

The evolution of harmonic mappings with free boundaries

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Abstract: We establish the existence of a global, partially regular weak solution to the evolution problem for harmonic maps with free boundaries on a suitable support hypersurface.

1. Let (M, g) be a m -dimensional manifold with boundary ∂M and let N be a compact ℓ -dimensional manifold, which for convenience we may regard as isometrically embedded in some Euclidean space $\mathbb{R}^n$. Also let Σ be a k -dimensional sub-manifold of $\mathbb{R}^n$, $S = \Sigma \cap N$. Finally, let $u_0 = (u_0^1, \dots, u_0^n) : M \rightarrow N$ with $u_0(\partial M) \subset S$ be given.

We study the existence of harmonic maps $u : M \rightarrow N \hookrightarrow \mathbb{R}^n$ solving the free boundary problem

$$(1.1) \quad -\Delta u = \Gamma(u)(\nabla u, \nabla u) \perp T_u(N) ,$$

$$(1.2) \quad u(\partial M) \subset S ,$$

$$(1.3) \quad \frac{\partial}{\partial n} u \perp T_u S \quad \text{on } \partial M ,$$

where n denotes a unit normal vector field along ∂M , $\Delta = \Delta_M$ is the Laplace-Beltrami operator on M , and Γ denotes a bilinear form related to the second fundamental form of the embedding $N \hookrightarrow \mathbb{R}^n$. Finally, $T_p N$ denotes the tangent space (in $\mathbb{R}^n$) of N at p , and $\perp$ means orthogonal (in $\mathbb{R}^n$). That is, we look for critical points of the energy

$$(1.4) \quad E(u) = \frac{1}{2} \int_M |\nabla u|^2 dM$$

on the space of maps

$$H_S^{1,2}(M; N) = \{u \in H^{1,2}(M; \mathbb{R}^n); u(M) \subset N, u(\partial M) \subset S\}.$$

Here, $H^{1,2}(M; \mathbb{R}^n)$ is the Sobolev space of L^2 -maps $u : M \rightarrow \mathbb{R}^n$ with $\nabla u \in L^2$; the norm $|\nabla u|^2$ is computed in the metric on M .

As in [13] for a related problem, we approach (1.1)-(1.3) by means of the evolution problem

$$(1.5) \quad u_t - \Delta u = \Gamma(u)(\nabla u, \nabla u) \quad \text{on } M \times [0, \infty[,$$

$$(1.6) \quad u(x, t) \in S, \quad \text{for } x \in \partial M, t \geq 0,$$

$$(1.7) \quad \frac{\partial}{\partial n} u(x, t) \perp T_{u(x, t)} S, \quad \text{for } x \in \partial M, t > 0,$$

$$(1.8) \quad u(\cdot, 0) = u_0 \quad \text{on } M.$$

If $m = 2$ this strategy has been successfully implemented by Ma Li [10]. See also Dierkes-Hildebrandt-Wohlrab [5] and Hildebrandt-Nitsche [7] for further material on the two-dimensional case. Here we confront the higher dimensional case $m \geq 3$. Assume all data are smooth. For simplicity, we consider only the case

$$M = B = B_1(0) = \{x \in \mathbb{R}^m; |x| < 1\}.$$

Moreover, we make the following assumption about Σ , the global “extension” of S to the ambient Euclidean space:

$$(1.9) \quad \begin{array}{l} \text{There exists a ball } U \subset \mathbb{R}^n \text{ containing } N, \text{ whose boundary } \partial U \\ \text{intersects } \Sigma \text{ orthogonally in the sense that the normal } \nu_U \text{ to } \partial U \text{ at} \\ \text{a point } p \in \Sigma \text{ lies in } T_p \Sigma. \end{array}$$

In addition assume that the nearest neighbor projection $\pi_\Sigma : U \rightarrow \Sigma \cap U$ is well-defined and smooth in U , and

$$(1.10) \quad |D^2 \pi_\Sigma| \cdot \text{diam}(U) < 1/2.$$

Let $R_\Sigma(p) = 2\pi_\Sigma(p) - p$ be the reflection of a point $p \in U$ in Σ . Also we suppose Σ is oriented by a smooth normal frame $\nu = (\nu_1, \dots, \nu_{n-k})$. An example of a configuration (N, Σ) satisfying (1.9-10) is $N = S^{n-1} \subset \mathbb{R}^n$, $\Sigma = \mathbb{R}^k \times \{0\}$, $k \leq n-1$, or a perturbation of $\mathbb{R}^k \times \{0\}$ by a diffeomorphism $\Phi = id + \varepsilon \tau$, with a smooth map

