

$$(3.2.2)_2 \quad f_{v_1 \dots v_i} = f_{v_1 \dots v_i}^{(m)} \cdot g_1^m + \sum_{\Delta=0}^{m-1} f_{v_1 \dots v_i}^{(\Delta)} g_1^\Delta$$

where $f_{v_1 \dots v_i}^{(\Delta)}$ ($\Delta=0, \dots, m-1$) does not contain

(2)

the term g_1 at all.

Define a vector y^{i+1} by

$$(3.2.3)_1 \quad y_{v_1 \dots v_i}^{i+1} = \begin{cases} f_{v_1 \dots v_i}^{(n)}, & \text{if } (v_1, \dots, v_{i+1}) \\ & = (1, v_1, \dots, v_i) \\ 0, & \text{otherwise.} \end{cases}$$

We put $y^{i+1} = y^i - K(m, i) \cdot y^{i+1}$ and

write the vector y^i in the form: $y^i = (g_{\mu_1 \dots \mu_i})$

It is clear that the equation

$$(3.2.3)_2 \quad g_{\mu_1 \dots \mu_i} = \sum_{\Delta=0}^{m-1} f_{v_1 \dots v_i}^{(\Delta)} \cdot g_1^\Delta$$

holds for $2 \leq v_1 < \dots < v_i \leq l$.

Taking account into the fact that $(\mu_1', \dots, \mu_{i-1}') -$

component $(2 \leq \mu_1' < \dots < \mu_{i-1}' \leq l)$ of the terms $f_{\mu_1' \dots \mu_{i-1}'}^{(\Delta)} \cdot g_{1, \mu_1' \dots \mu_{i-1}'}$

are divisible by g_1^m , we know that the following

equations are valid.

$$(3.2.3)_3 \quad \sum_{2 \leq \mu_1' < \dots < \mu_{i-1}' \leq l} f_{\mu_1' \dots \mu_{i-1}'}^{(\Delta)} \cdot g_{1, \mu_1' \dots \mu_{i-1}'} = 0.$$

Note that the above equality is equivalent to

$$(3.2.3)_4 \quad \sum_{2 \leq v_1 < \dots < v_{i-1} \leq l} f_{v_1 \dots v_{i-1}}^{(\Delta)} \cdot g_{1, v_1 \dots v_{i-1}} = 0 \quad (s=1, \dots, m-1)$$

On the otherhand, we note that if a vector $(g''_{v_1 \dots v_i})_{1 \leq v_1 < \dots < v_i \leq l}$ has the following properties

$$(3.2.4) \quad \sum_{1 \leq v_1 < \dots < v_i \leq l} g''_{v_1 \dots v_i} \cdot g_{v_1 \dots v_i} = 0, \quad g''_{v_1 \dots v_i} = 0 \quad (2 \leq v_1 < \dots < v_i \leq l)$$

then $(g''_{v_1 \dots v_i})_{1 \leq v_1 < \dots < v_i \leq l} = 0$.

From the relations $(3.2.3)_2$, $(3.2.3)_4$ and from

the above remark, we know that inductive

arguments (on the number l of the functions $g_{\mu}^{(i)}$)

lead to the following conclusion.

(3.2.I) The solution spaces of the equation

$$K(m, i) \cdot g_{v_1 \dots v_i}^{(i)} = 0 \quad : \quad \text{are spanned by the vectors}$$

$$\left\{ g_{v_1 \dots v_i}^{(i)} \right\}_{1 \leq v_1 < \dots < v_i \leq l}$$

Thus we obtain the following exact sequence

$$(3.2: I) \quad 0 \rightarrow \mathcal{O}_P^{(l)} \xrightarrow{K(m, 1)} \mathcal{O}_P^{(l)} \xrightarrow{K(m, 2)} \mathcal{O}_P^{(l)} \xrightarrow{K(m, 3)} \mathcal{O}_P^{(l)} \xrightarrow{m, 0} \mathcal{E} \rightarrow 0$$

This sequence will be called a canonical syzygy attached to $\mathcal{E}^{m, 0}$.
(Remark) (The exact sequence (3.2.I)' belongs,

the author believes, to the realm of well

known facts. Because, in an quantitative argument done

soon below, we make use of the above procedures

in details, we made a detailed argument as above.)

n° 1.1 Let us continue elementary arguments in local levels,

in this case quantitative properties will be included.

Given a domain $\mathcal{D}'$ in $\mathbb{C}^{\ell'} = (z'_1, \dots, z'_{\ell'})$ and a

polydisc $\Delta_r(0; (g)) = \{(g) = (g_1, \dots, g_{\ell'}) : |g_{\mu}| \leq r^{(\mu; \ell')}\}$

in $\mathbb{C}^{\ell}$, we assume that additional

holomorphic functions $(g_{\ell+1}, \dots, g_{\ell+\tilde{\ell}})$ in $\mathcal{D}' \times \Delta_r(0; (g))$ are

given. Furthermore we assume that $g_{\ell+\mu}$'s vanish

on the manifold defined by $g_1 = \dots = g_{\ell} = 0$ in $\mathcal{D}' \times \Delta_r^{(\ell; \ell')}(g)$.

We fix the following explicit expressions of $g_{\ell+j}$'s.

$$(3.2.5)_1, \quad g_{\ell+j} = \sum_{\mu=1}^{\ell} g'_{j\mu} \cdot g_{\mu}, \quad \text{where } g'_{j\mu} \text{'s}$$

are holomorphic in $\Delta_r(0; (g))$.

Let B be a positive number with which the inequality

$$(3.2.5)_2, \quad B \geq |g'_{j\mu}|, \quad \text{holds}$$

for each $g'_{j\mu}$ in $\Delta_r \times \Delta_r(0, (g))$.

In an analogous way to (n° 1.1), we let $\gamma_{r_1, \dots, r_{\ell}}^{m, \ell-1}$

$(1 \leq r_1 < \dots < r_{\ell} \leq \ell + \tilde{\ell})$, be the vector in $\mathbb{R}^{\Delta_r \times \Delta_r(0; (g))}$.

which is characterized in the following manner.

$$(3.2.6) \quad \mathcal{F}_{v_1 \dots v_i}^{m, i-1} (1 \leq v_1 \dots \leq v_{i-1}) = \begin{cases} 0, & \text{if } (v_1, \dots, v_{i-1}) \notin (v_1, \dots, v_i) \\ (-1)^{i-1} g_{v_i} (v_1, \dots, v_{i-1}, v_i) \end{cases}$$

Also the submodule $\mathcal{E}^{m, i-1}$ of $\mathcal{O}_{(v_1, \dots, v_i)}^{(l+1)}$ will be defined to be the one spanned by $\mathcal{F}_{v_1 \dots v_i}^{m, i-1}$'s.

Moreover, we define sheaf homomorphisms $K(m, i)$ from $\mathcal{O}_{(v_1, \dots, v_i)}^{(l+1)}$ to $\mathcal{O}_{(v_1, \dots, v_{i-1})}^{(l+1)}$ by

$$(3.2.7) \quad K'(m, i)(f_{v_1 \dots v_i}) = \sum_{1 \leq v_1 < \dots < v_{i-1} \leq l} f_{v_1 \dots v_i} \cdot \mathcal{F}_{v_1 \dots v_{i-1}}^{m, i-1}$$

It is clear that $K(m, i) \circ K(m, i+1) = 0$ is valid.

But it is not true in general that the exact sequence

(3.2.1)' is valid for homomorphisms $K(m, i)$'s.

We ask a simple condition in order that a given

vector $\mathcal{F}^{i-1} \in \Gamma(\mathcal{O}_{(v_1, \dots, v_i)}^{(l+1)}, \mathcal{E}^{m, i-1})$ is in $\Gamma(\mathcal{O}_{(v_1, \dots, v_{i-1})}^{(l+1)}, \mathcal{E}^{m, i-1})$. Assume that a

vector $\mathcal{F}^{i-1} = \{f_{v_1 \dots v_i}\}_{1 \leq v_1 < \dots < v_i \leq l}$ satisfies the following two conditions.

$$(3.2.7)_1 \quad K'(m, i)(\mathcal{F}^{i-1}) = 0, \quad (i=1, \dots, l-1),$$

$$(3.2.7)_2 \quad |f_{v_1 \dots v_i}| \leq \delta \cdot d^{v_i} \quad ; \quad \delta, d \in \mathbb{R}^+, \quad |v_1| = \dots = |v_{i-1}| = d$$

Now we show the following

Proposition. 3. 2. 1. There exist positive

numbers $(\beta'_{\mu}) = (\beta'_{x,i,\mu})$ ($\mu = 1, 2, 3, 4$) and linear

functions $L_{\mu} = \{L_{x,i,\mu}\}_{(x,i)}$ with which the following are

valid.

(3. 2. 8)₁ Data β'_{μ} 's as well as L_{μ} 's
depend on $(\ell, \tilde{\ell}, i)$ only, and so independent of (x)

(3. 2. 8)₂ For a vector $y^i \in P(\bar{\alpha}_x \bar{\Delta}_x(\omega, i), \theta^{(i)})$ satisfying
the conditions (3. 2. 7)₁, (3. 2. 7)₂ we find a vector

$y^{i+1} \in P(\bar{\alpha}_x \bar{\Delta}_x(\omega, i+1), \theta^{(i+1)})$ so that the following two conditions
are valid.

$$(3. 2. 8)_3 \quad K(m, i+1) (y^{i+1}) = (y^i)_{(i=1, \dots, K)}$$

$$(3. 2. 8)_4 \quad |y^{i+1}| \leq (\beta'_1)^{\beta'_2 \alpha'_3} \dots b \cdot \alpha^{\beta'_3 \alpha'_3 - K_2(m)} (|\beta_1| = \dots = |\beta_2| = d)$$

if $\alpha'_3 \geq L_3(m)$ holds.

Proof of Proposition 3.2.1. Our proof will be done

inductively on $\tilde{\ell}$: (i) $\tilde{\ell} = 0$: First we show our

assertion inductively on double indices $(\ell, i' = \ell - 1)$:

Concerning inductive procedures taken here, we note a simple quantitative fact: Let h be a holomorphic function in $\mathcal{O}(\Delta_r(0; \delta))$. We assume the following estimation.

$$(3.2.9) \quad |h|_{|\partial_1|=\dots=|\partial_d|=d} \leq b'' d^{n''}; \quad b'' \in \mathbb{R}^+, \quad n'' \in \mathbb{Z}^+.$$

We expand h in $\mathcal{O}(\Delta_r(0; \delta))$ in the following (unique) way

$$(3.2.9)_2 \quad h = h^{(n'')} \cdot g_1^{n''} + \sum_{j=0}^{n''-1} h_j \cdot g_1^j; \quad h_j (j=0, \dots, n''-1)$$

does not contain the term g_1 at all. At a point $Q \in \Delta_r(0; \delta)$ with coordinates $(j): |\partial_1| = |\partial_2| = \dots = |\partial_d| = d$,

estimations of $h^{(n'')}, h_j$'s are given by

$$(3.2.9)_3 \quad |h_0| \leq \int \frac{|h|}{|\partial_1|} d\partial_1, \quad h_1 \leq \int \frac{h-h_0}{|\partial_1|} d\partial_1, \dots, h_{n''} = \int \frac{h-h_0-h_1 g_1-\dots-h_{n''-1} g_1^{n''-1}}{|\partial_1|} d\partial_1.$$

Then, by a quite simple calculation, we obtain

the following estimation of $h^{(n'')}, h_j$.

$$(3.2.9)_4 \quad |h_j| \leq (4\pi F)^{n''-j} \cdot d^{n''-j} \cdot b'', \quad h^{(n'')} \leq (4\pi F)^{n''} \cdot b''.$$

From $(3.2.9)_4$, we obtain easily an expansion of h

of the following form

$$(3.2.9)_5 \quad h = \sum_{j \in \mathbb{N}} h_j \cdot g^j; \quad j = (j_1, \dots, j_d),$$

$$(3.2.9)_6 \quad |h_j| \leq c_j^{n''} \cdot b''; \quad c_j \in \mathbb{R}^+, \text{ depends on } l \text{ only.}$$

Now it is quite easy to deduce desired results

from (9.2.9)_{4,5,6} and procedures to find $y \in \mathbb{P}(\Delta \times \overline{\Delta^{(m)}}, \theta)^{(b)}$

to be sure, we proceed as follows : Define $y \in \mathbb{P}(\Delta \times \overline{\Delta^{(m)}}, \theta)^{(b)}$
by (9.2.6). Then the vector $y^{n,i+1}$ is defined in

$\Delta \times (\overline{\Delta^{(m)}}, \theta)^{(b)}$ and is estimated in the following way.

$$(9.2.10), \quad |y^{n,i+1}| \leq \ell_2 \cdot d^{d_3-n} \cdot b \quad (C_2 \in \mathbb{R}^+, \text{ depends on } b \text{ only})$$

$n_1 = \dots = n_l = d$

Next, we apply the induction hypothesis (for $l+1$)

to the vector $y^{n,i} = y^{n,i} - K(n,i) \cdot y^{n,i+1}$. Then we know

easily the desired facts (9.2.8)₃, (3.2.8)₄.

(ii) Next we show the assertions (3.2.8)₃

(9.2.8)₄ inductively on $\tilde{l}$.

For this purpose, we write the equation :

$K(m, i) \cdot y^i = 0$: explicitly in the following

form :

$$K(m, i) \cdot y^i = \sum_{1 \leq v_1 < \dots < v_i \leq l} f_{v_1, \dots, v_i} y^{i-1} + \sum_{1 \leq v_1 < \dots < v_i \leq l, u_1} f_{v_1, \dots, v_i, u_1} y^{i-1} + \dots + \sum_{1 \leq v_1 < \dots < v_i \leq l, u_1, \dots, u_{i-1}} f_{v_1, \dots, v_i, u_1, \dots, u_{i-1}} y^{i-1} = 0$$

$1 \leq u_1 < \dots < u_{i-1} \leq \tilde{l}$

Corresponding to the above explicit expression of the equation,

we say that a vector ψ^i is of cotype (a)

if the coefficients $(f_{\nu_1, \dots, \nu_{i-2}, \ell+1, \dots, \ell+M_i})'_s (d > a)$ are zero.

Our (elementary) discussion here will be done by diminishing the cotype (a) of the given vector.

We argue by dividing the cases to $a = 1$, or

$a < 1$. In both cases we assume: $|\psi^i| \leq \tilde{b} \cdot d^{\tilde{d}_i} (|\beta_1| = \dots = |\beta_i| = d)$

(ii)₁ $a = 1$. In this case, the equation is

written in the following way.

$$\sum_{1 \leq \nu_1 < \dots < \nu_{i-2} \leq \ell} f_{\nu_1, \dots, \nu_{i-2}} \cdot \psi_{\nu_1, \dots, \nu_{i-2}}^{i-1} + \dots + \sum_{1 \leq \mu_1 < \dots < \mu_{i-2} \leq \ell} f_{\mu_1, \dots, \mu_{i-2}} \cdot \psi_{\mu_1, \dots, \mu_{i-2}}^{i-1} = 0.$$

Then, from (3.2.9)_{5,6} the coefficients $(f_{\mu_1, \dots, \mu_{i-2}})_{(\mu_1, \dots, \mu_{i-2})}$

are expressed in the manner

$$f_{\mu_1, \dots, \mu_{i-2}} = \sum_j f_{\mu_1, \dots, \mu_{i-2}, j} \cdot \tilde{g}_j^m, \quad \text{in } \mathcal{O} \times \overline{\Delta_{\mathbf{r}}(10): (8)}$$

where the coefficients $f_{j, \mu}$ of $(f_{\mu_1, \dots, \mu_{i-2}})$ are estimated

in the following fashion.

$$(3.2.10), \quad |f_{\mu_1, \dots, \mu_{i-2}, j}| \leq \tilde{c}_{\mu_1}^{\tilde{d}_1} \cdot b \cdot d^{\tilde{d}_2 \cdot d_3 - \tilde{\beta}_{\mu_2} m} \quad (|\beta_1| = |\beta_2| = d)$$

In the above, $(\tilde{c}_{\mu_1}, \tilde{c}_{\mu_2})$ and $(\tilde{\beta}_{\mu_1}, \tilde{\beta}_{\mu_2})$ are depending on

μ only.

