

On the fields of moduli for FM-structures.

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We shall give a rough description of our subject because the problem discussed here will be considered more thoroughly in a separate paper.

Throughout the paper we fix a universal domain $\mathbb{K}$ and are concerned with algebraic geometry over $\mathbb{K}$.

DEFINITION 1.1. $(\mathcal{X}, F)$ is a pair of a field F and a collection $\mathcal{X}$ of geometric objects $S, S', \dots$, together with the following three laws:

i) Given $S \in \mathcal{X}$, the notion of fields of rationality for S is defined. (We consider only a field k containing F as a field of rationality for S .)

ii) Given S & $S' \in \mathcal{R}$, $S \cong S'$ or $S \not\cong S'$.

iii) Given $S \in \mathcal{R}$ and $\sigma \in \text{Aut}(\mathbb{K}/F)$ = the group of automorphisms of $\mathbb{K}$ over F , S^σ is defined and belongs to $\mathcal{R}$.

The pair $(\mathcal{R}, F)$ is called an FM-system and an object S in $\mathcal{R}$ is called an FM-structure (in $(\mathcal{R}, F)$) if the following conditions are satisfied:

fm i) S is rational over k and $k' \supset k$
 $\Rightarrow S$ is rational over k' .

fm ii) $\cong$ is an equivalence relation.

fm iii) S & $S' \in \mathcal{R}$ and $\sigma, \tau \in \text{Aut}(\mathbb{K}/F)$

1) $\sigma|_k = \text{identity}$, where k is a field of rationality for S . $\Rightarrow S^\sigma = S$

2) S is rational over $k \Rightarrow S^\sigma$ is rational over k^σ .

3) $S \cong S' \Rightarrow S^\sigma = S'^\sigma$.

4) $S^{\sigma\tau} = (S^\sigma)^\tau$.

DEFINITION 1.2. For an FM-system $(\mathcal{X}, F)$ and $S \in \mathcal{X}$, a field K_S containing F is called the "field of moduli for S " if the following two conditions are satisfied:

$$\text{FM 1)} \quad \text{For } \sigma \in \text{Aut}(\overline{K}/F), \\ S^\sigma \cong S \iff \sigma|_{K_S} = \text{identity}.$$

$$\text{FM 2)} \quad K_S = \bigcap K,$$

where K runs over the set of all fields of rationality for all $S' \cong S$.

EXAMPLE 2.1. Let V be a complete variety rational over F , non-singular in codimension 1, containing an F -rational simple point; let $\mathcal{D}_a(V)$ be the group of V -divisors algebraically equivalent to zero. The relation $\cong$ in $\mathcal{D}_a(V)$ is defined by the linear equivalence. Then $(\mathcal{D}_a(V), F)$ becomes an FM-system in a natural way and the field of moduli $K_{\text{el}}(X)$ for $X \in \mathcal{D}_a(V)$ exists and is generated by the point of the Picard variety $\hat{V}$ of V over F , which corresponds to the linear class

$Cl(X)$ determined by X , (if we choose a suitable model of the Picard variety of V).

EXAMPLE 2.2. Let V be a projective non-singular variety rational over F having an F -rational point. If we replace $\mathcal{D}_a(V)$ and the linear equivalence in Ex. 2.1. respectively by the group $\mathcal{Z}_a(V)$ of zero-cycles on V of degree zero and the regular equivalence in $\mathcal{Z}_a(V)$, we get the same kind of results for $(\mathcal{Z}_a(V), F)$ as those for $(\mathcal{D}_a(V), F)$ in Ex. 2.1., and the Albanese variety of V works as the Picard variety of V did in the previous case.

EXAMPLE 2.3. For a polarized abelian variety $\mathcal{P} = (A, p(X))$, the field of moduli $\mathbb{K}_p$ exists.

EXAMPLE 2.4. Let V be a complete non-singular curve. For the biregular isomorphism

class determined by V , the field of moduli k_V exists.

REMARK. In the case of positive characteristic, if we consider a birational class determined by a curve (may be having singularities), the field of moduli for the class never exists. This is a consequence of the following easy lemma.

LEMMA 2.5. Let k' be a purely inseparable extension of a field k . If $k'(x)$ is a regular extension of k' , there exists a subfield K of $k'(x)$ which is a regular extension of k such that $k'(x)$ is the composite of K and k' .

(Proof) Let $(t_1, \dots, t_n) = (t)$ be a set of independent variables in $k'(x)$ over k' such that $k'(x)$ is separable algebraic over $k'(t)$. If we define a subfield K of $k'(x)$ by

$$K = \{ y \in k(x) \mid y \dots \text{separable algeb. over } k(t) \}$$

K satisfies all the conditions we want.