$\tau : \mathbb{R}^n \rightarrow \mathbb{R}^n$ having compact support, and $|\varepsilon| < \varepsilon_0 = \varepsilon_0(\tau)$. Then we obtain the following result reminiscent of the results in [2] for the evolution of harmonic maps on closed domains, that is, with $\partial M = \emptyset$.

Theorem 1.1: Suppose $M = B, N, S, u_0$ are as above and S satisfies conditions (1.9-10). Then there exists a global weak solution u of problem (1.5-8) satisfying the energy inequality

$$\int_0^T \int_B |u_t|^2 dx dt + E(u(T)) \leq E(u_0),$$

for all $T \geq 0$, and smooth off a singular set of codimension ≥ 2 . As $t \rightarrow \infty$ suitably, $u(t)$ converges weakly in $H^{1,2}(B; N)$ to a weak solution u_∞ of (1.1-3) which is smooth off a set of codimension ≥ 2 .

Remark 1.1: (i) If the range $u(B \times [0, \infty))$ lies in a convex neighborhood of a point p on N , u is globally smooth and converges uniformly on $\bar{B}$ to a smooth solution u_∞ of (1.1-3) homotopic to u_0 .

(ii) Conversely, for instance in the case of a sphere as target manifold, it is known that solutions to (1.5) may develop singularities in finite time, see [4], [1].

(iii) A result like Theorem 1.1 should also hold without the hypotheses (1.9-10) on S ; however, for a general support manifold S - already in the Euclidean case $N = \mathbb{R}^n$ and in contrast to the two-dimensional case - in higher dimensions $m \geq 3$ the problem of boundary regularity for (1.5) poses considerable difficulties and the construction of global, partially regular solutions to (1.5-8) or (2.1), (1.6-8) below is not yet within reach.

(iv) Similar results should hold on a general compact domain with boundary. In fact, much of what follows is true for such general domains and we keep the notation M in that case.

2. Let $U_\delta(N)$ be the δ -tubular neighborhood of N in $\mathbb{R}^n$. We may choose $\delta > 0$ such that $U_\delta(N) \subset U$, see (1.9), and such that the nearest neighbor projection $\pi : U_\delta(N) \rightarrow N$ is well-defined and smooth in $U_\delta(N)$. Let $\chi \in C_0^\infty(\mathbb{R})$ be a non-decreasing function satisfying $\chi(s) = s$ for $0 < s < \frac{\delta^2}{2}$, $\chi(s) = \delta^2$ for $s \geq \delta^2$.

Following the approach of [2], we approximate (1.5-8) by the following evolution problem for maps with range in $\mathbb{R}^n$:

$$(2.1) \quad u_t - \Delta u + K\chi'(\text{dist}^2(u, N)) \frac{d}{du} \left(\frac{\text{dist}^2(u, N)}{2} \right) = 0$$

in $M \times [0, \infty[$, with boundary and initial conditions (1.6-8). (2.1) is the evolution equation for the functional

$$(2.2) \quad E_K(u) = \frac{1}{2} \int_M \left[|\nabla u|^2 + K\chi(\text{dist}^2(u, N)) \right] dM$$

for maps $u : M \rightarrow \mathbb{R}^n$.

Lemma 2.1: Let u be a smooth solution to (2.1), (1.6-8). Then we have

$$\int_0^T \int_M |u_t|^2 dM dt + E_K(u(T)) \leq E_K(u_0) = E(u_0)$$

for all $T \geq 0$.

Proof: Multiply (2.1) by u_t and integrate by parts. The boundary term vanishes on account of (1.6-7). □

For the following result hypotheses (1.9-10) on S are essential.

Lemma 2.2: Suppose $u \in C^1(\overline{M} \times [0, T[; \mathbb{R}^n)$ is a smooth solution to (2.1), (1.6-8) on $\overline{M} \times [0, T[$; then u and its first spatial derivatives are uniformly bounded and u extends to a smooth solution of (2.1), (1.6-8) on $\overline{M} \times [0, T]$.