(ii) $a < 1$: In this case the explicit

(10)

Define a vector $\mathcal{Y}^{i+1} \in \Gamma(\mathcal{D}' \times \overline{(\Delta_{\mathbf{x}(0)}(0))}, \mathcal{O}^{(l+2)}_{i+1})$

by

$$(\mathcal{Y}, \mathcal{Q}, \mathcal{I}) \quad \mathcal{Y}^{i+1}(\tau_1, \dots, \tau_{i+1}, \mu_1, \dots, \mu_{i+1}) = \begin{cases} \frac{1}{2^{i+1}} \dots \frac{1}{2^{i+1}} \delta & \text{for } (i, l+1, \dots, l+1) \\ 0, & \text{otherwise} \end{cases}$$

Then it is clear that the vector $\mathcal{Y}^{i+1} = \mathcal{Y}^i - \mathbb{I}_{(i, i+1)} \cdot \mathcal{Y}^{i+1}$

$\in \Gamma(\overline{\mathcal{D}' \Delta_{\mathbf{x}(0)}(0)}, \mathcal{O}^{(l+2)}_{i+1})$ is of cotype $(a-1, \dots)$. Quantita.

properties of the vector $\mathcal{Y}^{i+1}$ is easily

known from the above explicit expression of $\mathcal{Y}^{i+1}$.

The above procedure provides a necessary

information in the case of $i = a$.

(11) $a < i$: In this case

the following desired explicit.

(11)

expression of the equation $K(m, i)(\psi^i) = 0$ is as follows

$$(3.2.10)' \quad K(m, i)(\psi^i) = \sum_{1 \leq v_1 < v_2 \leq l} f_{v_1 v_2} \psi_{v_1 v_2}^{i-1} + \dots + \sum_{\substack{1 \leq v_1 < v_2 < \dots < v_a \leq l \\ 1 \leq \mu_1 < \dots < \mu_a \leq l}} f_{v_1 v_2 \dots v_a \mu_1 \mu_2 \dots \mu_a} \psi_{v_1 v_2 \dots v_a \mu_1 \mu_2 \dots \mu_a}^{i-1} = 0.$$

Consider the $(v_1, \dots, v_{i-a-1}, l+\mu_1, \dots, l+\mu_a)$ - components of

the above equality. Then we obtain

$$(3.2.10)' \quad \sum_{v_0 < v_1} f_{v_0 v_1} \psi_{v_0 v_1}^m - \sum_{v_1 < v_2} f_{v_1 v_2} \psi_{v_1 v_2}^m + \dots + (-1)^{i-1} \sum_{v_{i-1} < v_i} f_{v_{i-1} v_i} \psi_{v_{i-1} v_i}^m = 0.$$

On the other hand, note that the equation $K(m, i-a)((\tilde{f}_{v_1 \dots v_{i-a}}))$

$(1 \leq v_1 < v_2 \leq l)$ is equivalent to the following

$$(3.2.10)'' \quad \sum_{v_0 < v_1} \tilde{f}_{v_0 v_1} \psi_{v_0 v_1}^m - \sum_{v_1 < v_2} \tilde{f}_{v_1 v_2} \psi_{v_1 v_2}^m + \dots = 0 \quad (\text{for each } 1 \leq v_1 < v_2 \leq l).$$

Thus we obtain, for each set of indices

$(\mu_1, \dots, \mu_a)$, a vector $(f_{v_1 \dots v_{i-a+1}, l+\mu_1, \dots, l+\mu_a})_{1 \leq v_1 < v_{i-a+1} \leq l}$

satisfying the equation: $K(m, i-a+1)(\tilde{f}_{v_1 \dots v_{i-a+1}, l+\mu_1, \dots, l+\mu_a}) = (\tilde{f}_{v_1 \dots v_{i-a+1}, l+\mu_1, \dots, l+\mu_a})_{1 \leq v_1 < v_{i-a+1} \leq l}$

Quantitative properties of this vector $(\tilde{f}_{v_1 \dots v_{i-a+1}, l+\mu_1, \dots, l+\mu_a})_{1 \leq v_1 < v_{i-a+1} \leq l}$

known from the case of $\tilde{f} = 0$: (i.e. the

case of $a = 0$) : Namely the vector $(\tilde{f}_{v_1 \dots v_{i-a+1}, l+\mu_1, \dots, l+\mu_a})_{1 \leq v_1 < v_{i-a+1} \leq l}$ is

estimated in the manner

$$(3.2.10)''' \quad |\tilde{f}_{v_1 \dots v_{i-a+1}, l+\mu_1, \dots, l+\mu_a}| \leq \tilde{\beta}_1'' \tilde{\beta}_2'' \tilde{\beta}_3'' \dots \tilde{\beta}_d'' \quad (|\beta_1| = \dots = |\beta_d| = d)$$

Here data $(\tilde{\beta}_1'', \tilde{\beta}_2'', \tilde{\beta}_3'', \dots, \tilde{\beta}_d'')$ are those ones depending

on (l, i, a) only.

We define a vector $\mathcal{F} \in \Gamma_{(0, \Delta_x(0,0))}^{(k)}(\mathcal{O})$ by

$$(3.2.10)' \quad \mathcal{F}^{i+1}(x_1, \dots, x_{i+1}, l + \mu_1, \dots, l + \mu_{i+1}) = \begin{cases} f_{x_1, \dots, x_{i+1}, l + \mu_1, \dots, l + \mu_{i+1}} & \text{for } (x_1, \dots, x_{i+1}, l + \mu_1, \dots, l + \mu_{i+1}) \\ 0, & \text{otherwise} \end{cases}$$

It is clear that the vector $\mathcal{F}^{i+1}_{K(m, i+1)}$ is of cotype $(a-1)$.

(iii) Now it is easy to see that the results in (i) and (ii) yield the desired results $(3.2.8)_{3,4}$. q.e.d.

Remark. Assume that one of functions g_i , for example g_1 , is not zero (at any points) in $\mathcal{D} \times \overline{\Delta}_x((0), (q))$. In this case, a similar consideration as in Proposition 3.2.1 becomes quite simple: Namely, let $\mathcal{F}^i$ be a vector in $\Gamma(\mathcal{D} \times \overline{\Delta}_x((0), (q)), \mathcal{O}^{(k+i)})$ which satisfies the equation

$$K(m, i) \mathcal{F}^i = 0.$$

Define a vector $\mathcal{F}^{i+1} \in \Gamma(\mathcal{D} \times \overline{\Delta}_x((0), (q)), \mathcal{O}^{(k+i+1)})$ by

$$(3.2.11) \quad \mathcal{F}^{i+1}(x_1, \dots, x_{i+1}) = \begin{cases} g_1 \times f_{x_1, \dots, x_i} & \text{for } (x_1, \dots, x_i), \\ 0, & \text{otherwise.} \end{cases}$$

Then we know that the following equation

$$(3.2.11) \quad \mathcal{F}^i - K(m, i+1) \mathcal{F}^{i+1} = 0,$$

holds.

n. 2. In this numero we shall formulate

our basic problems in this section. Let U be

a domain in $\mathbb{C}^n$, and let P_0 be a point in

U . These data (U, P_0) are assumed to be given

throughout in this section. When there is no fear

of confusions, we omit these data (U, P_0) .

A basic datum in this section will be explained as

follows. By X we mean a variety $(\ni P_0)$ in U .

The variety X_s stands for the singular locus of X

(in U). Also assume that a proper subvariety

$V(\ni P_0)$ of X is given. The irreducible

decomposition of $\bar{X}_{P_0}$, $X_{P_0, \lambda}$ as well as $V_{P_0}, \dots$ will be

written in the following form.

$$X_{P_0} = U_{\delta} X_{P_0, \delta}, \quad \bar{X}_{P_0} = U_{\delta} \bar{X}_{P_0, \delta}, \quad V_{P_0} = U_{\delta} V_{P_0, \delta}$$

the conditions () and () are assumed i.e.

$$V_{P_0, \delta} \neq \bar{X}_{P_0, \delta}' \text{ for a pair } (V_{P_0, \delta}, \bar{X}_{P_0, \delta}') \text{ and } V_{P_0, \delta} \neq \bar{X}_{P_0, \delta}$$

for a $V_{P_0, \delta}$) Recall that the conditions (3.2.)

and (3.1.) are valid for each point r near

r_0 . Such a datum (X, V) will be fixed,

3-2-14,

As before, our arguments will be done not only with the datum (X, V) but also with a set of functions associated with varieties in question. Namely assume that a sets $\mathcal{M}_X = \{g_1, \dots, g_r\}$ of functions defining X and $\mathcal{M}_V = \{g_1, \dots, g_s\}$ defining V on X are given. The functions $g_1, \dots, g_s$ restricted on X will be denoted by $\tilde{g}_1, \dots, \tilde{g}_s$. We assume the following condition on $\tilde{g}_1, \dots, \tilde{g}_s$.

$$(\tilde{g}) \quad \tilde{g} \not\equiv 0, \text{ on each component}$$

$X_{P_0, i}$ of I_{P_0} .

Corresponding to notations given in n° 1, define

$$\text{vectors } \tilde{g}_{v_1, \dots, v_i}^{i-1} \in \Gamma(X, \mathcal{O}_X^{(l)}) \text{ over } X \text{ by}$$

$$(3.2.12)_1 \quad \tilde{g}_{v_1, \dots, v_i}^{i-1} = \begin{cases} (-1)^k \cdot g_k^m, & \text{for } (v_1, \dots, v_i, \dots, v_i), \\ 0, & \text{otherwise} \end{cases}$$

Also define sheaf homomorphisms $\tilde{K}(m, i)$ over X by

$$(3.2.12)_2 \quad \tilde{K}(m, i)(\tilde{f}) = \sum_{\substack{1 \leq v_1, \dots, v_i \leq l \\ 1 \leq v_1, \dots, v_i \leq l}} \tilde{f}_{v_1, \dots, v_i} \tilde{g}_{v_1, \dots, v_i}^{i-1}.$$

The image $\tilde{K}(m, i) \cdot \mathcal{O}_X^{(l)}$ will be denoted by $\tilde{\mathcal{E}}^{m, i-1}$.

Our main purpose is to give a type of a

vanishing theorem for such sheaves $\tilde{\mathcal{E}}^{m, i-1} (i=0, \dots, l)$. An

another problem, which is closely related to our

basic problem , will be also studied . The

basic problem will be called a vanishing theorem

with algebraic division and algebraic growth condition

On the otherhand , we call the second problem

a weak syzygy with quantity . These two

results will be abbreviated as v.a.d.g (or v.a.) and w. s.

q. in the sequel . (n.d.) We begin by formulating

the first problem : in this case we consi-

-der a divisor D defined by $h = 0$, besides

data $\{ X, V, \mathcal{V}_X, \mathcal{V}_V \}$. The irreducible decomposition

of the germs D_{P_0} and $\tilde{D}_{P_0}$ ($\tilde{D} = D \wedge X$)

will be written as follows.

$$D_{P_0} = \bigcup_i D_{P_{0,i}} , \quad \tilde{D}_{P_0} = \bigcup_i \tilde{D}_{P_{0,i}}$$

We call the collection $(X, V, D, \mathcal{V}_X, \mathcal{V}_V, (u))$

an underlying datum for a vanishing theorem with

algebraic division and algebraic growth condition. (u.d.v.g)

if the following conditions are valid:

$$(U.D.V.A)_{P_0} \quad D_{P_0} \supset X_{P_0}$$

(U. D. V. A)₂ $D_{P_0, i} \nrightarrow X_{P_0, i'}$ for a pair $(D_{P_0, i}, X_{P_0, i'})$.

(U. D. V. A)₃ $\tilde{D}_{P_0, i} \nrightarrow V_{P_0, i'}$ for a pair $(\tilde{D}_{P_0, i}, V_{P_0, i'})$.

Let us consider a point P (near P_0)^{*}

in $D \cap V$. A cochain $\psi^d \in C^d(N(\alpha(\tilde{U}_0 - \tilde{D}), \mathbb{K}))$ is

called of algebraic growth (d_1, d_2) and

algebraic division (d_3) (or simply of algebraic

growth condition (d_1, d_2, d_3)) if ψ^d

has the following properties.

(A. D. G) For each set $\Delta_s(q_0, \dots, q_s; \tilde{D}) \neq \emptyset$ ($q_0, \dots, q_s$

$\in \tilde{U}_0 - \tilde{D}$), the element $\psi_{(q_0, \dots, q_s; \tilde{D})}^d$ has the

following expansion

$$(A. D. G)_1 \quad \psi_{(q_0, \dots, q_s; \tilde{D})}^d = \sum_{1 \leq i_1 < \dots < i_s \leq s} \psi_{(q_0, \dots, q_s; \tilde{D})}^d \cdot \frac{q_{i_1}^{d_1} \dots q_{i_s}^{d_s}}{q_{i_1}^{d_1} \dots q_{i_s}^{d_s}}, \quad \text{where}$$

$\psi_{(q_0, \dots, q_s; \tilde{D})}^d \in \Gamma(\Delta_s(q_0, \dots, q_s; \tilde{D}), \mathbb{K})$ is estimated in the form

$$(A. D. G)_2 \quad |\psi_{(q_0, \dots, q_s; \tilde{D})}^d| \leq d_1 \cdot d(q, \tilde{D})^{-d_2} \\ d(q, \tilde{D})^{d_3}.$$

Take a neighbourhood $\mathcal{N}_0$ of P_0 and a point P

in $\mathcal{N}_0$. A pair (P, ψ^d) where ψ^d is a

cocycle in $C^d(N(\alpha(\tilde{U}_0 - \tilde{D}), \mathbb{K}))$ with

we use the phrase ' a point P near P_0 ' as the synonym of ' a point P in a neighbourhood of P_0 '. Here the existence of a neighbourhood of P_0 , in which assertions in question are valid , is important . But informations (such as measures of the neighbourhood etc.....) are not asked. because detailed informations concerning neighbourhoods of P are not used .

algebraic growth condition $(\alpha_1, \alpha_2, \alpha_3)$
 (for a suitable $(Q) = (\alpha_1, \alpha_2, \alpha_3)$) is called a
testifying datum for v.a. (with quantities
 $r, (\delta), (\alpha)$ (t.d.v.a.)

We mean by $T_{v.a.}((U.D.), \mathcal{N})$, the set
 set $\{P, \mathcal{Y}_i^d\} : P \in \mathcal{N}$ of all the t.d.v.a.'s. In

the sequel informations about $\mathcal{N}$ are not so
 important and we shall omit a neighbourhood $\mathcal{N}$ in
 our formulation. Let us consider the following

datum $Q_{v.a.}^i((U.D.)_{v.a.}) = (\phi(\delta^0), M_1, M_2, P_1, P_2, E_1, E_2, E_3, E_4, \mathcal{L})$

($i=1, \dots, l$) depending on the u.d.v.a. $(U_0)_{v.a.}$ only. Such a datum will

be called an expressing quantitative datum for v.a.
(ex. q. d. v.a.) if the following conditions are
 valid.

(E.Q.V.A) For a t.d.v.a. $(P, \mathcal{Y}_i^d)$ with
 quantities $(r, (\delta), (\alpha))$ there exists a
 $(s-1)$ - cocchain $\mathcal{Y}_i^{s-1} \in (N(\mathcal{U}(\tilde{\alpha}_i - \tilde{\alpha}), \mathcal{E})^{m_i})$ of algebraic growth
 $(\alpha'_1, \alpha'_2, \alpha'_3)$ so that the following are valid:

$$(E.Q.V.A)_1 \quad \delta_{\text{Cech}}(\mathcal{Y}_i^{s-1}) = (\mathcal{Y}_i^d),$$

$$(E, Q, V, A)_2 \quad \gamma' = M_1(\gamma), \quad (\delta') = d(\delta), \quad (\alpha'_1 = M_2(\alpha)^{-1} \cdot P(\alpha, \delta) e^{R_2(\alpha, \delta)} R_3(\alpha, \delta))$$

$$\text{So far as } \gamma < \gamma^0, \quad \delta < (\delta')_{\frac{1}{2}, \frac{1}{3}} \text{ are valid} \\ \delta'_2 = L_2(\alpha_2, \delta_2), \quad \alpha'_3 = L_3(\alpha_3) - L_3(m)$$

If there exists an ex. q. d. v.a. for a given

datum $\mathcal{T}_{\nu, \lambda}((U, D))$, we say that v.a. (vanishing theorem with algebraic division and algebraic growth condition) is valid for (U, D) v.a.

Our basic assertion in this section is formulated in the following manner.

Theorem 3.2.1. (A vanishing theorem with algebraic division and algebraic growth condition)
For any u.d. v.a. (U, D) = \{X, V, D, \mathcal{T}_r, \mathcal{T}_v, (k)\}
the vanishing theorem with algebraic division and algebraic growth condition is valid.

we shall make some remarks on the above theorem.

(C) A basic fact is that data $(\delta'_2), (\alpha'_3)$
are independent of the integer m in question.

This fact will be made an essential use of in

the sequel ((c.f) § 4. Theorem 4.1). concerning the

above comment, we shall add one another remark.

As we remarked previously, it is not true in general that the following sequence is valid.

$$(3.2.I) \quad 0 \rightarrow \tilde{\mathcal{E}}^{m,0} \xrightarrow{\tilde{K}(m,1)} \tilde{\mathcal{E}}^{m,1} \xrightarrow{\tilde{K}(m,2)} \tilde{\mathcal{E}}^{m,2} \xrightarrow{\tilde{K}(m,3)} \tilde{\mathcal{E}}^{m,3} \rightarrow 0$$

If we work with a resolution of $\tilde{\mathcal{E}}^{m,0}$ (without entering into dependences of a chosen resolution on integers m carefully) it seems not possible to assure the independenceness of $(\delta_1), (\delta_2)$ of m .

This is the essential reason why we work with series of sheaves $\tilde{\mathcal{E}}^{m,i}$ to which dependences of quantities on the integers m are rather easily examined. The author believes that our discussion of series of sheaves $\tilde{\mathcal{E}}^{m,i}$ in this section owes its own interest besides the above explained fact.

(N°. 2.)₂ In addition to the above theorem 3.2.1 we shall make one another argument which we call a weak syzygy with quantity (w.sy. q) both theorems v.a and w.sy. q. are, as will

be shown in later, related mutually. We formulate our second problem as follows : in this case we do not consider the divisor D . Instead we consider a subvariety V^* of V so that the condition

$$V^* \supset V \cap X_\Delta.$$

holds. The cases $V^* = V \cap X_\Delta$ or $V^* = V$ are included. Also the possibility $V^* = \phi$ is admitted (of course, in this case, $X_\Delta = \phi$). When the above condition is satisfied, we call the collection (X, V, V^*, q, q_r) an underlying datum for a weak syzygy with quantities (u.d.w, sy. q.)

and we will write the above collection as $(U.D)_{u.s.s.}$

Functions related to the variety V^* do not appear in our formulation of problems. Here our arguments will be done in a neighbourhood of P_c (informations of neighbourhoods are not important in the later arguments and we shall omit to

write a neighbourhood explicitly .) now

our consideration will be divided to the following

two cases (1) $V^* \neq V$, (2) $V^* = V$.

In a little while we are concerned with a

nontrivial case (1) . A pair of $P \in V - V^*$

and of an element $\tilde{f}_{i,0} \in P(\Delta(P; \mathcal{O}_X^{(P)}))$ will be called a

testifying datum for w. sy. q. . (t. d. w. sy. q)

(w. sy. q. stands for the abbreviation of

weak syzygy with quantity) if the

following condition is true.

$$(W. sy. Q)_1 \quad \tilde{K}(m, i) \cdot \tilde{f}_{i,0} = 0$$

for the case of $i=0$, we do not assume any

algebraic conditions for $\tilde{f}_{0,0}$.

We say that $\mathcal{F}_Q^{m,i}$ is of q.p. (quantitative property)

(r, b, d_3) if the following conditions are valid.

$(W.s/Q)_2$ There exists an element $\mathcal{F}^{m,i} \in \mathcal{F}(\tilde{\Delta}_x(p), \mathcal{O}_x^{(d)})$

in such a manner that

$(W.s/Q)_{2,1}$ the stalk $\mathcal{F}_{p,Q}^{m,i}$ of $\mathcal{F}^{m,i}$ at p is $\mathcal{O}_p^{m,i}$.

and moreover,

$(W.s/Q)_{2,2}$ In $\tilde{\Delta}_x(p)$, the element $\tilde{\mathcal{F}}^{m,i} = (\tilde{f}_\mu^{m,i})_{\mu=1, \dots, (l)}$ is

estimated in the following way

$$(W.s/Q)_{2,3} \quad |\tilde{f}_\mu^{m,i}(p')| \leq b \cdot d(Q', v)^{d_3}.$$

In the sequel the estimations $(W.s/Q)_{2,3}$ will be

abbreviated as ; $|\tilde{\mathcal{F}}^{m,i}| \leq b \cdot d(Q', v)^{d_3}.$

By $(\mathcal{T}.Q)_{u,y}^i$, we mean the set of all the (t.d.w. sy:

$(p, \mathcal{O}_p^{m,i}); p \in (V - V^*),$ where $\mathcal{O}_p^{m,i}$ has a q.p.

(r, b, d_3) with suitable quantities (r, b, d_3) .

A datum $Q_{w.s/y}^{*,i} = \{(\tilde{\mathcal{F}}_1^i, \tilde{\mathcal{F}}_2^i), \tilde{\mathcal{M}}_1^i, \tilde{\mathcal{M}}_2^i, \tilde{\mathcal{M}}_3^i, \tilde{\mathcal{P}}^i, \dots, \tilde{\mathcal{M}}_{31}^{*,i}, \tilde{\mathcal{M}}_{32}^{*,i}\} (i \geq 0)$

will be called an expressing quantitative datum for $(W.$

Sy. Q) (an ex.q. d. w. sy. q.) if the following state-
ments are valid with such a datum $Q_{w.s/y}^{*,i}$.

(Ex.Q.W.s/Q) For any pair $(Q, \mathcal{F}_Q^{m,i}) \in (\mathcal{T}.Q)_{w.s/y}^i (i \geq 0)$

of a quantitative property (r, b, d_3) , there exists

an element $\mathcal{P}'_{a,i+1} \in \mathcal{O}_{X,Q}^{(m_{i+1})}$ of a q.p. (E', b', d') so

that the equation

$$(Ex. q. w. sy. q.)^i_1 \tilde{K}^{m_{i+1}}(\mathcal{P}'_{a,i+1}) = \mathcal{P}_{a,i}^{m_i} \quad (i=0, \dots, l-1)$$

with the data

$$(Ex. q. w. sy. q.)^i_2 \quad r' = \mathbb{P}_1^i(r) \quad , \quad b' = M_{2,3}^{-1} \mathbb{P}_3^i(b) \quad \cdot \quad (e)^{P(d_3, m)}$$

$$d'_3 = \mathbb{L}_{3,1}^i(d_3) - \mathbb{L}_{3,2}^i(m)$$

is valid, so far as the inequalities

$$(Ex. q. w. sy. q.)^i_3 \quad \mathbb{L}_{3,1}^i(d_3) - \mathbb{L}_{3,2}^i(m) > 0 \quad , \quad r \leq \tilde{\delta}_1^i d(Q, V^*)^{\tilde{\delta}_1^i}$$

hold.

(Remark) If the variety V^* is empty then the above assertion is understood as follows. In the ex. q. d. w. sy. q. we replace $(\tilde{\delta})$ by $\tilde{\tau}$, and the other materials are unchanged. In the statement, the part $r \leq \delta_1^i d(Q, V^*)^{\delta_1^i}$ is changed to $r \leq \tilde{\tau}$. Remaining parts are unchanged.

We say that a weak syzygy with quantities holds

for an u. d. w. sy. q. $(U. D)_{w. sy. q.}$ if we can find an

ex. q. d. w. sy. q. $\mathcal{Q}_{w. sy. q.}$ for $(\mathcal{Q})_{w. sy. q.}$.

(Remark) Arguments hitherto are done for the case : $V \neq V^*$.

If $V = V^*$, we regard that w. sy. q. holds.

Now our second assertion is formulated in the

following manner.

Theorem 3.2.2 (A weak syzgy with quantities)

For any u.d.w.sy.q. $(U.D)_{w.sy.q} = \{X, V, V^*, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}\}$

the w.sy. q. holds.

We mean by Theorem 3.1.aⁿ and Theorem 3.w.sy.qⁿ

statements which assert the validity of assertions in

Theorem 3.1.aⁿ and Theorem 3.w.sy.qⁿ for the cases where

the dimension of X is at most n .

Remaining parts of this section will be devoted to investigations of theorems 3.2.1.a and 3.2.2.w.sy.q.

n. 3. There exist certain relations between theorem 3.2.1.a and theorem 3.2.2.w.sy.q. We shall begin by clarifying them and by reducing theorems 3.1.aⁿ and Theorem 3.w.sy.qⁿ to more weaker problems. Reductions will be done in several ways.

(1) Assume that an u.d. v.a. $(U.D)_{v.a.} = \{X, V, D, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, h\}$ is given. Instead of the original assertion (v.a.), Theorem 3.2.1.v.a., we shall consider the following weaker assertion.

A datum $Q^i((U.D.)_{v,a}) = \{M_1^i, M_2^i, P_1^i, P_2^i, P_3^i, L_2^i, L_3^i, \mathcal{A}^i\}$

is said to be an ex. q. d. pre. v. a. (an expressing quantitative

datum for pre-vanishing theorem with algebraic growth con

dition) if the following statement, weaker than v. a.,

is valid.

(EX.Q.P.V.A) For a pair $(Q, \tilde{y}^d) \in (TD)_{v,a}$ with

quantitative conditions $(T, (s), (\tilde{\alpha}) = (\tilde{\alpha}_1, \tilde{\alpha}_2, \tilde{\alpha}_3))$ $(\tilde{\alpha}_1, \tilde{\alpha}_2)$, there exists

a $(s-1)$ -cochain $\tilde{c}^{s-1}(N(\hat{U}_1^s(U_1(P)) - D) \tilde{\mathcal{A}}^{n_i})$ with

algebraic growth $(d'_1, d'_2, d'_3): d'_3 = 0$ in such a manner that

the following relations

$$(EX.Q.P.V.A)_1 \quad \tilde{y}^d = \delta_{\text{cech}}(\tilde{y}^{d-1}),$$

$$(EX.Q.P.V.A)_2 \quad r' = M_1^i(r), \quad (\mathcal{A}') = \mathcal{A}'(s), \quad d'_1 = M_2^i(r)^{-1} P_1^i(\mathcal{A}_1) \cdot P_2^i(\mathcal{A}_2) \cdot P_3^i(\mathcal{A}_3)$$

$$d'_2 = L_2^i(d_2, s_2),$$

holds so far as $d_3 > L(n)$ is true.

In a similar way to the case of v. a., we say that

the pre v. a. holds for an u. d. v. a. $(U.D.)_{v,a}$ if

there exists an ex q d. pre. v. a. for $(U.D.)_{v,a}$.

A weaker assertion than the original theorem Theorem 3.2 is

formulated in the following way.

Lemma 3.21 pre. v. a. (Pre vanishing theorem with algebraic

growth condition)

For a given u.d.v.a. $(U.D)$, pre v.a. is valid.

By lemma 3.2.1ⁿ, we mean the validity of the above assertion for the cases where the dimension of X is at most n . Our first assertion of the theorem 3.2.1.v.a will be done in the following two steps.

Lemma 3.2.2 Assume that an u.d.v.a. $(U.D)_{v.a} = \{ X, V, D, \mathcal{U}_X, \mathcal{U}_V, \mathcal{H} \}$ is given and that the w.sy. q. is valid

for the u.d. w.sy. q. $(U.D)_{w.sy.q} = \{ X, V, V^*, \mathcal{U}_X, \mathcal{U}_V \}$

: $V^* = V \cap D$. Then we know that

$(3.2.13)_1$ pre v. a. holds for $(U.D)_{v.a}$.

Moreover, we know that

$(3.2.13)_2$ the assumptions that, for $(U.D)_{v.a}$, the w.sy. q. holds and the assumption that pre v. a. is valid for $(U.D)_{v.a}$ imply the theorem 3.2.1.v.a.

Concerning the second assertion of the w. sy. q., our reduction will be done as follows: Assume that an u. d. w. sy. q. $(U. D)^*_{w. sy. q.} = \{X, V, V^*, \pi_X, \pi_V\}$ as well as $(U. D)^*_{w. sy. q.} = \{X, V, V^*, \pi_X, \pi_V\}$ are given. Moreover, we assume the following condition.

$$(3.2.14) \quad V_P \supset V_P^* \supseteq V_{P_0}^* .$$

Here we also admit the possibility of $V = V^*$.

Then our reduction step for theorem $w. sy. q.$ is stated in the following ways.

Lemma 3.2.3ⁿ w. sy. q. Assume that an u. d. w. sy. q. $(U. D)^*_{w. sy. q.}$ and $(U. D)^*_{w. sy. q.}$ are given so that the condition () is valid. Moreover, assume the validity of theorem 3.2.1ⁿ⁻¹ w. sy. q. as well as the w. sy. q. for $(U. D)^*_{w. sy. q.}$. Then there exists a subvariety V''^* of V which satisfies the following conditions.

$$(3.2.15)_1 \quad V''^* \supseteq V'^* \supset V^*$$

$$(3.2.15)_2 \quad \text{The w. sy. q. holds for } (U. D)^*_{w. sy. q.} = \{X, V, V''^*, \pi_X, \pi_V\}$$

Proof of reduction steps: Lemma 3.2.1, 3.2.2 and Lemma 3.2.3ⁿ w. sy. q.

will be given in the later numeros.

Here we shall see that the above lemmas imply basic as

assertions Theorems 3.1.2 and 3.2.1 w. sy. q. if we know the validity

of Theorem 3ⁿ⁻¹_{w.s.f.}. At first it is quite clear that the lemmaⁿ_{w.s.f.}, applied inductively to V^* , leads to the following implication.

(3.2/6), Theorem 3ⁿ⁻¹_{w.s.f.} $\Rightarrow$ Theorem 3ⁿ_{w.s.f.}.

On the otherhand, it is also obvious that

(3.2/6)₂ Theorem 3ⁿ_{w.s.f.} implies Theorem 3ⁿ_{v.a.}.

Thus we know that lemmas and Theorem is sufficient to show Theorem and Theorem.

Remaining parts of this section will be devoted to verifications of lemmas 3.2.1, 3.2.2, 3.2.3 and theorem 3.

we quickly indicate our devices ; (I) Lemmas 3.2, 3.2.2 have many similarities, and will be discussed paralelly (in one place) . On the otherhand, Our verification of lemma 3_{w.s.f.} is reduced to further weaker assertions :

n. 4 . In order to discuss Lemma 3.2. w.s.y.g. , we shall make further reductions of problems. we begin by fixing certain notation used here and by pointing out simple facts used in our discussion of Lemma 3. w.s.y.g. In the first place , let $(U, D)_{w.s.y.g.} = \{X, V, V^*, \mathcal{U}_X, \mathcal{U}_V\}$ be an underlying datum for a weak synergy. Let us consider the following condition on V :

(3.2.14) At the origin P_0 , the relation () is valid.

$$(3.2.14)' \quad V_{P_0, i}^* \neq V_{P_0, i}' ,$$

for any pair $(V_{P_0, i}^* , V_{P_0, i}')$ of irreducible components of $V_{P_0}^*$ and $V_{P_0, i}'$ of V_{P_0} at P_0 .

We quickly show the following

Proposition 3.2.2 The validity of w.s. q. for all the u. d. sy. q. $(U, D)_{w.s.y.g.}$ satisfying the condition (3.2.14') leads to the validity of w.s.y. q. as in theorem 3.2.2.w.s.y.g. .

Proof . Let us write the irreducible decomposition of V and V^* at P_0 in the following manner.

$$V = \bigcup_i V_{P_0, i} , \quad V^* = \bigcup_i V_{P_0, i}^* .$$

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Divide irreducible components of V^* in two types : (i) Irreducible components V_{P_0, i_1}^* 's which do not coincide with any irreducible component V_{P_0, i_2} of V_{P_0} . (ii) Irreducible components which coincide with an irreducible component V_{P_0, i_2} of V_{P_0} . Define germs $V_{P_0, i}^*$ ($i = 1, 2, \dots$) by $V_{P_0, 1}^* = \bigcup_{i_1} V_{P_0, i_1}^*$ and $V_{P_0, 2}^* = \bigcup_{i_2} V_{P_0, i_2}$, where V_{P_0, i_1}^* 's as well as V_{P_0, i_2} 's exhaust components of the first and of the second types respectively. Let $\tilde{V}_{P_0, 1}^*$ be the germ defined by $\tilde{V}_{P_0, 1}^* = V_{P_0, 1}^* \cup (V_{P_0}^* \cap X_{P_0, 1})$. Take a variety $\tilde{V}_1$ in a neighbourhood of P_0 where the stalk $(\text{of } \tilde{V}_1 \text{ at } P_0) \cong \tilde{V}_{P_0, 1}^*$ the condition (3.2.16) imposed on X, V shows the validity of w. sy. q. for the u.d. w. sy. q. $(U, D)_{w. sy. q.} = \{X, V, \tilde{V}_1^*, \mathcal{H}_X, \mathcal{H}_V\}$. Because the germ $\tilde{V}_{P_0, 1}^*$ is contained in $V_{P_0}^*$, the validity of w. sy. q. for the u.d. w. sy. q. $(U, D)_{w. sy. q.}$ implies the validity of the w. sy. q. for the $(U, D)_{w. sy. q.} = \{X, V, V^*, \mathcal{H}_X, \mathcal{H}_V\}$. q.e.d.