Proof: The interior estimates easily follow from the energy estimate Lemma 2.1 and the interior regularity estimates for the heat equation; see for instance [9]. To obtain the estimates at the boundary we argue as follows. Note that by the maximum principle for the heat equation and (1.6-7), (1.9) the image of u satisfies $u(x, t) \in U$ for all (x, t) , and by (1.10) the reflection of u in Σ is defined. Thus, in the special case $M = B$, for $x \in \mathbb{R}^m, t \geq 0$ we may let

$$\tilde{u}(x, t) = \begin{cases} u(x, t) & , \text{ if } |x| < 1, \\ R_\Sigma(u(x/|x|^2)) & , \text{ if } |x| > 1. \end{cases}$$

Then $\tilde{u}$ is of class C^1 on $\mathbb{R}^m \times [0, T[$ and satisfies

$$(2.3) \quad |\tilde{u}_t + A\tilde{u}| \leq \begin{cases} CK & , \text{ if } |x| < 1 \\ CK + \Gamma_\Sigma(\tilde{u})(\nabla\tilde{u}, \nabla\tilde{u}) & , \text{ if } |x| > 1, \end{cases}$$

where A is an elliptic operator in divergence form with Lipschitz coefficients, $A = -\Delta$ for $|x| < 1$, and where Γ_Σ is a bilinear form related to the second fundamental form of $\Sigma \subset \mathbb{R}^n$.

In fact, from

$$\begin{aligned} (\tilde{u}_t + A\tilde{u}) \left(\frac{x}{|x|^2} \right) &:= \left(2(\partial_t - \Delta)\pi_\Sigma(u) - (\partial_t - \Delta)u \right)(x, t) = \\ &= \left(2[D\pi_\Sigma(u) - id][(\partial_t - \Delta)u] - 2D^2\pi_\Sigma(u)(\nabla u, \nabla u) \right)(x, t), \end{aligned}$$

we can read off the precise form of A and Γ_Σ . (2.3) is a parabolic system of the type

$$u_t + Au = f(\cdot, u, \nabla u),$$

on any ball $B_\rho = B_\rho(0)$, where

$$|f(\cdot, u, p)| \leq a|p|^2 + b$$

with constants $a, b \in \mathbb{R}$. Moreover, by (1.10), for $\rho > 1$ sufficiently close to 1 there holds

$$a \cdot \sup|u| < \lambda,$$

where $\lambda > 0$ denotes the ellipticity constant of the operator A on B_ρ . By the results of [6] for such systems, $\tilde{u}$ is locally Hölder continuous on $B_\rho \times]0, T]$. Higher regularity $|\nabla^2 \tilde{u}| \in L^2_{loc}(B_\rho \times [0, T])$, $|\nabla \tilde{u}| \in L^4_{loc}(B_\rho \times [0, T])$ then follows as in [9]. Finally, by [9; p. 593f.] we also obtain uniform bounds for $\nabla \tilde{u}$ in L^{2p}_{loc} and hence $\tilde{u}_t$ and $\nabla^2 \tilde{u}$ in L^p_{loc} for all $p < \infty$. By the Sobolev embedding theorem [9; Lemma II. 3.3] this then implies the desired bound. □

The a-priori bounds of Lemma 2.2 now yield the following global existence result.

Proposition 2.1: Under the hypotheses of Theorem 1.1, for any $K \in \mathbb{N}$ there exists a global solution $u = u_K \in C^1(\overline{B} \times [0, \infty[; \mathbb{R}^n)$ to (2.1), (1.6-8). The solution u is smooth in $\overline{B} \times [0, \infty[$ and satisfies the energy inequality Lemma 2.1.

Proof: Local existence follows from a fixed point argument as in [13]. For completeness we sketch the argument. Extend u_0 to $\mathbb{R}^n$ by letting

$$(2.4) \quad u_0(x) = R_\Sigma \left(u \left(\frac{x}{|x|^2} \right) \right)$$

for $x \notin \overline{B}$, and fix $\rho > 0$, $T > 0$ sufficiently small. Let

$$V_\rho(T) = \left\{ u \in C^{1,1/2}(\overline{B}_\rho \times [0, T]; \mathbb{R}^n); u(0) = u_0 \right\},$$

where $C^{1,1/2}(\dots)$ is the space of functions u which are continuously differentiable in the spatial variable x and uniformly Hölder continuous in time with Hölder exponent $\frac{1}{2}$. A norm is given by the Hölder constant and $\|\nabla u\|_{L^\infty}$. - In [9;p.7f.] this space is introduced as $H^{1,1/2}$.