Here we remark that, for an u.d. v. a. $(U, D)_{v. a.} = \{X, V,$

$D, \mathcal{H}_X, \mathcal{H}_V, (h), \}$ the intersection $V^* = V \cap D$ satisfies

the conditions () and (). We restrict, henceforth, our attention to those u.d. w.sy.q. (U.D)_{w, sy, q} = { $X, V, V^*, \mathcal{U}_X, \mathcal{U}_V$ } in which the variety V^* satisfies the condition (3.2.16) besides (). More precisely, we shall mean by Theorem 3.2.2_{w, sy, q}' the one which says that the w, sy. q. is valid for any u.d. w.sy.q. (U.D)_{w, sy, q} in which the condition (3.2.16) on V^* is true (except the case $V^* = V$). Also by Lemma 3.2.2_{w, sy, q}' we understand the assertion which is obtained from Lemma 3.2.2_{w, sy, q} in the following manner : (1) In the statement of Lemma 3.2.2_{w, sy, q}' we impose the further condition (3.2.16) on $V^*, V^*(2)$. Also we replace the induction hypothesis of the validity of Theorem 3.2.2_{w, sy, q}ⁿ⁻¹ by Theorem 3.2.2_{w, sy, q}^{n-1'}. Finally we impose the condition (3.2.16) on the variety V^{**} to be found out. Then the following implications are preserved.

(3.2.16)'₁ Theorem 3.2.2_{w, sy, q}' $\Rightarrow$ Theorem 3.2.v.a,

(3.2.16)₂' Theorem 3.2.2_{w, sy, q}^{n-1'}, Lemma 3.2.2_{w, sy, q}^{n'} $\Rightarrow$ Theorem 3.2.2_{w, sy, q}ⁿ

The first implication is valid because , for u.d. v.a

$(\bar{U}, D)_{v,a} = (X, V, D, \mathcal{U}_X, \mathcal{U}_V, (h))$, the condition (3.2.17)

is valid for $V^* = V \cap D$ because of (3.1.). On

the otherhand, the implication (3.2.16)₂' is seen immediately.

In later numeros, we work with theorem 3.2.2_{w.s.y.f} and lemma 3.2.3_{w.s.y.f} rather than theorem 3.2. and lemmaⁿ_{w.s.y.f}.

this restriction is not of an essential nature. But

our ideas will be fixed in several places by this restriction.

Now, for our purposes, we shall need further reductions:

Let $(\bar{U}, D)_{v,a} = \{X, V, D, \mathcal{U}_X, \mathcal{U}_V, (h)\}$ be an

u.d. v.a. We assume that the w. sy. q. is valid for the

w. sy. q. $(\bar{U}, D)_{v,a} = \{X, V, D, \mathcal{U}_X, \mathcal{U}_V\}$. A pair $(P, \tilde{\varphi})$ of

a point $P \in V \cap D$ and of an element $\tilde{\varphi} \in P(\bar{U}, \tilde{D})^{(n)}$ is called

a meromorphic testifying datum for the weak synergy with

quantities $(r, b, \alpha_2, \alpha_3, \dots)$ if the

following conditions are satisfied.

$$(M_{e.w.s.y.q}), \quad \tilde{\kappa}(m, i) \cdot \tilde{\varphi} = 0 \quad (\text{if } i \neq 0)$$

and is estimated in the way

$$(M_{e.w.s.y.q})_2 \|\tilde{\varphi}\| \leq b \cdot d(Q, \tilde{D})^{-d_2} d(Q, V)^{d_3} : Q \in \bar{U}_X(Q) \cap \tilde{D}.$$

In a similar way to w. sy. f, we mean by $\tilde{\kappa}_{\text{mero.w.s.y.f}}$ the

set of all the t.d. mero. w. sy. q. (with suitable quantities $((r, b, \alpha_2, \alpha_3))$). For this set $\mathcal{T}_{\text{mero. w. sy. q.}}^i$, we

consider the following datum $Q_{\text{mero. w. sy. q.}}^i = \{\delta_{ne}^i, M_{ne}^i, M_{ne}^i, P_{ne}^i, P_{ne}^i, P_{ne}^i, L_{ne}^i,$

$L_{ne}^i, L_{ne}^i\}$ associated with $(U.D.)_{v. q. v. a.}$. Such a datum

$Q_{\text{mero. w. sy. q.}}^i$ is called an weak quantitative datum for mero-

-morphic weak syzgy with quantity (ex. q. d. mero. w. sy.

q.) if the following statements, which is weaker

and is regarded as a meromorphic form of w. sy. q., are valid.

$(EX.O.D.ME.W.SY)_1$, For a t.d. mero. w. sy. q. $(P, \gamma^{n,i})$ with (x, b, d)

we find an element $\gamma^{n,i+1} \in \Gamma(\overline{U_r(P-D)}, \mathcal{O}_I^{(n)})$ so that the equation

$$(EX.O.D.ME.W.SY)_1' \quad \tilde{K}(m, i+1) \gamma^{n,i+1} = \gamma^{n,i},$$

and the estimation

$$(EX.O.D.ME.W.SY)_2' \quad |\gamma^{n,i+1}| \leq b.d(\alpha', \tilde{D}) d(\alpha', \gamma) \alpha_3'; \quad \alpha' \in \overline{U_r(P-D)};$$

is valid with quantities $(\alpha', b, \alpha_2, \alpha_3)$ determined by

$$r' = M_{ne,1}^i(r), \quad b' = M_{ne,2}^i(r) M_{ne,2}^i(b),$$

$$\alpha_2' = L_{ne,2}^i(\alpha_2), \quad \alpha_3' = L_{ne,3,1}^i(\alpha_3) - L_{ne,3,2}^i(m).$$

if there exists an exd,mero. w. sy. q. for the datum

then we say that the mero. w. sy. q. is valid for $(U, D)_{n,2}$.

As two steps to Lemma 3.2 ⁿ w. sy. q., we show the following

two assertions. in later.

Lemma 3.2.4 ^{mero, n} w. sy. q. For an u.d.v. a. $(U, D)_{n,2} \equiv (\bar{x}, \bar{v}, D, \mathcal{M}_x,$

$\mathcal{M}_v, (h))$, where the w. sy. q. is valid for

$(U, D)_{w. sy. q.} = \{ \bar{x}, \bar{v}, \bar{v}^* \bar{v} \wedge D, \mathcal{M}_x, \mathcal{M}_v \}$, the meromorphic

weak syzygy with quantities is valid for $(U, D)_{n,2}$.

Lemma 3.2. Lemma 3.2. ^{mero, n} w. sy. q. and Lemma 3.2 ⁿ⁻¹ w. sy. q. implies

Lemma 3.2. ^{n'} w. sy. q.

Remaining parts of this section will be devoted

to verifications of the aforementioned lemmas :

lemma 3.2.1. lemma 3.2.2, lemma 3.2.3 and lemma 3.2.4.

We first discuss the lemma 3.2.1 and after that we enter into lemmas 3.2.2, 3 and lemmas 3.2.4...

n.5 Inorder to discuss lemma 3.2.1, we begin

with a simple remark. Assume that an u.d. w.sy. q.

$(U. D)_{w. sy. q.} = \{ \mathbf{x}, \mathbf{v}, \mathbf{v}^*, \mathcal{U}_I, \mathcal{U}_V \}$ is given. We assume that

the w. sy. q. is valid for $(U. D)_{w. sy. q.}$.

Furthermore, we assume that a variety V^* satisfying the condition $V^* \supset T^* \supset V^*$ is given. (We include the case $T^* = V^*$) We generalize the notion of w.s.f. given in n° 3 in the following manner : Consider a point $Q \in (X \leftarrow X_0) - V^*$ and an element $\mathcal{F}_Q^i \in \mathcal{O}_{X,Q}^{(1)}$. A pair $(Q, \tilde{\mathcal{F}}_Q^i)$ is called a (t. d. w. sy. q') with quantities (r, b, d_3) if the following condition, similar to , is valid.

(w. sy. q') There exists an element $\mathcal{F}^i = \begin{pmatrix} f_i \\ 1 \end{pmatrix} \in \Gamma(\overline{\Delta}_1(Q), \mathcal{O}_1^{(q)})^*$, whose stalk at Q is $\tilde{\mathcal{F}}_Q^i$, so that the equation $(w.s.f. q')_i \quad \tilde{K}(m, i)(\mathcal{F}^i) = 0 \quad (i = 1, \dots, l)$

and the estimation

$$(w.s.f. q')_2 \quad |\mathcal{F}^i| \leq b \cdot d(\mathcal{F} : V)^{d_3} ; \quad Q' \in \Delta_x(Q)$$

are valid.

By $T_{w.s.f.}((U, D) V^*)$, we mean the set of all the

(t. d. w. sy. q') with quantities (r, b, d_3)

(for suitable quantities (r, b, d_3) . A datum

$$T_{w.s.f.}((U, D) V^*) = \{ \tilde{s}_1^{x_1}, \tilde{s}_1^{x_2}, \tilde{M}_1^{x_1}, \tilde{M}_2^{x_1}, \tilde{M}_3^{x_1}, \tilde{E}_{n_1}^{x_1}, \tilde{E}_{s_1}^{x_1}, \tilde{E}_{s_2}^{x_1} \}$$
 is said

to be an (ex. w. sy. q') (for $\tilde{\mathcal{E}}^{m, i}$) if the following

is valid.

(EX.w.sy.q)' If $d(q, \tau) < \tilde{\delta}_1^i d(q, x_1^v v^*)^{\tilde{\delta}_2^i}$ ($q \in (X - X_s)$

$-v^*$) is true, then there exists an element

$\Delta_x(q) : \mathcal{O}_x^{(m)} (i=0, \dots, k-1)$ so that the equation

(EX.w.sy.q)' $\tilde{K}(\frac{m}{i}) = \frac{m}{i}$ in $\Delta_x(q)$

as well as the estimation

(EX.w.sy.q)' $|\frac{m}{i}| \leq b' \cdot d(q, \tau)^{d_3'}$; $q' \in \Delta_x(q)$

hold. where quantities r' , b' , and d_3' are determined by

(3.2.18) $r' = M_1'(\tau)$, $b' = M_1'(\tau) \cdot M_2'(b) \cdot P_{\eta}(c)^{P_{12}(\omega)}$

$d_3' = L_{31}'(d_3) - L_{32}'(m)$,

so far as $\tau < \sim$

Now, the following elementary proposition will be made use of in the later arguments.

Proposition 3.2.3 . If the w. sy. q. is valid for

(U.D)_{w.sy.q} = (X, V, V*, $\frac{m}{i}$, $\frac{m}{j}$) and if a subvariety

V'^* ; $V \supset V'^* \supset V^*$; is given, then there exists

an {ex.d.w. sy. q}. $Q(\frac{m}{i}, \frac{m}{j}, V^*) = \{ \tilde{\delta}^i, \tilde{c}^k, \tilde{M}_1^i, \tilde{M}_2^i, \tilde{M}_3^i, \tilde{L}_1^i, \tilde{L}_2^i, \tilde{L}_3^i \}$ for

$\{ (U.D)_{w.sy.q}, V^* \}$.

Proof. It is obvious that, for a point $q \in V - V'^*$,

assertions (EX.w.sy.q)'₁, (EX.w.sy.q)'₂ are valid with an

ex. w. sy. d. q. $Q_{w, sy, \delta}^i((U, D)_{w, sy, \delta})$.

(i i) Next we consider the case where a point Q is not in V . Choose a sufficiently small couple $(\tilde{\delta}_0')$. Our argument will be done

separately according to whether Q is in $N_{\tilde{\delta}_0'}(V, V^*)$ or Q is not in $N_{\tilde{\delta}_0'}(V, V^*)$. Recall that if Q is not in $N_{\tilde{\delta}_0'}(V, V^*)$ then the following inequality

$$(3.2.10)_1^1 \quad d(Q, V^*) \leq \tilde{c}_1' d(Q, V)^{\tilde{\delta}_0'}$$

holds with a suitable $(\tilde{c}_1', \tilde{c}_2')$ (cf. §1).

Thus, for a sufficiently small $(\tilde{\delta}_0'')$, the following facts are valid in view of (1.) and the Lojasiewicz's inequality

(3.2.10)₂¹ There exists a function $\tilde{g}_1$ (for example $\tilde{g}_1$)

$$\text{so that} \quad |\tilde{g}_1(Q, Q')|^{-1} \leq \tilde{h}_1' d(Q, V)^{-\tilde{\delta}_0''}$$

with suitable constants $(\tilde{h}_1', \tilde{h}_2')$ so far as Q' is in $\tilde{\Delta}_{\tilde{\delta}_0''}(Q, V'^{\tilde{\delta}_0''})$.

Then, from (), our assertion of the existence

of a datum $Q'' = \{ \tilde{s}_1'', \tilde{M}_1'', \tilde{M}_2'', \tilde{P}_1'', \tilde{P}_2'', \tilde{L}_1'', \tilde{L}_2'' \}$, with which

$(\text{Ex.w.s/q})'$ holds for any points $Q \notin N_{\tilde{s}_0}(V : V^*)$, follows easily.

(iii) Consider a point $Q \in N_{\tilde{s}_0}(V : V^*)$ and take a point $Q_T \in V$ so that $d(Q, Q_T) = d(Q, V)$ holds.

From the existence of an ex.w.sy. $Q'((U, D)_{w.s.f.})$ for

$(U, D)_{w.s.f.}$, we know the existence of a datum $Q'' = \{ (\tilde{s}_1''), (\tilde{c}_1''), (\tilde{c}_2''), \tilde{M}_1'', \tilde{M}_2'', \tilde{P}_1'', \tilde{P}_2'', \tilde{L}_1'', \tilde{L}_2'' \}$, depending

the datum $(U, D)_{v, d}$ only, in such a manner that the following fact is valid.

(3.2.19) If $r \geq \tilde{c}_1'' d(Q, Q_T)^{\tilde{c}_2''}$, then, for a (t.d.w. sy.) $(Q, \mathcal{V}_a^{m,i})$, there exists an element $\mathcal{V}_1^{m,i} \in \Gamma(\Delta_{\mathcal{V}_1}(\mathcal{Q}_T, \mathcal{Q}))$

) so that the relations below are valid.

$$(3.2.19)_1^2 \quad \tilde{K}(m, i+1) \cdot \mathcal{V}_1^{m,i} = \mathcal{V}_1^{m,i},$$

$$(3.2.19)_2^2 \quad |\tilde{\mathcal{V}}_1^{m,i}| \leq b.d(Q, V)^{\tilde{c}_3''} (Q' \in \Delta_{\mathcal{V}_1}(Q_T))$$

with quantities

$$(3.2.19)_3^2 \quad r'' = M_1''(r), \quad b'' \leq M_2''(T)^{-1} M_3''(B) \cdot P_{11}''(L)^{P(2)}$$

Moreover, we can assume the inequality

$$r'' \geq 4 \cdot d(Q, Q_T)$$

(Remark) For the above purpose, take $(\tilde{c}_1^{m,i}, \tilde{c}_2^{m,i})$

so that $\tilde{m}_{12}'' \tilde{c}_{21}'' \tilde{m}_{11}'' \tilde{c}_{11}'' > 4$ where $M_{11}(r) = \tilde{m}_{11}'' r \tilde{m}_{12}''$.

Thus we know that, so far as $r \geq \tilde{c}_1 d(Q, Q_V)^{\tilde{c}_2}$

holds, there exists an element $\tilde{q}^{n_{i+1}} \in \Gamma(\tilde{\Delta}_r(Q), \mathcal{O}_{i+1}^{(n)}): \tilde{K}(\tilde{q}^{n_{i+1}})$

$\tilde{q}^{n_i}$ whose quantities (r'', b'', d_3''') are determined

by $(3.2.19)_3^2$. Next consider the case in

which the inequality $r \leq \tilde{c}_1^i d(Q, Q_V)^{\tilde{c}_2^i}$ holds. For

an arbitrary (small enough) couple $(\tilde{\delta}''')$, there

exists a monomial M'' , depending on $(\tilde{\delta})_{w, sy, \tilde{q}}(U, D)$ only,

so that $r'' = M''(r) < \tilde{\delta}_1'' d(Q, V)^{\tilde{\delta}_2''}$ holds.

In such a situation we know, from (),

the existence of a datum $Q = \{\tilde{L} \tilde{M}_1'', \tilde{M}_2'', \tilde{M}_3'' \tilde{P}_{11}'', \tilde{P}_{12}'', \tilde{L}_{31}'' \tilde{L}_{32}''\}$

depending on M'' and $(\tilde{v}_0)_{w, sy, \tilde{q}}$ only, in such a manner that,

for a pair $(Q, \tilde{q}^{n_i})$: $\tilde{q}^{n_i} \in \Gamma(\tilde{\Delta}_r(Q), \mathcal{O}_X^{(n)})$

the assertions $(EX.w.q_0)_1'$, $(EX.w.q_0)_2'$ are true with quantities

$$r'' = \tilde{M}_1''(\Gamma), \quad b'' = \tilde{M}_2''(r) \cdot \tilde{M}_3''(b) \cdot \tilde{P}_{11}''(\tilde{c}''), \quad d_3 = \tilde{L}_{31}''(d_2)$$

- $\tilde{L}_{32}''(n)$.

It is clear that (i), $(3.2.19)'$ and $(3.2.19)^2$ lead to

our assertion of the existence of an ex w. sy. q. d.

$\tilde{q}^{n_i}$ for $\{(U, D), v^*\}$. q. e. d.

Now we prove lemma 3.2.1 and lemma 3.2.1₂ in

a parallel way .

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Proof of lemma 3.2.1 (i) . Let $(H, D)_{v.a} = (X,$

$V, D, \mathcal{U}_X, \mathcal{U}_V(h)$) be an u.d.v.a. so that

the w. sy. q. holds for the w. sy. q. $(U, D)^*_{w.sy} = (X, V, \mathcal{V}^* = \mathcal{V} \cap D,$

$\mathcal{U}_X, \mathcal{U}_V$) . By lemma 3.2.1₁, lemma 3.2.1₂ we mean the

validity of lemma 3.2.1₁ and lemma 3.2.1₂ for the sheaf $\mathcal{E}^{n_i}(i \leq l)$

Both assertions lemma 3.2.1₁ and lemma 3.2.1₂ are proven

in an inductive way on i . We discuss problems

in the following manner.

(3.2.20)_{1,0} To show that

(3.2.20)₁, the lemma 3.1. implies the assertion

lemma 3.2.1₁ ,

and that

(3.2.20)₂ the assertion lemma 3.1₁⁰ implies lemma 3.2.1₂⁰.

(3.2.20)₁' To show that

(3.2.20)₁' the assertions lemma 3.2.1_{1,2}ⁱ are reduced

to the assertions lemma 3.2.1_{1,2}ⁱ⁺¹ ($0 \leq i \leq l$) .

Our inductive devices lemma 3.2.1_{1,2}ⁱ ($0 \leq i \leq l$) are done

parallelly in both cases lemma 3.2.1₁ⁱ and lemma 3.2.1₂ⁱ . On the

otherhand the assertions lemma 3.2.1₁^l and lemma 3.2.1₂^l separatedly

(ii) we fix a datum $Q^0 = \{S^0, M_1^0, M_2^0, P_1^0, P_2^0, L^0, \mathcal{L}^0\}$ with which

lemma 3.1 is valid for the structure sheaf $\mathcal{O}_X$ and

an ex. w. sy. q. d. $Q_{w,q,\delta}^i = \{M_1^{i*}, M_2^{i*}, M_3^{i*}, \tilde{P}_{21}^{i*}, \tilde{P}_{22}^{i*}, \tilde{L}_{21}^{i*}, \tilde{L}_{22}^{i*}\}$
for $\mathcal{E}^{n,i}$.

(ii)₁ First we consider the inductive steps ()_{l,1} and

()_{l,2}. In both cases, we fix a cocycle $\tilde{\gamma}^d \in$

$Z^d(N(\hat{\alpha}^d(\overline{U}_r(D)), \tilde{\mathcal{E}}^{n,l}))$ with algebraic growth condition

$(\tilde{\alpha}_1, \tilde{\alpha}_2, \tilde{\alpha}_3) : P \in D_h V^*$ Also we fix an expression of a co

cocycle $\tilde{\gamma}^d$ in the following manner

$$(3.2.21)_1 \quad \tilde{\gamma}^d(Q_0, \dots, Q_d) = \gamma_{1,l}^d(Q_0, \dots, Q_d) \cdot \frac{\tilde{\gamma}^{n,l}}{\tilde{\gamma}^{n,l}}, \quad ()$$

where $\tilde{\Delta}_i(Q_0, \dots, Q_d) = \bigcap_{j=0}^d \tilde{\Delta}_i(Q_j) \neq \emptyset$ and $\gamma_{1,l}^d(Q_0, \dots, Q_d)$ is in $\Gamma(\tilde{\Delta}_i(Q_0, \dots, Q_d), \tilde{\mathcal{O}}_X)$

Moreover, the following estimation of $\gamma_{1,l}^d(Q_0, \dots, Q_d)$ is assumed.

$$(3.2.21)_2 \quad |\gamma_{1,l}^d(Q_0, \dots, Q_d; Q')| \leq \tilde{\gamma}_1 d(Q', \tilde{\mathcal{O}}) \cdot d(Q', V)^{\tilde{\gamma}_3}.$$

Remark that the fact $\tilde{\gamma}^d$ is a s-cocycle is equivalent

to say that $\left\{ \gamma_{1,l}^d(Q_0, \dots, Q_d) \right\}_{Q_0, \dots, Q_d} : \Delta_i(Q_0, \dots, Q_d) \neq \emptyset;$

is in $Z^d(N(\pi_{\Delta}^s(\overline{U}_r(D)), \mathcal{O}_X))$.

(ii)_{1,1} In the first case the assertion ()^l is

obviously an immediate consequence of Lemma 3.1.

We assume that P is in a neighbourhood M_0 of P_0 (fixed once and for all). (5)

(ii)_{4,2} In the second case, our situation is more subtle. We use the fact that $\tilde{\mathcal{E}}^{m'}$ is a subsheaf of

$\mathcal{E}^{m,0}$ ($m' \geq m$) . Note that the existence

of an ex.d.pre. v.a. q , $Q^a = \{ \tilde{s}_1^a, \tilde{M}_1^a, \tilde{M}_2^a, \tilde{P}_{11}^a, \tilde{P}_{12}^a, \tilde{L}_2^a, \tilde{L}_3^a, \tilde{L}_3^a, \tilde{L}^a \}$

for the sheaf $\tilde{\mathcal{E}}^{m'}$ leads to the following.

(3.2.25), There exists a datum $Q''^0 = \{ \tilde{s}_1^0, \tilde{M}_1^0, \tilde{M}_2^0, \tilde{P}_{11}^0, \tilde{P}_{12}^0, \tilde{L}_2^0, \tilde{L}_3^0, \tilde{L}_3^0, \tilde{L}^0 \}$, depending (U. D)_{ra} only, so that the following

modified form of Theorem 2.2.1. is valid with the

datum Q''^0 .

(3.2.25)₂ For a given cocycle $\tilde{y}^s \in \hat{A}^1(N(\tilde{\alpha}(\tilde{Q}^0-\tilde{D}), \tilde{\mathcal{E}}^{m'})$

with algebraic growth condition $(\tilde{J}_1, \tilde{J}_2, 0)$, there

exists a (s - 1) - cochain $y^{s-1} \in C^{s-1}(N(\tilde{\alpha}(\tilde{Q}^0-\tilde{D}), \tilde{\mathcal{E}}^{m'})$

of algebraic growth $(\tilde{J}_1'', \tilde{J}_2'', 0)$ satisfying the

condition

$$\delta_{\text{Coch}}(y^{s-1}) = y^s \text{ in } C^s(N(\tilde{\alpha}^s(\tilde{Q}^0-\tilde{D}), \tilde{\mathcal{E}}^{m'}))$$

where quantities (r'') , (s'') , m' , $(\tilde{J}_1'', \tilde{J}_2'')$ are

given in the following way.

(3.2.25)' $r'' = M_1^0(\mathcal{Y})$, $(s'') = L''(\delta)$, $m' = [L(m)]$

$$\tilde{Q}_1'' = M_2^0(\mathcal{Y})^{-1} \cdot P_{11}^0(\alpha, \delta_1) \cdot t \cdot t^{-1} \cdot P_{12}^0(\alpha, \delta_2) \cdot L_2^0(\alpha, \delta_2).$$

The above fact is deduced from () () in a quite elementary way and we shall omit its details.

Next we remark the following : Assume that a cocycle

$\tilde{\gamma}^A \in Z^A(N(\tilde{\mathcal{U}}(\tilde{\mathcal{U}}(E)-\tilde{\mathcal{D}})), \tilde{\mathcal{E}}^{m,0})$ of algebraic growth
 $(\tilde{\alpha}) = (\tilde{\alpha}_1, \tilde{\alpha}_2, \tilde{\alpha}_3,)$ is given . Then the validity

of the w. sy. q. for the u.d. w. sy. q. (U. D)_{w.s.f.} =
 $\{X, V, V \wedge D, \mathcal{U}_V, \mathcal{U}_X\}$ leads to the following statement.

(3.2.22) There exists a s-cocycle $\tilde{\gamma}^A \in Z^A(N(\tilde{\mathcal{U}}(\tilde{\mathcal{U}}(E)-\tilde{\mathcal{D}})), \tilde{\mathcal{E}}^{n,0})$

of algebraic growth $(\tilde{\alpha}_1'', \tilde{\alpha}_2'', 0)$ so that

(3.2.22)₁, $\tilde{\gamma}^A = \tilde{\gamma}^A$ (as elements in $Z^A(N(\tilde{\mathcal{U}}(\tilde{\mathcal{U}}(E)-\tilde{\mathcal{D}})), \tilde{\mathcal{E}}^{n,0})$, so far as $\tilde{\alpha}_3 > \tilde{\alpha}_3'(m)$.

where quantities $(\tilde{\delta}'), \tilde{m}', (\tilde{\alpha}_1', \tilde{\alpha}_2')$ are given by

(3.2.22)₂ $\tilde{\delta}' = \tilde{\alpha}'(\tilde{\delta})$, $\tilde{m}' = [\tilde{L}_3(\tilde{\alpha}_3)] + m$,

$\tilde{\alpha}_1' = \tilde{P}_1(\tilde{\alpha}_1, \tilde{\alpha}_2)$, $\tilde{\alpha}_2' = \tilde{L}_2(\tilde{\alpha}_2, \tilde{\alpha}_3)$, where $\tilde{\alpha}', \tilde{L}'$ and $\tilde{P}'$ depend on (U, D) only.

Now it is easy to see that (3.2.22)₁ and (3.2.22)₂ imply

the following conclusion.

(3.2.22) Under the condition of the validity of w.

sy. q. for $(X, V, V \wedge D, \mathcal{U}_X, \mathcal{U}_V)$ and the pre v.a. for (U, D) _{v.a.}

the sheaf $\tilde{\mathcal{E}}^{m,0}$ implies the v.a. for the sheaf $\tilde{\mathcal{E}}^{m,0}$.

(iii) Next we shall consider implications (). In this step, both cases of the pre. v. a. and v. a. are argued in a parallel way. Remark the following.

(3.2.23)₁ IF the relation $2 \cdot \delta_1 > \max(\delta_1', \delta_2', \delta_1'', \delta_2'')$, $\delta_2 < \min(\delta_2', \delta_2'')$ are valid for couples (δ) , (δ') and (δ'') , then, for any points $Q_0, \dots, Q_s, Q$ satisfying the condition

$$\Delta_\delta(Q_0, \dots, Q_s; \tilde{D}) \supseteq Q,$$

the inclusion relation

$$\Delta_{\delta_j}(Q; \tilde{D}) \subset \Delta_\delta(Q_j; \tilde{D}) \quad (j = 0, \dots, s)$$

is true.

(3.2.23)₂ Assume that the following conditions are valid for couples $(\tilde{\delta}')$, $(\tilde{\delta}'')$ and $(\tilde{\delta}''')$

$$(2.2.23)'_2 \quad 2 \cdot \tilde{\delta}_1''' > (\tilde{\delta}_1', \tilde{\delta}_2', \tilde{\delta}_1'', \tilde{\delta}_2'') \quad \tilde{\delta}_2''' < (\tilde{\delta}_2', \tilde{\delta}_2'')$$

Then, for any points $Q_0, \dots, Q_s, Q \in ()$ satisfying the condition

$$\Delta_{\tilde{\delta}'''}(Q_0, \dots, Q_s) \supseteq Q.$$

the inclusion relation

$$(3.2.23)_3 \quad \Delta_{\tilde{\delta}'''}(Q) \supset \Delta_{\tilde{\delta}'''}(Q_j) \quad (j = 0, \dots, s)$$

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holds.

Now let us take up a s -cocycle $\mathcal{Y}^s \in Z^s(N(\tilde{\Delta}^s(\tilde{Q}_1, \dots, \tilde{Q}_s), \tilde{D}), \tilde{E}^{\pi_s})$ ($s=0, \dots, l-1$) with algebraic growth condition $(\tilde{\Delta}_1, \tilde{\Delta}_2, \tilde{\Delta}_3)$. For

each non empty intersection $\tilde{\Delta}_s(q_1, \dots, q_s, \tilde{D}) = \bigcap_{j=1}^s \tilde{\Delta}_j(q_j, \tilde{D})$

we express $\mathcal{Y}^s(q_1, \dots, q_s)$ explicitly in the following form.

$$\mathcal{Y}^s(q_1, \dots, q_s) = \sum_{1 \leq i_1 < \dots < i_s \leq l} \mathcal{Y}^s(q_1, \dots, q_s) \tilde{F}_{i_1, \dots, i_s}^{\pi_s}$$

where $\mathcal{Y}^s(q_1, \dots, q_s)$ is in $\Gamma(\tilde{\Delta}_s(q_1, \dots, q_s, \tilde{D}), \mathcal{O}_{\tilde{D}})$.

We assume the following estimation.

$$|\mathcal{Y}^s(q_1, \dots, q_s, q)| \leq d_1 \cdot d(q, \tilde{D})^{-d_1} \cdot d(q, v)^{d_2}.$$

Then one can choose maps $f_{i,j}$ of (e, b) -types depending on the fixed ex. d. w. sy. $\mathcal{Y}^s$ only with which we know the following.

(3.2.24) If the intersection $\tilde{\Delta}_s(q_1, \dots, q_s, \tilde{D})$, then, for any point $q \in \tilde{\Delta}_s(q_1, \dots, q_s, \tilde{D})$, there exists a neighbourhood $\tilde{\Delta}_s(q, \tilde{D}) \supset \tilde{\Delta}_s(q_1, \dots, q_s, \tilde{D})$ and a vector $\tilde{\mathcal{Y}}^{s+1} \in \Gamma(\tilde{\Delta}_s(q, \tilde{D}), \mathcal{O}_{\tilde{D}}^{(i_s)})$ so that the equation

$$(3.2.24)_1 \quad \left\{ \sum_{\mu} (-1)^{\mu-1} \mathcal{Y}^s(q_1, \dots, q_s, q_\mu) \right\} = K^{s+1}(\tilde{\mathcal{Y}}^{s+1}) \cdot (\tilde{\mathcal{Y}}^{s+1}) = (\tilde{\mathcal{Y}}^{s+1})_{q_1, \dots, q_s, q_\mu}$$

holds.