For $u \in V_\rho(T)$ let v solve

$$(2.5) \quad v_t + A v = \begin{cases} K \chi'(\text{dist}^2(u, N)) \frac{d}{du} \left(\frac{\text{dist}^2(u, N)}{2} \right), & \text{if } |x| < 1 \\ K \chi'(\dots) \frac{d}{du}(\dots) + \Gamma_\Sigma(u)(\nabla u \nabla u), & \text{if } |x| > 1, \end{cases}$$

on $B_\rho \times [0, T]$ with boundary and initial data u . By the interior estimates for the heat equation we can bound v and its first and second derivatives in Hölder norm on $\partial B_{1/\rho} \times [0, T]$ in terms of the $C^{1,1/2}$ -norm of u on $B_\rho \times [0, T]$ and u_0 . Define new C^2 -Dirichlet data by letting

$$w(x, t) = R_\Sigma \left(v \left(\frac{x}{|x|^2}, t \right) \right), x \in \partial B_\rho,$$

and let $\bar{u}$ solve (2.5) with initial data u_0 and boundary data w . By (2.4) w and u_0 are compatible. Moreover, by the linear estimates for the heat equation (see [7; Theorem IV. 9.1]) the map $F : u \mapsto \bar{u}$ is bounded from $C^{1, \frac{1}{2}}(\overline{B}_\rho \times [0, T])$ into the space

$$W_p^{2,1} = \left\{ u \in L^p(B_\rho \times [0, T]); u_t, \nabla^2 u \in L^p \right\}$$

for all $p < \infty$, which for $p > m + 2$ is compactly embedded into $C^{1, \frac{1}{2}}(\overline{B_\rho} \times [0, T])$; see [9; Lemma II.3.3]. Finally, if $T > 0$ is sufficiently small, F maps a convex $C^{1, \frac{1}{2}}$ -neighborhood of the function $u(t) \equiv u_0$ to itself. Hence F has a fixed point $u = F(u)$, satisfying (2.5) and the condition

$$u(x, t) = w(x, t) = R_\Sigma \left(v(x/|x|^2, t) \right)$$

on $\partial B_\rho \times [0, T]$. But then also $u_1(x, t) = R_\Sigma \left(u \left(\frac{x}{|x|^2}, t \right) \right)$ is a solution of (2.5) in $\{(x, t); 1/\rho < |x| < \rho\}$ with the same initial and boundary data. It follows that $u = u_1$ and thus u satisfies (2.1), (1.6-8). The local solution can be continued globally on account of Lemma 2.2. □

To derive uniform interior estimates independent of K we need the following analogue of the monotonicity formula from [14]. Fix $z_0 = (x_0, t_0) \in \overline{M} \times]0, \infty[$. Let

$$G(x, t) = \frac{1}{\sqrt{4\pi|t|}^m} \exp \left(-\frac{|x|^2}{4|t|} \right)$$

be the fundamental solution to the heat equation. Then let

$$\Phi_{z_0}(R) = \Phi_{z_0}(R; u, K) = \frac{1}{2} R^2 \int \left[|\nabla u|^2 + K \chi(\text{dist}^2(u, N)) \right] G(\cdot - z_0) dx,$$

where we integrate over $B \times \{t_0 - R^2\}$. On a general domain we would need to localize Φ in coordinate charts via suitable cut-off functions, as in [2].