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(Remark) As maps $\mathcal{L}_{(1,2)}$ of e - $\mathcal{L}$ type, we can choose such ones explicitly: Let $\mathcal{L}_1$ be a map $\mathcal{L}(\delta_1, \delta_2) \rightarrow (\delta'_1) = (\delta_1^{-2\delta_2}, 2\delta_2)$, and let $\mathcal{L}_M: (\delta'_1) \rightarrow (\delta'_2)$ be an $(e-\mathcal{L})$ -map satisfying the condition $\delta''_1 d(q, \mathcal{D}) < M(\delta'_1 d(q, \mathcal{D}))$ for any $q \in I-\mathcal{D}$. Then the composition

$$\mathcal{L}_2 = \mathcal{L}_1 \circ \mathcal{L}_M \circ \mathcal{L}_1 \quad \text{surfaces for our purpose.}$$

Now, by (3.2), the following estimation

$$(3.2.26) \quad \left| \tilde{\mathcal{P}}_{\nu_1, \dots, \nu_{s+1}}^{\Delta+1}(q_0, \dots, q_{s+1}; q) \right| \leq \tilde{\mathcal{Q}}_1^* \cdot d(q; \mathcal{D}) \cdot d(q, \mathcal{V})^{\tilde{\mathcal{Q}}_3^*}$$

is valid, where quantities $(\tilde{\mathcal{Q}}_1^*, \tilde{\mathcal{Q}}_2^*, \tilde{\mathcal{Q}}_3^*)$ are determined by

$$\begin{aligned} \tilde{\mathcal{Q}}_1^* &= \mathbb{P}_1^{\Delta+1}(\tilde{\mathcal{Q}}_1, \tilde{\mathcal{Q}}_1) \cdot \mathbb{P}_2^{\Delta+1}(\tilde{\mathcal{Q}}_2, \tilde{\mathcal{Q}}_2) \cdot \mathbb{P}_3^{\Delta+1}(\tilde{\mathcal{Q}}_3, \tilde{\mathcal{Q}}_3) \quad \tilde{\mathcal{Q}}_2^* = \mathbb{L}_2^{\Delta+1}(\delta_2 + \tilde{\mathcal{Q}}_2), \\ \tilde{\mathcal{Q}}_3^* &= \mathbb{L}_{31}^{\Delta+1}(\tilde{\mathcal{Q}}_3) - \mathbb{L}_{32}^{\Delta+1}(M). \end{aligned}$$

In the above the datum $(\mathbb{P}_1^{\Delta+1}, \dots, \mathbb{L}_{32}^{\Delta+1})$ is of course depending on $(U, \mathcal{D})_{u, \mathcal{D}, q}$ only. We write the right side of

$$(3.2.24)_1 \quad (\text{i.e. the term } \tilde{\mathcal{K}}^{(s+1)} \cdot \tilde{\mathcal{P}}_{q_0, \dots, q_{s+1}}^{\Delta+1}) \quad \text{as } \tilde{\mathcal{P}}_{q_0, \dots, q_{s+1}}^{\Delta+1}.$$

This element $\{\tilde{\mathcal{P}}_{q_0, \dots, q_{s+1}}^{\Delta+1}\}$ is in $\Gamma(\Delta_{\delta_1, \dots, \delta_{s+1}}(q_0, \dots, q_{s+1}), \tilde{\mathcal{K}}^{(s+1)})$.

The collection $\{\tilde{\mathcal{P}}_{q_0, \dots, q_{s+1}}^{\Delta+1}\}_{q_0, \dots, q_{s+1}}$ defines a $(s+1)$ -

$$\text{cocycle} \quad : \quad \tilde{\mathcal{P}}^{\Delta+1} \in \mathbb{Z}^{\Delta+1}(N(\hat{\mathcal{M}}(\bar{\mathcal{Q}}_1, \mathcal{D}) - \mathcal{D}), \tilde{\mathcal{K}}^{(s+1)}) \quad (i = 0, 1, \dots,$$

$\ell-1$). Apply the induction hypothesis for $(i+1)$ (of the

both assertions pre. v. a. and v. a.) to $\tilde{\mathcal{P}}^{\Delta+1}$. Then we know

$$(3.2.27)'' \quad \left\{ \psi_{v_1 \dots v_k}^{\Delta} (a_0, \dots, a_k) \right\}_{1 \leq v_1 < \dots < v_k \leq l} = \left\{ \psi^{\Delta} (a_0, \dots, a_k) \right\} \\ + \left\{ \psi_{v_1 \dots v_k}^{\Delta} (a_0, \dots, a_k) \right\}_{1 \leq v_1 < \dots < v_k \leq l}.$$

it is clear that the following equations are valid.

$$(3.2.28)_1 \quad K(m, i) \left\{ \psi_{v_1 \dots v_k}^{\Delta} (a_0, \dots, a_k) \right\}_{1 \leq v_1 < \dots < v_k \leq l} = K(m, i) \left\{ \psi_{v_1 \dots v_k}^{\Delta} (a_0, \dots, a_k) \right\}_{1 \leq v_1 < \dots < v_k \leq l} \\ (3.2.28)_2 \quad K(m, i) \cdot \left\{ \psi_{v_1 \dots v_k}^{\Delta} (a_0, \dots, a_k) \right\}_{1 \leq v_1 < \dots < v_k \leq l} = K(m, i) \left\{ \psi_{v_1 \dots v_k}^{\Delta} (a_0, \dots, a_k) \right\}_{1 \leq v_1 < \dots < v_k \leq l}$$

Of course the left side of the above equations (3.2.28),

and (3.2.28)₂ are nothing else than the given ψ^{Δ}

after taking a refinement map $\tilde{r}: \hat{N}(\tilde{u}_i(\mathbb{R}) - \mathbb{D}) \hookrightarrow \hat{N}(\tilde{u}_i(\mathbb{R}) - \mathbb{D})$, $(\tilde{r}: \hat{N}(\tilde{u}_i(\mathbb{R}) - \mathbb{D}) \hookrightarrow \hat{N}(\tilde{u}_i(\mathbb{R}) - \mathbb{D}))$

because $\left\{ \psi_{v_1 \dots v_k}^{\Delta} \right\}_{1 \leq v_1 < \dots < v_k \leq l}$ (resp. $\left\{ \psi_{v_1 \dots v_k}^{\Delta} \right\}_{1 \leq v_1 < \dots < v_k \leq l}$) are

cocycles (with coefficient in $\mathcal{O}^{(l)}$), we apply Lemma 3.2.

to $\left\{ \psi_{v_1 \dots v_k}^{\Delta} \right\}_{1 \leq v_1 < \dots < v_k \leq l}$ and () to $\left\{ \psi_{v_1 \dots v_k}^{\Delta} \right\}_{1 \leq v_1 < \dots < v_k \leq l}$.

Then we obtain the following decomposition of $\{\psi^{\Delta}\}$ (resp. $\{\psi^{\Delta}\}$)

$$(3.2.29) \quad \delta_{\text{cech}}(\psi^{\Delta-1}) = \psi^{\Delta} \text{ in } G^*(N(\hat{N}(\tilde{u}_i(\mathbb{R}) - \mathbb{D}), \mathcal{O}^{(l)})) \\ (\text{resp. } \delta_{\text{cech}}(\psi^{\Delta-1}) = \psi^{\Delta} \text{ in } G^*(N(\hat{N}(\tilde{u}_i(\mathbb{R}) - \mathbb{D}), \mathcal{O}^{(l)})))$$

in which cochains $\psi^{\Delta-1}$ (resp. $\psi^{\Delta-1}$) are estimat-

-ed in the following ways

$$(3.2.30)_1 \quad |\psi^{\Delta-1}(a)| \leq \bar{\alpha}_1' \cdot d(a, \mathbb{D})^{-\bar{\beta}_2'} \cdot d(a, \mathbb{D})^{\bar{\beta}_3'}$$

$$(3.2.30)_2 \quad |\psi^{\Delta-1}(a)| \leq \bar{\alpha}_1'' \cdot d(a, \mathbb{D})^{-\bar{\beta}_2''}$$

and take a positive number r . Moreover, assume that a

vector $\tilde{y}^{m,i} \in F(\tilde{\Delta}_r(P) - \tilde{\omega}, \mathcal{O}^{(i)})$ so that the equation

$$(3.2.31)_1, \quad \tilde{K}(m, i) \tilde{y}^{m,i} = 0 \quad (i \neq 0, 1)$$

and the estimation

$$(3.2.31)_2, \quad |\tilde{y}^{m,i}| \leq b \cdot d(a', \tilde{\omega})^{-d_2'} d(a', \tau)^{d_3'}; \quad a' \in \tilde{\Delta}_r(P) - \tilde{\omega};$$

hold.

Fix a couple $(\tilde{\delta})$ of positive numbers in such a way

that the couple $(\tilde{\delta})$ is chosen in an independent

manner from the choice of P . Then the w.s.y. q.

for (U, ν) is applied for any point P (near P_0) and

for $\tilde{y}^{m,i} \in F(\tilde{\Delta}_r(P) - \tilde{\omega}, \mathcal{O}^{(i)})$, and so we find a vector

$$(i = 0, 1, \dots, l-1) \quad \tilde{y}_a^{m,i+1} \text{ in } F(\Delta_{\tilde{\delta}}(a, \tilde{\omega}), \mathcal{O}^{(i+1)}); \quad a \in \tilde{\Delta}_r(P) - \tilde{\omega}; \text{ so that}$$

the equation

$$(3.2.32)_1, \quad \tilde{K}(m, i+1) \cdot \tilde{y}_a^{m,i+1} = \tilde{y}_a^{m,i} \text{ in } F(\Delta_{\tilde{\delta}}(a, \tilde{\omega}), \mathcal{O}^{(i)})$$

and the estimation

$$(3.2.32)_2, \quad |\tilde{y}_a^{m,i+1}| \leq b' \cdot d(a', \tilde{\omega})^{-d_2'} d(a', \tau)^{d_3'}; \quad a' \in \Delta_{\tilde{\delta}}(a, \tilde{\omega}),$$

are valid, where b', d_2', d_3' , are given by

$$(3.2.32)_3, \quad b' = M(b) \cdot P(a, a) e^{P(a, a)}, \quad d_2' = L_2(a_2 + a_1), \quad d_3' = L_3(a_3) - L_2(a_1),$$

3-2-5/ with data $\{M, P, P, P, L_1, L_2, L_3\}$ depending on (U, ν) only.

If $\Delta_s(q, \tilde{d}) \cap \Delta_s(q', d) \neq \emptyset$, then the equation

$$(3.2.32)_4 \quad \tilde{K}(m, i+1) (\gamma_a^{n_{i+1}} - \gamma_{a'}^{n_{i+1}}) = 0,$$

holds.

If $i+1 = \ell$ the equation () shows that the equation

$$(3.2.32)_5 \quad \gamma_a^{n_{i+1}} - \gamma_{a'}^{n_{i+1}} = 0,$$

is true.

In view of the equation (), the assertion ()

of the present lemma is verified in the case of $i+1 = \ell$.

We assume that $i+1 \neq \ell$. Apply the w. sy. q. for

the equation (): namely there exists a couple $(\tilde{\gamma})$

depending on $Q_{w, \ell, 2}$ only with which we find an

element $\tilde{\gamma}_{a, a'}^{n_{i+2}} \in \Gamma(\Delta_s(q, d; d), Q^{(i+2)})$ satisfying

the following.

$$(3.2.33)_1 \quad \tilde{K}(m, i+2) \cdot \tilde{\gamma}_{a, a'}^{n_{i+2}} = \tilde{\gamma}_a^{n_{i+1}} - \tilde{\gamma}_{a'}^{n_{i+1}},$$

$$(3.2.33)_2 \quad |\tilde{\gamma}_{a, a'}^{n_{i+2}}| \leq b'' d(q, \tilde{d})^{-d_2''} a(q', \tau)^{d_3''} \quad (q' \in \Delta_s(q, d))$$

In the above $\{b'', \alpha_2'', \alpha_3''\}$ are given by

$$(3.2.33)_3 \quad b'' = M(b) \cdot P(\delta_1) \cdot e^{P(\alpha_2, \delta)} \cdot e^{P(\alpha_3, m)}$$

$$\alpha_2'' = L(\alpha_2 + \delta_2), \quad \alpha_3'' = L_3(\alpha_3) - L_{3,2}(m).$$

(Of course, polynomials M, P 's and linear functions

L' are determined by $Q((\sigma, D)_{w, s, f, n, q, i})$ only.)

For any pair $(Q, Q') \in \overline{\Delta} \otimes \mathcal{O}$, we fix solutions $\gamma_{Q, Q'}^{n, i+2}$ ($i = 0$

$l-2$) . Denoting the right side of (

by $\gamma_{Q, Q'}^{(n, i+2), 1} \in \Gamma(\Delta_{(Q, Q')}(X))$ we obtain a 1-cocycle $\{\gamma_{Q, Q'}^{(n, i+2), 1}\}$ by the

equation

$$(3.2.34) \quad \gamma_{Q, Q'}^{(n, i+2), 1} = \gamma_{Q, Q'}^{(n, i+2), 1}$$

Now our position is to apply the v. a. to the above

cocycle $\gamma_{Q, Q'}^{(n, i+2), 1}$. Then we find a 0-cocycle $\gamma_{Q, Q'}^{(n, i+2), 0} \in C^0(N(\overline{u}(D), X))$

with algebraic growth condition $(\alpha_1, \alpha_2, \alpha_3)$. so that

the equation

$$(3.2.35) \quad \delta_{\alpha \alpha_1}(\gamma_{Q, Q'}^{(n, i+2), 0}) = \gamma_{Q, Q'}^{(n, i+2), 1}$$

Here quantities $\{b', \alpha_1', \alpha_2', \alpha_3'\}$ are expressed as follows.

$$(3.2.35)' \quad b = M^{-1}(\gamma) \cdot P(\quad), \quad = L(\quad)$$

$$= L(\quad) - L(\quad).$$

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AS in (), we define an element ().

() (P) by

$$(3.2.36) = ,$$

it is clear that we obtain the following equations.

$$(3.2.36)' , \text{ for } Q, Q$$

(P) if (w, u) is not empty.

$$(3.2.36)'' h(m, i+1) = .$$

Finally, we obtain the following estimation of the element .

$$(3.2.36)''' b d (u, v) d (u, v)$$

with (u, , : u = =

$$= .$$

in the above data are depending on

the datum only.

the equations (), () and the estimation

() gives an answer for our problem. q.e.d.

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n. 6 . In this numero we shall consider the final stage of proving lemma 3.2 : our arguments here are divided into three steps. Because each step requires certain detailed arguments, we shall outline our method here : We start with a pair $\{ (U, D)_{w, s, g}^*, (U, D)_{w, s, g}^* \}$ of two u.d.w. sy. q. as in lemma . Such a pair will be called an admissible pair of u. d. w. sy. q. Our first step is to find a suitable divisor $D \in V^*$ defined by (h) so that $(X, V, D, \mathcal{Q}_1, \mathcal{Q}_2, h)$ turns out to be an underlying datum for v. a . Then we can apply the mero. v.a for a point $P \in D \cap V^*$ (except a proper subvariety V^* of V^*)... Secondly we reduce our result, which is obtained in a meromorphic mean by making use of mero. v.a .. to a problem in $\tilde{D} = D \cap X \cap D$. This is a key step in reducing the dimension of the ' ambient space X ' and the theorem of Artin - Ree (c.f. M. Nagata) with a care about quantitative properties) enters into . Finally we combine results done in the first and the second steps .

(1) Let $(U, D)_{w, y, \xi}^*$, $(U, D)_{w, y, \xi}^*$ be an admissible pair of u.d. w. sy. q. We express the irreducible

decomposition of $V_{P_0}^*$ in the following form:

$$V_{P_0}^* = \bigcup_{i_1} V_{P_0, i_1}^* \cup \bigcup_{i_2} V_{P_0, i_2}^*, \quad \text{where the first}$$

components V_{P_0, i_1}^* exhaust all the irreducible components of

$V_{P_0}^*$ which do not coincide with any irreducible

components of V_0^* while the second components

V_{P_0, i_2}^* are irreducible components of $V_{P_0}^*$ which are

at the same time irreducible components of $V_{P_0}^*$. Germs

V_{P_0, i_1}^* and V_{P_0, i_2}^* are called of first and second type

respectively. Let D^* be a divisor defined by h^*

(in a small neighbourhood of P_0). The irreducible

decomposition of $D_{P_0}^*$ is written in the following form.

$$D_{P_0}^* = \bigcup_i D_{P_0, i}^* .$$

Moreover, we mean by $\tilde{D}^*$ the intersection $\tilde{D}^* = D^* \cap I$
and by $\tilde{\mathcal{D}}_{P_0}^*$ the germ of $\tilde{D}^*$ at P_0 .

in the following form.

$$\tilde{D}_{P_0}^* = \bigcup_{\delta} \tilde{D}_{P_0, \delta}^* .$$

Let us consider the following conditions on $D_{P_0}^*$ and $\tilde{D}_{P_0}^*$.

(3.2.)_{1.1} There exists no relations : $D_{P_0, \delta}^* \supset X_{P_0, \delta}'$

between irreducible components $D_{P_0, \delta}^*$ of $D_{P_0}^*$

and $X_{P_0, \delta}'$ of X_{P_0} .

Note that the above condition implies the following

(3.2.37)'_{1.1} There is no identities : $D_{P_0, \delta}'^* = X_{P_0, \delta}'$

between irreducible components $D_{P_0, \delta}'^*$ of $D_{P_0}'^*$ and $X_{P_0, \delta}'$

of X_{P_0} .

Moreover, we assume the following conditions.

(3.2.37)_{1.2} $\tilde{D}_{P_0}^* \supset X_{P_0}$.

(3.2.37)_{2.1} There is no inclusion relations ; $\tilde{D}_{P_0, \delta}^* \supset V_{P_0, \delta}'$,

$V_{P_0, \delta}' \supset \tilde{D}_{P_0, \delta}^*$ between irreducible components $\tilde{D}_{P_0, \delta}^*$ of $\tilde{D}_{P_0}^*$ and

$V_{P_0, \delta}'$ or V_{P_0} .

(3.2.37)_{2.2} $\tilde{D}_{P_0}^* \supset V_{P_0}'^*$

(3.2.37)_{2.3} Functions $\tilde{g}_i$ ($i = 1, \dots, l$) do not

vanish on components $\tilde{D}_{P_0, \delta}^*$ or $\tilde{D}_{P_0}^*$, where $\tilde{D}_{P_0, \delta}^*$ satisfy

the condition

$$\hat{D}_{\mathbb{A}, \mathbb{P}_0}^* \nsubseteq X_{\mathbb{A}, \mathbb{P}_0} .$$

(3.2.37)₃ For each irreducible component $V_{\mathbb{A}, \mathbb{P}_0}^{**}$ of first type, there exists a proper subgerm $V_{\mathbb{A}, \mathbb{P}_0}^{**}$ of $V_{\mathbb{A}, \mathbb{P}_0}^{**}$ in such a manner that the conditions below are true.

(3.2.37)₃' Germs of varieties $X_{\mathbb{P}_0}$ and $\hat{D}_{\mathbb{P}_0}^*$ are smooth^{*} in $V_{\mathbb{A}, \mathbb{P}_0}^{**} - V_{\mathbb{A}, \mathbb{P}_0}^{**}$ and the ideal sheaf $\hat{\mathcal{I}}(\hat{D}_{\mathbb{P}_0}^*) \subset \mathcal{O}_X$ is generated by $\hat{h}^*$ in $V_{\mathbb{A}, \mathbb{P}_0}^{**}$; $\hat{h}^*$ is the restriction of h on X .

(In the above, varieties $V_{\mathbb{A}, \mathbb{P}_0}^{**}$'s are not determined uniquely by $(D^*, V_{\mathbb{A}, \mathbb{P}_0}^{**})$. In the actual arguments, done in later, varieties $V_{\mathbb{A}, \mathbb{P}_0}^{**}$'s will be fixed for purposes of arguments.) It is not difficult to see the existence of a divisor D^* satisfying the above conditions: () ~ ()_{1,1} ₃.

But, for completeness we shall show a method to assure the existence of a divisor D^* quickly.

Let us write the irreducible decomposition of $X_{\mathbb{A}, \mathbb{P}_0}$ as follows.

$$X_{\mathbb{A}, \mathbb{P}_0} = \bigcup_{\mathbb{P}_0} X_{\mathbb{A}, \mathbb{P}_0, \mathbb{P}_0} .$$

* By this we mean that $X_{\mathbb{P}_0}$ is smooth at each point in $V_{\mathbb{A}, \mathbb{P}_0}^{**}$.

Fix sets of generators $\{ \tilde{f}_{\lambda, j, \tau_i} \}$ of the ideals

$$\mathcal{O}(X_{\lambda, P_0})' \quad \text{and} \quad \{ f'_{\lambda, j, \tau_i} \} \quad \text{of} \quad \mathcal{O}(V_{P_0}^*) .$$

Also fix generators $\{ \tilde{f}_{\lambda, j, \tau_i}'' \}$ of $(V_{P_0}^*)'$, where

germs V_{P_0, i_1}^* 's and V_{P_0, i_2}^* 's are of first and of second types

respectively. By F_{τ, τ_1, τ_2} , we mean the product $F_{\tau, \tau_1, \tau_2} = \prod_{\lambda} \tilde{f}_{\lambda, j, \tau_i} \cdot \tilde{f}_{\lambda, j, \tau_i}''$

$\prod_{\lambda} f_{\lambda, j, \tau_i} \cdot \prod_{\lambda} f'_{\lambda, j, \tau_i}$. It is clear that the germ determined by

$\{ F_{\tau, \tau_1, \tau_2} \}_{\tau, \tau_1, \tau_2}$ is nothing else than $X_{\lambda, P_0} \cup V_{P_0}^*$. we shall

define a divisor D_λ^* to be the zero locus of the

holomorphic function h_λ^* of the form

$$(3.2.38) \quad h_\lambda^* = \sum_{\tau, \tau_1, \tau_2} c_{\tau, \tau_1, \tau_2} F_{\tau, \tau_1, \tau_2}$$

In a little while we consider a divisor D_λ^* parametrized

by $(\tau) = (c_{\tau, \tau_1, \tau_2})$: It is clear that the zero locus

of h_λ^* contains X_{λ, P_0} and $V_{P_0}^*$. Take an irreducible

component V_{P_0, i_1}^* of $V_{P_0}^*$ of first type. Let the codimension

of $V_{P_0, i_1}^* \cap U$ be d . Then, from the fact that V_{P_0, i_1}^* is of first type, the following is obvious.

(3.2.39) There exist functions $F_{\tau, \tau_1, \tau_2}^1, \dots, F_{\tau, \tau_1, \tau_2}^d$ so that

$(F_{\tau, \tau_1, \tau_2}^1, \dots, F_{\tau, \tau_1, \tau_2}^d)$ generate the ideal sheaf of V_{P_0, i_1}^*

(except a proper subvariety $V_{\beta_1}^*$ of $V_{\beta_1}^*$.)

Now , in a little while , $\Sigma^* \cong \{ (c) \}$ denotes the parameter space of the function h_c^* . We

consider parameters (c) near the origin 0 of Σ^* . To show the existence of a

divisor D_L^* satisfying the conditions (3.2.37)_{1,1} ~

(3.2.37)₃, it is of course enough to

show the following.

(3.2.40) For each condition (3.2.37)_{1,1},

(3.2.37)_{2,1}, (3.2.37)_{2,2}, (3.2.37)_{2,3}, (3.2.37)₃ , there exists a proper

subgerm Σ_0' of Σ_0^* (= the germ of the whole

space Σ^* at 0), so that , for $\Sigma_0^* - \Sigma_0'$,

each condition (3.2.37)_{1,1}, (3.2.37)_{2,1}, , is true.

For conditions (3.2.37)_{1,1}, (3.2.37)_{2,1}, the above is

clear from the conditions (3.1.), (3.1.) .

On the other hand , it is clear that () leads

easily () for the condition ()₃. Finally

consider the condition () . This follows from the

following simple observation : Let $\tilde{D}_{\mu, P_0}^*$ be the germ of the intersection $\tilde{D}_{\mu, P_0}^* \cap X$, and let $\tilde{D}_{\mu, P_0}^*$ be an irreducible component of $\tilde{D}_{\mu, P_0}^*$ (satisfying the condition $\tilde{D}_{\mu, P_0}^* \not\subset X_{\mu, P_0}$). Two possibilities occur :

(i) $\tilde{D}_{\mu, P_0}^* \not\subset V_{P_0}^*$ for any irreducible component $V_{P_0, i}$ of V_{P_0} (i i) For a component $V_{P_0, i}$ of V_{P_0} the relation $V_{P_0, i} \subset \tilde{D}_{\mu, P_0}^*$ holds. In the second case, the consideration for (3.2.37)_{2,1} assures () for $\tilde{D}_{\mu, P_0}^*$. On the otherhand, in the first case there is no relation of the form $V_{P_0, i}^* \supset \tilde{D}_{\mu, P_0}^*$ (with an irreducible component $V_{P_0, i}^*$ of $V_{P_0}^*$). This is an immediate consequence of the altitude theorem of Krull ((c. f)) M. Nagata []) $\tilde{D}_{\mu, P_0, i}^*$ is of codimension one in $\overline{X}_{P_0}$, while the condition () shows that V^* are at least two dimensional codimension in X .

It is quite clear that () follows for the case of (i i) from the above consideration.

we say that a divisor D^* defined by $h^* = 0$, is called adequate (for an admissible pair $(U, D)_{u, y, z}^*$ and $(U, D)_{u, y, z}'^*$). Also a set of germs $\{V_{P_{i_1}}'', V_{P_{i_2}}''\}_{i_1, i_2}$

here $V_{P_{i_1}}^*, (V_{P_{i_2}}^*)$ associated with each irreducible component $V_{P_{i_1}}^*, (V_{P_{i_2}}^*)$ of $V_{P_0}^*$ will be called to be attached to $\{(U, D)_{u, y, z}^*, (U, D)_{u, y, z}'^*, D^*\}$ if the following conditions are satisfied.

(3.2.4) For each component $V_{P_{i_1}}^*$ of $V_{P_0}^*$ (either of first or of second type)

$$V_{i_1}''^* \subset V_{i_1}^*$$

holds.

Moreover, if $V_{i_1}^*$ is of first type then $V_{i_1}''^*$ is a proper subgerm of $V_{i_1}^*$.

(3.2.4)' For each $V_{i_1}^*$ of first type, the condition (3.2.3)' is satisfied by $V_{i_1}''^*$.

The arguments done here is nothing else than to say that there exists an adequate divisor

$$D^* \text{ to } \{(U, D)_{u, y, z}^*, (U, D)_{u, y, z}'^*\} \text{ and a set of}$$

subgerms $\{V_{i_1}^*, V_{i_2}^*\}$ attached to $\{(U, D)_{u, y, z}^*, (U, D)_{u, y, z}'^*\}$ and D^* . Henceforth, we fix such data $(D^*, \{V_{i_1}^*, V_{i_2}^*\})$.

(i i) Take an irreducible component $V_{P_{\delta_1}}^*$ of first type. Then there are uniquely determined irreducible components $\tilde{D}_{P_{\delta_1}}^*$ of $\tilde{D}_{P_{\delta_1}}^*$ and $X_{P_{\delta_1}}$ of $X_{P_{\delta_1}}$ so that the relations $\tilde{D}_{P_{\delta_1}}^* \supset V_{P_{\delta_1}}^*$ and $X_{P_{\delta_1}}^* \supset V_{P_{\delta_1}}^*$ hold. The germ $\tilde{D}_{P_{\delta_1}}^*$ is not contained in $X_{P_{\delta_1}}$ and the relation $X_{P_{\delta_1}}^* \supset \tilde{D}_{P_{\delta_1}}^*$ holds. Moreover, we state that the dimension of $\tilde{D}_{P_{\delta_1}}^*$ is at least 2. Remark that there

exists a monomial $M_{\delta_1}^*$ and a couple $(\delta_{\delta_1}^*)$,

which are depending on $(U, D)_{u, y, \delta}$ and $(V_{P_{\delta_1}}^*, V_{P_{\delta_1}}^{**})$ only, in such a manner that the following is valid.

(3.2.42) For a point $P_{\delta_1}^* \in V_{P_{\delta_1}}^* - V_{P_{\delta_1}}^{**}$ and for a positive number $r < \delta_{\delta_1}^*(P_{\delta_1}^*, V_{P_{\delta_1}}^{**})$, there

exists an analytic map $\pi_{P_{\delta_1}^*} : \Delta_r(P_{\delta_1}^*) \longrightarrow \Delta_r(P_{\delta_1}^*)_{\wedge} \tilde{D}^*$

so that $\pi_{P_{\delta_1}^*}$ is identity on $\Delta_r(P_{\delta_1}^*)_{\wedge} \tilde{D}^* : \tilde{r} = M_{\delta_1}^*(\omega)$.

This fact is shown quickly as follows : From the condition

(), we can choose functions $(\tilde{h}_1^{i_1}, \dots,$

$\dots, \tilde{h}_{d-1}^{i_{d-1}}, \tilde{h}^*)$, where $\tilde{h}_1^{i_1}, \dots, \tilde{h}_{d-1}^{i_{d-1}}$ are in $\mathcal{O}(X_{P_{\delta_1}})$

and a set of coordinates $(x_1, \dots, x_{d-1}, x_d)$ so that

the following Jacobian conditions are true.

$$(J)_1 \quad J(h_1^{i_1}, \dots, h_{i_1}^{i_1}) = \det \frac{\partial(h_1^{i_1}, \dots, h_{i_1}^{i_1})}{\partial(x_1, \dots, x_d)} \neq 0 \text{ in}$$

$$(J)_2 \quad J(h_1^{i_1}, \dots, h_{i_1}^{i_1}, h) = \det \frac{\partial(h_1^{i_1}, \dots, h_{i_1}^{i_1}, h)}{\partial(x_1, \dots, x_d, x_{d+1})} \neq 0, \text{ in}$$

In the above statements, $\tilde{d} = \text{codim}(X_{P_0, j_1})$ in U .

From the above conditions $(J)_1$ and $(J)_2$, the

assertion () follows immediately. in view of prop. 12.

Here we choose $(\tilde{x}_\lambda) = (x) - (x_1, \dots, x_d) = (x_{d+1}, \dots, x_r)$

as coordinates of X_{P_0, j_1} around a neighbourhood of V_{P_0, j_1}^* .

Also from (), we may assume that

(x_λ) are coordinates of X_i in $\Delta_i(P, V_{i_1}^*)$

for each point $P^{*i} \in V_{i_1}^* - V_{i_1}^{**}$. In $\Delta_i(P, V_{i_1}^*)$, $\tilde{h}$ is

regarded as holomorphic function of coordinates $(\tilde{x}_\lambda)$.

Assume that an element $f_{P^{*i}} \in (\Gamma(\tilde{\Delta}_i(P^*), \tilde{\mathcal{E}}^{n_i}))$ is given

$(r < \tilde{\delta}_i(P^{*i}, V_{i_1}^*)^{\tilde{d}_i})$. We assume the following explicit

expansion of $f_{P^{*i}}$.

$$(3.2.43) \quad f_{P^{*i}} = \sum_{\mu=0}^l \tilde{g}_\mu \cdot \tilde{g}_\mu^n, \quad \text{where } \tilde{g}_\mu \text{ is in}$$

$\Gamma(\tilde{\Delta}_i(P^{*i}), \tilde{\mathcal{O}}_i)$ and is estimated in the following

following way.

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$$(3.2.44) \quad |k_\mu| \leq \tilde{b} \cdot d(Q, V)^{\tilde{d}_3}.$$

Moreover, we assume that $\tilde{f}_{P^{*i_1}}$ is divisible by $\tilde{h}$ in

$\Delta_r(P^{*i_1})$. Now we show the following fact.

(3.2.45) there exist $\{\tilde{M}_1^{*i_1}, \tilde{M}_2^{*i_1}, \tilde{M}_3^{*i_1}, \tilde{L}^{*i_1}, \tilde{P}\}$ with which

two conditions below are true.

(3.2.45)₁ So far as $\tilde{d}_3 \geq \tilde{L}^{*i_1}(m)$, the function $\tilde{f}_{P^{*i_1}}$

is written in the following form in $\tilde{\Delta}_r(P^{*i_1}) \cap \mathbb{D}^+(r)$.

$$(3.2.45)_1 \quad \tilde{f}_{P^{*i_1}} = \tilde{h} \cdot \{\tilde{\sum} \tilde{k}_\mu \cdot g_\mu^m\}, \text{ where } \tilde{k}_\mu \text{ is}$$

holomorphic in $\tilde{\Delta}_r(P^{*i_1})$ and is estimated as

$$(3.2.45)_{1,2} \quad |k'_\mu| \leq \tilde{M}_2^{*i_1}(r)^{-1} \tilde{M}_3^{*i_1}(b) \cdot e^{\tilde{M}_3^{*i_1}(\tilde{d}_3, m)}.$$

(remark) The above fact is, in its algebraic form, nothing

nothing else than a special case of the Artin-Ree's

theorem (c.f. M. Nagata [1]). A quantitative examination;

which is quite simple, will be made use of in later.

We quickly show the above statement. The restricted

functions of $\tilde{f}$, $\tilde{g}_\mu$, $\tilde{k}_\mu, \dots$, on $\Delta_r(P^{*i_1}) \cap \mathbb{D}^+$ are

denoted by $\tilde{f}^*, \tilde{g}_\mu^*, \tilde{k}_\mu^*, \dots$. Then the conditions () and ()

imply the following equation.

$$(3.2.46) \quad \sum_{\mu=0}^l \tilde{k}_\mu^* \cdot \tilde{g}_\mu^m = 0.$$

From the Lojasiewicz's inequality, we obtain the following estimation of $\tilde{k}_\mu^*$.

$$|\tilde{k}_\mu^*| \leq b \cdot c_1^{\tilde{d}_3} \cdot d (Q, V \cap D^*)^{\tilde{d}_2 \tilde{d}_3}; \quad Q \in \tilde{\Delta}_F(P^{*,i}) \cap \tilde{D}_{E, \tilde{d}_1}$$

where (c_1, c_2) are positive numbers determined by (D^*, h^*) only.