Lemma 2.3: There exist constants depending only on M and N such that for all $z_0 = (x_0, t_0)$ and $0 \leq R \leq R_0 \leq \sqrt{t_0}$ there holds

$$\Phi_{z_0}(R) \leq \exp(c(R_0 - R)) \Phi_{z_0}(R) + c E(u_0)(R_0 - R).$$

Proof: At interior points this result was obtained in [2; Lemma 4.2]. At the boundary, for simplicity we present the proof only for a half-space $M = \mathbb{R}_+^m$, where

$$\mathbb{R}_+^m = \{x = (x', x_m) \in \mathbb{R}^m; x_m > 0\},$$

and $z_0 = (0, 0)$. (The general case then follows as in [2].) Consider the family of scaled maps

$$u_R(x, t) = u(Rx, R^2 t).$$

Note that u_R satisfies (2.1) with $R^2 K$ instead of K , and also satisfies (1.6), (1.7). Moreover,

$$\Phi_0(R; u, K) = \Phi_0(1; u_R, R^2 K),$$

whence (at $R = 1$, say)

$$\begin{aligned} \frac{d}{dR} \Phi_0(R; u, K) &= \frac{d}{dR} \Phi_0(1; u_R, R^2 K) \\ &= \int_{S_+} \left\{ \nabla u \cdot \nabla \left(\frac{d}{dR} u_R \right) + K \chi(\text{dist}^2(u, N)) \right. \\ &\quad \left. + K \chi'(\dots) \frac{d}{du} \left(\frac{\text{dist}^2(u, N)}{2} \right) \frac{d}{dR} u_R \right\} G \, dx, \end{aligned}$$

where $S_+ = \mathbb{R}_+^m \times \{-1\}$. Integrating by parts in the first term, on account of (2.1) and the fact that $\nabla G = \frac{x}{2t} G$, this gives

$$= \int_{S_+} \frac{|x \cdot \nabla u + 2tu_t|^2}{2|t|} G \, dx + \int_{S_+} K \chi(\text{dist}^2(u, N)) G \, dx \geq 0,$$

as desired. Note that by (1.6-7) no boundary terms appear. □

Denote by

$$e_K(u) = \frac{1}{2} \left\{ |\nabla u|^2 + K \chi(\text{dist}^2(u, N)) \right\}$$

the energy density for the penalized equation. For a point $z_0 = (x_0, t_0) \in \mathbb{R}^m \times \mathbb{R}$, $\rho > 0$ also denote

$$P_\rho(z_0) = \{z = (x, t); |x - x_0| < \rho, t_0 - \rho^2 < t < t_0\}$$

the parabolic cylinder of radius ρ centered at z_0 , $P_\rho = P_\rho(0)$ for brevity, and let

$$P_\rho^+(z_0) = P_\rho(z_0) \cap \{x_m > 0\},$$

$$P_\rho^-(z_0) = P_\rho(z_0) \cap \{x_m < 0\},$$

respectively.

Lemma 2.4: There exists a constant $\varepsilon_0 > 0$ depending only on M and N with the following property: If for some $z_0 = (x_0, t_0) \in \overline{M} \times]0, \infty[$ and $R < \varepsilon_0$ the inequality

$$\Phi_{z_0}(R; u_K, K) < \varepsilon_0$$

is satisfied, then

$$\sup_{P_{\delta R}(z_0)} e_K(u_K) \leq c(\delta R)^{-2},$$

with constants c depending only on M and N and $\delta > 0$ possibly depending also on $E(u_0)$ and $\min\{R, 1\}$.

Proof: The proof for interior points $x_0 \in M$ is the same as that of Lemmas 2.4, 4.4 of [2]. We sketch the modifications at a boundary point x_0 . Again assume for simplicity that $M = \mathbb{R}_+^m$ and shift z_0 to 0. By reflection we may extend u to a solution $\tilde{u}$ of

$$(2.6) \quad \tilde{u}_t - \Delta \tilde{u} = \begin{cases} K\chi'(\text{dist}^2(\tilde{u}, N)) \frac{d}{du} \left(\frac{\text{dist}^2(\tilde{u}, N)}{2} \right), & \text{if } x_m > 0 \\ K\chi'(\dots) \frac{d}{du}(\dots) + \Gamma_\Sigma(\tilde{u})(\nabla \tilde{u}, \nabla \tilde{u}), & \text{if } x_m < 0 \end{cases}$$

on a full neighborhood of x_0 . Scaling as in [2; p. 92], we obtain a solution v of problem (2.6) for some $\tilde{K} = \frac{K}{e_0}$ on P_1 , satisfying

$$e_{\tilde{K}}(v) \leq 4$$

and

$$e_{\tilde{K}}(v)(0) = 1.$$

Moreover, we have the differential inequality

$$(2.7) \quad (\partial_t - \Delta)e_K(v) + |\nabla^2 v|^2 \leq C e_K(v),$$

separately in P_1^+ and P_1^- . (The proof of this Bochner-type estimate can be conveyed very easily from [2; p. 90].) Let us for brevity write $e_{\tilde{K}}(v) = e(v)$ in the sequel. Our aim is to extend (2.7) to P_1 .

Due to the structure of (2.6), $\Delta e(v)$ may have a singular component on the hypersurface $\{x_m = 0\}$ - in our old coordinates. As in [13], we may control this component in the following way.

Given $\varphi \in C_0^\infty(B)$, $-1 < t < 0$, we have

$$-\int \Delta e(v) \varphi^2 dx = \int_{\{x_m=0\}} [\partial_{x_m} e(v)]_-^+ \varphi^2 dx' + 2 \int \nabla e(v) \nabla \varphi \varphi dx,$$

where $\int \dots$ denotes integration over $B \times \{t\}$, and where we denote

$$[f(x', 0)]_-^+ = \lim_{x_m \searrow 0} f(x', x_m) - \lim_{x_m \nearrow 0} f(x', x_m)$$

for any function f .

To estimate the boundary integral we decompose

$$\begin{aligned}
 [\partial_{x_m} e(v)]_-^+ &= \frac{1}{2} \left[\partial_{x_m} (|\nabla v|^2) \right]_-^+ + \frac{\tilde{K}}{2} \left[\partial_{x_m} \chi(\text{dist}^2(v, N)) \right]_-^+ \\
 &= \frac{1}{2} \left[\partial_{x_m} (|\nabla v|^2) \right]_-^+ \\
 &= [\partial_{x_m}^2 v \partial_{x_m} v]_-^+ + [\partial_{x_m} (\nabla_{x'} v) \nabla_{x'} v]_-^+ \\
 &= [\Delta v \partial_{x_m} v]_-^+ - 2[\Delta_{x'} v \partial_{x_m} v]_-^+ + [\nabla_{x'} \cdot (\partial_{x_m} v \nabla_{x'} v)]_-^+ .
 \end{aligned}$$

But by (1.6), (1.7)

$$\partial_{x_m} v \nabla_{x'} v = 0 .$$

Hence, and on account of (2.6), (1.6), we have

$$[\partial_{x_m} e(v)]_-^+ = \langle \Gamma_\Sigma(v)(\nabla v, \nabla v), \partial_{x_m} v \rangle - 2[\Delta_{x'} v \partial_{x_m} v]_-^+ ,$$

where for clarity we now denote $\langle \cdot, \cdot \rangle$ the scalar product in $\mathbb{R}^n$. Using the normal frame $\nu = (\nu_1, \dots, \nu_{n-k})$ for Σ , the last term by (1.7) may be more conveniently written

$$\begin{aligned}
 \Delta_{x'} v \partial_{x_m} v &= \sum_j \langle \Delta_{x'} v, \nu_j(v) \rangle \langle \nu_j(v), \partial_{x_m} v \rangle \\
 &= - \sum_j \langle \nabla_{x'} v, \nabla_{x'} (\nu_j(v)) \rangle \langle \nu_j(v), \partial_{x_m} v \rangle .
 \end{aligned}$$

Smoothly extend ν_j to $\mathbb{R}^n$. Then by the divergence theorem

$$\begin{aligned}
 \int_{\{x_m=0\}} [\partial_{x_m} e(v)]_-^+ \varphi^2 dx' &= \int_{P_1^-} \text{div} \left(\langle \Gamma_\Sigma(v)(\nabla v, \nabla v), \nabla v \rangle \varphi^2 \right) dx \\
 &\mp \sum_j \int_{P_1^\pm} \text{div} \left(\langle \nabla_{x'} v, \nabla_{x'} (\nu_j(v)) \rangle \langle \nu_j(v), \nabla v \rangle \varphi^2 \right) dx \\
 &\leq C \int_{P_1} (|\nabla^2 v| |\nabla v|^2 + |\nabla v|^4) \varphi^2 dx + C \int_{P_1} |\nabla v|^3 |\nabla \varphi| |\varphi| dx \\
 &\leq \varepsilon \int_{P_1} |\nabla^2 v|^2 \varphi^2 dx + C(\varepsilon) \int_{P_1} |\nabla v|^4 \varphi^2 dx \\
 &\quad + C(\varepsilon) \int_{P_1} |\nabla v|^2 |\nabla \varphi|^2 dx ,
 \end{aligned}$$

and - choosing $\varepsilon > 0$ sufficiently small - it follows that the inequality (2.7) - up to a factor - holds on P_1 in the distribution sense. But then the remainder of the proof of [2] applies also in this case. □