Apply the induction hypothesis : Lemmas $_{n-1, \mu, \gamma, \delta}$ for (U, D) $_{\delta}^{*, n-1}$

$= (X_{\tilde{f}}, V \cap X_{\tilde{f}}, V \cap X_{\tilde{f}}^*, \tilde{u}_X, \tilde{u}_V)$. Then , so far as $\tilde{d}_3 \geq \tilde{L}^{*, i_1}(m)$, we find a vector $\tilde{y}_{P^{*, i_1}}^* \in \Gamma (\tilde{\Delta}_F(P^{*, i_1}) \cap$

$\tilde{D}^*, \tilde{O}_{\tilde{D}^*}^{(2)})$ $r'' = \tilde{M}_1^{*, i_1}(r)$ so that the equation

$$(\tilde{K}^*(m, 2) \cdot \tilde{y}_{P^{*, i_1}}^* = (\tilde{k}_\mu^*)_{\mu=1, \dots, l},$$

as well as the estimation

$$|\tilde{y}_{P^{*, i_1}}^*| \leq \tilde{M}_2^{*, i_1}(r) \cdot \tilde{M}_3^{*, i_1}(b) \cdot \tilde{P}^{*, i_1}(\tilde{d}_3, n) \cdot d(Q^*, V \cap \tilde{D}^*)^{\tilde{L}_{2,1}^{*, i_1}(\tilde{d}_3) - \tilde{L}_{2,1}^{*, i_1}(m)} : Q^* \in \tilde{\Delta}_F^{*, i_1}(P^{*, i_1}),$$

holds. Here $\tilde{M}_1^{*, i_1}, \tilde{M}_2^{*, i_1}, \dots, \tilde{L}_2^{*, i_1}, \tilde{P}^{*, i_1}$ are determined by ($(U, D)^*, (U, D)^{*, i_1}, D^*, V^*, h^*$) only.

Combining () and (), we know the assertion ().

The statement () was given in such a manner that the exponent $\tilde{\alpha}_j$, originally given, is decreased to 0. Now it is easily deduced that the assertion () leads to the following statement. (This step is completely similar to the step : $\text{prev. } a \Rightarrow v. a$), and so we shall omit any details.)

(3.2.49) There exists a datum ()

depending on (U.D), (U.D) D, only

with which the following is valid.

(3.2.49), For a given element $\tilde{f}_{P_{i_1}^*} \in [\tilde{\Delta}_x(P_{i_1}^*), \tilde{\mathcal{E}}^{\infty}]$

with the following estimation:

$$\tilde{f}_{P_{i_1}^*} = \sum_{\mu} \tilde{K}_{\mu} g_{\mu}^n, \quad \tilde{K}_{\mu} \in P(\tilde{\Delta}_x(P_{i_1}^*), \tilde{\mathcal{E}}_x); |\tilde{K}_{\mu}| \leq b,$$

as well as the following divisibility condition

consequently we find another expression of $\tilde{f}_{P_{i_1}^*}$,

$$\tilde{f}_{P_{i_1}^*} = \hat{h}^* \cdot \left(\sum_{\mu=0}^l \tilde{K}'_{\mu} g_{\mu}^n \right), \quad \tilde{K}'_{\mu} \in \tilde{\Delta}_x(P_{i_1}^*)$$

where $\tilde{K}'_{\mu}$ are estimated in the manner

$$(3.2.49)_2 \quad |\tilde{k}'_\mu| \leq M(r) M(b)^i, \quad m' = [L(m)]$$

The above estimation is a direct consequence of

() and will be used in (iii) .

(iii) Our position here is to

combine arguments in (i) and (ii)

with the zero. w. sy. q. Start with

a given admissible pair $(U.D)_{w,y,t}^*$ $(U.D)_{w,y,t}'^*$.

Also we start with a fixed adequate divisor

and a set of germs V_P^{**} attached to $(U.D)_{w,y,t}^*$

$(U.D)_{w,y,t}'^*$ and D^* . Furthermore, we fix an ex.

.zero. w. sy. q. once and for all. Arguments

here will be divided into several steps.

(iii) We first show the existence of

= $\{M, M, M, D, L, L, L\}$ with which

the assertion () is valid for any pair

$(P^*, \gamma_{P^*}^i)$, where P^* is in $V_{P_2}^* - V_P^{**}$. For this,

it is easily seen that it is enough to show the existence of

$$\text{data } Q_i^{*} ((U, D), (U, D), D^{*}, V, V_{\bar{z}}^{*}) = (\tilde{\gamma}_0^{*,i}, M_1^{*,i}, M_2^{*,i}, M_3^{*,i}, P_1^{*,i}, P_2^{*,i}, L^{*,i}, L^{*,i}) \}$$

for any components $V_{\bar{z}_1}^{*}$ of first type, with which the assertions () and () are true for any elements $(P_{\bar{z}}^{*}, V_P^{*}) \in$ satisfying the further condition : $P_{\bar{z}}^{*} \in V_{\bar{z}}^{*} - V_{\bar{z}}^{*}$.

(i i i) First we consider the case of $i = 0$.

Take a positive number $r < \tilde{\gamma}_1 d(P_i^{*}, V_{\bar{z}}^{*})^{\tilde{\gamma}_2}$ and also take an element $\tilde{f} \in \Gamma(\Delta_r(P_{\bar{z}}^{*}), \mathcal{O}_r)$, which is estimated in the following way.

$$|\tilde{f}(Q)| \leq \text{b.d.} (Q, V_{\bar{z}_1}^{*})^{d_3} : Q \in \Delta_r(P_{\bar{z}}^{*})$$

By the zero. w. sy. q , $f(Q)$ is expressed in the following manner.

$$(3.2.50) \quad f(Q) = h^{-1} (\sum_{\mu} \tilde{f}_{\mu} \cdot \tilde{g}_{\mu}^m) \quad \text{in } \Delta_r(P_{\bar{z}}^{*})$$

where $r' = M(r)$, $m = [L(Q_3)]$ and $\tilde{f}_{\mu} \in \Gamma(\Delta_r(P_{\bar{z}}^{*}), \mathcal{O}_r)$

is estimated in the following way.

$$(3.2.50)' \quad |f_{\mu}| \leq M(r)' M(b) e^{P(Q_3)}$$

Here $M, \dots, P$, are depending on $(U, D)_{v,d} =$

$\{X, v, D^*, \mathcal{H}_X, \mathcal{H}_V(h)\}$ only.

Moreover, we point out that the order of the pole $\tilde{d}$ is independent of the choice of P_i^* and of f .

Next we apply the assertion () to the function

$$\left\{ \sum_{\mu} \tilde{f}_{\mu} \cdot \tilde{g}_{\mu}^m \right\} = \tilde{h}^{\lambda} \cdot \tilde{f}. \quad \text{Then the function } f \text{ is}$$

expressed in the following manner.

$$(3.2.5/) \quad f = \tilde{h}^{(d)} \left(\sum_{\mu} \tilde{f}_{\mu}^{(d)} \cdot \tilde{g}_{\mu}^{n'} \right) \text{ in } \tilde{\Delta}_T(P).$$

In the above, r'' , m'' are determined as follows.

$$(3.2.5/) \quad r'' = M(r), \quad m'' = [L(\mathcal{M}')].$$

Furthermore, $\tilde{f}_{\mu}^{(d)}$'s are estimated in the manner

$$(3.2.5/)_2 \quad |\tilde{f}_{\mu}^{(d)}| \leq M(r') \cdot M(b') \cdot e^{P(m)}$$

Repeat the above procedure $\tilde{d}$ times. Because the order $\tilde{d}$

of poles is independent of P_i^* and of f , we

know that the assertion () holds. This gives a proof of

the case of $i = 0$.

(iii)_{1,2} We discuss the case $i \neq 0$. Start with a pos

positive number $r < \tilde{\delta}_i(P_i^*, v_{\tilde{\delta}_i}^{**})$ and a vector $\tilde{v} \in L^{(4/5)}(P_i^*)$

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Of course, in the above, $M_{A_1, M}, R, L$ are determined by $(U, D)_{v, a}$ only.

Extend the vector $\tilde{y}^{i+2}$ by the following equation

$$(3.2.52) \quad \tilde{y}^{i+2} = \sum_{\mu} W^*(\tilde{y}^{i+2}) \cdot \tilde{g}_{\mu}^{i+2},$$

to the set $\tilde{\Delta}_r(P^*)$ with a projection π_{P^*} .

(c. r. (i) in) Furthermore, define

a vector $\tilde{y}^{i+1}_{\tilde{y} \in \tilde{P}(A_1, 0)}$ by

$$(3.2.53)_1 \quad \tilde{y}^{i+1} = \tilde{y}^{i+1} + \tilde{K}(m, i+2) \cdot \tilde{y}^{i+2}.$$

Of course

$$\text{Of course } \tilde{K}(m, i+1) \tilde{y}^{i+1} = \tilde{K}(m, i+1) \cdot \tilde{y}^{i+1} = \tilde{h}^i \cdot \tilde{y}^i$$

Furthermore, the estimations () and ()

shows that the vector $\tilde{y}^{i+1}$ is expanded in the

following way.

$$(3.2.53)_2 \quad \tilde{y}^{i+1} = \sum_{\mu} \tilde{y}_{\mu}^{i+1} \cdot g_{\mu}^{m''}.$$

Here m'' is expressed in the fashion

$$m' = [L_{3,1}(\alpha_4') - L_{3,2}(m)]$$

and $\tilde{y}_{\mu}^{i+1}$ are estimated in the manner

$$(3.2.53)_3 \quad |\tilde{y}_{\mu}^{i+1}| \leq M_1(r) M_1(b) e^{P(\alpha_4, m)}$$

moreover, the equations (), () show that

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by

$$r'' = M(r), \quad \alpha_3'' = [L(\alpha_3')].$$

here M, L depend on $(U, v)_{v, \alpha}$ only.

moreover $\gamma_{\alpha}^{i+1} \in \Gamma(\tilde{\Delta}(p_{\alpha}^*), \mathcal{O}_{\tilde{\Delta}}^{(k)})$ is estimated,

with M, P depending on $(U, v)_{v, \alpha}$ only, as

follows.

$$|\gamma_{\alpha}^{i+1}| \leq M(r) M(b) \cdot e^{P(d_{\alpha})}.$$

If $i = l - 1$, then the expressions (), () and the

equation () reduce the expressions (), () and

assertion to the case of $i = 0$. Therefore we assume

that $i \neq 0, l - 1$. As in the case of $i = 0$, we

determine the order $\tilde{d}_i$ of the pole $\tilde{h}_i^*$ in the right side

of (). Our argument is completely similar to the

argument of (i i) and will be done in the following

divides. Restrict a vector $\tilde{\gamma}^{i+1}$, a mapping $\tilde{K}(m, i+1) \dots$ on

$\tilde{D}^*$. Then the equation

$$(\tilde{K}^*(m, i+1))(\tilde{\gamma}^{i+1}) = 0,$$

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$\psi^i \in (\Delta_r(P_i^*), \mathcal{O}^{(l)})$ with quantities

(r, b, d_3) , Assume the following equation.

$$(\quad), \quad \tilde{K}(m, i) \cdot \psi^i = 0 \quad (i = 1, 2, \dots, l-1)$$

The mere, w. sy. q. claims the existence of a vector

$\psi^{i+1} \in \Gamma(\Delta_r(P), \mathcal{O}_r^{(l)})$ so that the following equation

$$(\quad) \tilde{K}(m, i+1) \cdot (\psi^{i+1} \cdot h^i) = \psi^i,$$

and the estimation

$$(\quad), \quad |\psi^{i+1}| \leq M(r)^{-1} M(b) e^{P(d_3, m)} \cdot d(a, v)^{d_3'}$$

hold.

In the above quantities r', d_3' are given by

$$(\quad) \quad r' = M(r), \quad d_3' = L(d_3)$$

$$- L(m).$$

Here, of course, M, m, P, L 's are determined by $(U, D)_{v, a}$ only.

From the argument in $(\quad)$, ψ^{i+1} is further expressed

(in $\Delta_r(P^*)$) in the following manner

$$(\psi^{i+1}) = \sum_{\mu} f_{\mu}^{i+1} g_{\mu}^{i+1},$$

with quantities r'' and d_3'' are, which are given

holds. We estimate the vector $\tilde{y}^{*,i+1}$ in the form

$$(\quad) \quad |\tilde{y}^{*,i+1}| \leq c_1 M(r) \cdot M(b) \cdot e^{P(d_1, n)} \cdot \tilde{\sigma}^{d_2, d_3} : \tilde{\sigma} \in \Delta_r^*(P_1) \tilde{\sigma}^*$$

with constants (c_1, c_2) depending on $(U, D)_{\kappa, a}$ only.

Apply the theorem $\pi-1$ to obtain a vector $\tilde{y}^{*,i+2} \in F(\Delta_r(P), \tilde{\sigma}_{\tilde{\sigma}^*}^{(i+2)})$,

satisfying the equation

$$(\quad) \quad K(m, i+2) \cdot \tilde{y}^{*,i+2} = \tilde{y}^{*,i+1}.$$

From theorem $\pi-1$ we know that $\tilde{y}^{*,i+2}$ is estimated in

the following fashion,

$$(\quad) \quad |\tilde{y}^{*,i+2}| \leq M(r) \cdot M(b) \cdot e^{P(d_1, n)} \cdot \tilde{\sigma}^{d_2, d_3} : \tilde{\sigma} \in \Delta_r^*(P_1) \tilde{\sigma}^*$$

where M, P, U are depending on $(U, D)_{\kappa, a}$,

only.

Moreover, we apply the theorem $\pi-1$ (for $i=0$) to the vector

$\tilde{y}^{*,i+2}$. Then the vectors $\tilde{y}^{*,i+2}$ are expressed in the following way.

$$(\quad) \quad \tilde{y}^{*,i+2} = \sum_{\mu} \tilde{y}_{\mu}^{*,i+2} \tilde{\sigma}_{\mu}^{d_1, d_2}, \text{ in } \tilde{\Delta}_r(P_1) \tilde{\sigma}_{\tilde{\sigma}^*}^*$$

in the above, $r, \tilde{y}_{\mu}^{*,i+2}$ are estimated as

$$(\quad) \quad r = M(n), |\tilde{y}_{\mu}^{*,i+2}| \leq M(r) \cdot M(b) \cdot e^{P(d_1, n)}$$

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()₄ the vector $\tilde{y}^{i+1}$ is zero on $\tilde{D}^*$.

Needless to say that, the above M, P

L 's depend on $(U, D)_{v, a}$ only. Apply the assertion

() to the vector $\tilde{y}^{i+1}$. Then the equation

() can be written, inductively on the

order of in the following way.

$$() \quad \tilde{K}(m, i+1) \left(\tilde{y}_{d-1}^{i+1} \right) = h^{i-1} \tilde{y}^i \text{ in } \Delta(P_i^*); r=M(\alpha),$$

where $\tilde{y}_{d-1}^{i+1}$ is expressed as follows.

$$() \quad \tilde{y}_{d-1}^{i+1} = \sum_{\mu} \tilde{y}_{d-1, \mu}^{i+1} \tilde{g}_{\mu}^{m''},$$

with $m'' = L_{\beta}(\alpha_3) - L_{\beta, 2}(m, D)$.

Moreover, $\tilde{y}_{d-1, \mu}^{i+1}$'s are estimated in the fashion

$$() \quad |\tilde{y}_{d-1, \mu}^{i+1}| \leq \{M(r)\}^{-1} M(\beta) e^{P(\alpha_3)}.$$

Because M, P, L 's are depending on $(U, D)_{v, a}$

and the degree of the pole $\tilde{\alpha}_i$ is independent of P_i^* and

$\tilde{y}^i$, we conclude (by repeating procedures

$\tilde{d}_i$ times) arguments of this part.

(iii)₂ NOW, it is easy to derive our assertion

in this Lemma (), from the results in ():

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What is remained here is to show the existence
of a datum with which the assertion
() is valid for any point in
. - V . But deducing the desired results
from the conclusion in (i i) is
done in an entirely same way as in the
proof of proposition . We, therefore, leave
this part untouched. q.e.d.