As in [2], we may now pass to the limit $K \rightarrow \infty$. Let u_K be a sequence of smooth solutions to (2.1), (1.6-8). We may assume that u_K converges weakly to u in the sense

$$\begin{aligned} \nabla u_K &\rightharpoonup \nabla u \quad \text{weakly} - * \text{ in } L^\infty([0, \infty[; L^2(M)), \\ \frac{\partial}{\partial t} u_K &\rightharpoonup \frac{\partial}{\partial t} u \quad \text{weakly in } L^2(M \times [0, \infty[), \\ u_K &\rightarrow u \quad \text{strongly in } L^2_{loc}(M \times [0, \infty[), \end{aligned}$$

and almost everywhere, where $u : \overline{M} \times [0, \infty[\rightarrow N$.

Proposition 2.2: The limit u weakly solves problem (1.5-8). Moreover, u is smooth and solves (1.5) classically on a dense relatively open set $Q_0 \subset \overline{M} \times [0, \infty[$ whose complement Q' has locally finite $(m-2)$ -dimensional Hausdorff measure on each time slice $\overline{M} \times \{t = \text{const.}\}$. Moreover, u satisfies the energy inequality

$$\int_0^T \int_M |u_t|^2 dM dt + E(u(T)) \leq E(u_0) ,$$

for all $T > 0$. Finally, as $t \rightarrow \infty$ suitably, a sequence $u(\cdot, t)$ converges weakly in $H^{1,2}(M; N)$ to a solution u_∞ of (1.1-3) with $E(u_\infty) \leq E(u_0)$ and smooth away from a closed set Q'' of finite $(m-2)$ -dimensional Hausdorff measure.

Proof: All proofs except (1.3), (1.7) are identical with those of [12; Theorem 6.1], resp. [2; Theorem 1.5] in the case of harmonic maps on domains without boundary. See [3] for an estimate of $H^{m-2}(Q' \cap \{t = \text{const.}\})$. To see (1.3), (1.7) in the case of a half-plane we extend u_K by reflection to solutions $\tilde{u}_K$ of equations (2.6), converging weakly locally to a function $\tilde{u}$. On Q_0 , as in [2; p. 94], we have C^1 -convergence $u_K \rightarrow u$, and (1.7) holds on Q_0 . Moreover, there holds $K \cdot \text{dist}(u, N) \rightarrow \lambda$ weakly in $L^2_{loc}(Q_0)$, whence

$$(2.7) \quad \tilde{u}_t - \Delta \tilde{u} \in L^2_{loc}(Q_0) .$$

Now let φ be an arbitrary testing function and let $\eta \in H^{1,\infty}$, $0 \leq \eta \leq 1$, $\eta = 0$ in a neighborhood of Q' , as in [2; p. 95]. Multiplying (2.7) by $\varphi\eta$, we obtain that

$$\int_0^\infty \int_{\mathbb{R}^m} (\tilde{u}_t - \Delta \tilde{u}) \varphi \eta dx dt = \int_0^\infty \int_{\mathbb{R}^m} \{ \tilde{u}_t \varphi + \nabla \tilde{u} \nabla \varphi \} \eta dx dt + F ,$$

where

$$|F| \leq \int |\nabla u| |\nabla \eta| |\varphi| dx dt \leq C(\eta) \left(\int_{\text{supp}(\nabla \eta)} |\nabla u|^2 \varphi^2 dx dt \right)^{1/2} .$$

As in [2] we may choose a sequence of maps η as above with a uniform constant $C(\eta) = C$ such that $\eta \rightarrow 1$ almost everywhere and $(\text{supp}(\nabla \eta)) \rightarrow 0$ in measure. By absolute continuity of the Lebesgue integral, thus $F \rightarrow 0$, and (1.7) also holds in the distribution sense. The proof of (1.3) is similar. □

Theorem 1.1 is an immediate consequence of Proposition 2.2. Remark 1.1 follows by adapting the argument of [8] to our problem. Since this technique is by now well-known we may omit the details.

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