

Global asymptotic stability for a class of difference equations

早稲田大学理工学部 室谷義昭 (Yoshiaki Muroya)

Department of Mathematical Science, Waseda University

東京理科大学理学部 石渡恵美子 (Emiko Ishiwata)

Department of Mathematical Information Science, Tokyo University of Science

Nicola Guglielmi

Dipartimento di Matematica, Pura ed Applicata, Università de L'Aquila

1 Introduction

Consider the following nonlinear difference equation with variable coefficients:

$$x_{n+1} = qx_n - \sum_{j=0}^m a_j f_j(x_{n-j}), \quad n = 0, 1, 2, \dots, \quad (1.1)$$

where $0 < q \leq 1$, $a_j \geq 0$, $0 \leq j \leq m$ and $\sum_{j=0}^m a_j > 0$. We now assume that

$$\begin{cases} f(x) \in C(-\infty, +\infty) \text{ is a strictly monotone increasing function,} \\ f(0) = 0, \quad 0 < \frac{f_j(x)}{f(x)} \leq 1, \quad x \neq 0, \quad 1 \leq j \leq m, \quad \text{and} \\ \text{if } f(x) \neq x, \text{ then } \lim_{x \rightarrow -\infty} f(x) \text{ is finite, otherwise } f(x) = x. \end{cases} \quad (1.2)$$

The above difference equation has been studied by many literatures (see for example, [1]-[9] and references therein).

Definition 1.1 The solution y^* of (1.1) is called *uniformly stable*, if for any $\epsilon > 0$ and non-negative integer n_0 , there is a constant $\delta = \delta(\epsilon) > 0$ such that $\sup\{|y_{n_0-i} - y^*| \mid 0 \leq i \leq m\} < \delta$, implies that the solution $\{y_n\}_{n=0}^\infty$ of (1.1) satisfies $|y_n - y^*| < \epsilon$, $n = n_0, n_0 + 1, \dots$.

Definition 1.2 The solution y^* of (1.1) is called *globally attractive*, if every solution of (1.1) tends to y^* as $n \rightarrow \infty$.

Definition 1.3 The solution y^* of (1.1) is called *globally asymptotically stable*, if it is uniformly stable and globally attractive.

In this paper, we study "semi-contractive" functions and global asymptotic stability of difference equations. In Section 2, we first define semi-contractivity of functions and show the related results on the global asymptotic stability of difference equations.

2 Semi-contractive function

Assume that

$$g(z_0, z_1, \dots, z_m) \in C(R^{m+1}) \quad \text{and} \quad g(y, y, \dots, y) = y \text{ has a unique solution } y = y^*. \quad (2.1)$$

Definition 2.1 The function $g(z_0, z_1, \dots, z_m)$ is said to be semi-contractive at y^* , if

- (i) for any constants $\underline{z} < y^*$ and $z_i \geq \underline{z}$, $0 \leq i \leq m$, there exists a constant $y^* < \bar{z} < +\infty$ such that $g(z_0, z_1, \dots, z_m) \leq \bar{z}$, and for any $\underline{z} \leq z_i \leq \bar{z}$, $0 \leq i \leq m$, there exists a constant $\tilde{z} > \underline{z}$ such that $\tilde{z} \leq g(z_0, z_1, \dots, z_m)$, or
- (ii) for any constants $\bar{z} > y^*$ and $z_i \leq \bar{z}$, $0 \leq i \leq m$, there exists a constant $y^* > \underline{z} > -\infty$ such that $g(z_0, z_1, \dots, z_m) \geq \underline{z}$, and for any $\underline{z} \leq z_i \leq \bar{z}$, $0 \leq i \leq m$, there exists a constant $\tilde{z} < \bar{z}$ such that $\tilde{z} \geq g(z_0, z_1, \dots, z_m)$.

Lemma 2.1 If $g(y) \in C(R)$ is a strictly monotone decreasing function such that $g(g(y)) > y$ for any $y < y^*$, then $g(z)$ is semi-contractive for y^* .

Lemma 2.2 Assume (2.1) and that each $g_i(z_0, z_1, \dots, z_m)$, $0 \leq i \leq m$ is semi-contractive for y^* . Then for any $b_{n,i} \geq 0$, $n \geq 0$, $0 \leq i \leq m$ such that $\sum_{i=0}^m b_{n,i} = 1$ and $\lim_{n \rightarrow \infty} b_{n,i} = b_i$, $0 \leq i \leq m$, it holds that $\sum_{i=0}^m b_{n,i} g_i(z_0, z_1, \dots, z_m)$ is semi-contractive for y^* .

Corollary 2.1 Assume (2.1) and that $g(z_0, z_1, \dots, z_m)$ is semi-contractive for y^* . Then for any $0 \leq q_n < 1$, $g_n(z_0, z_1, \dots, z_m)$ and k such that

$$\begin{cases} \lim_{n \rightarrow \infty} q_n = q < 1, & \text{and } 0 \leq k \leq m, \\ \lim_{n \rightarrow \infty} g_n(z_0, z_1, \dots, z_m) = g(z_0, z_1, \dots, z_m) & \text{for any } z_0, z_1, \dots, z_m \in (-\infty, +\infty), \end{cases} \quad (2.2)$$

it holds that $q_n z_k + (1 - q_n) g_n(z_0, z_1, \dots, z_m)$ is semi-contractive for y^* .

Corollary 2.2 Assume that each $g_i(z) \in C(R)$ and $g_i(y) = y$ has a unique solution $y = y^*$, $0 \leq i \leq m$, and each $g_i(z_i)$, $0 \leq i \leq m$ is semi-contractive for y^* , then for any $b_{n,i} \geq 0$, $n \geq 0$, $0 \leq i \leq m$ such that $\sum_{i=0}^m b_{n,i} = 1$ and $\lim_{n \rightarrow \infty} b_{n,i} = b_i$, $0 \leq i \leq m$, it holds that $\sum_{i=0}^m b_{n,i} g_i(z_i)$ is semi-contractive for y^* . In particular, for any $0 \leq q_n < 1$ and k such that $\lim_{n \rightarrow \infty} q_n = q < 1$ and $0 \leq k \leq m$, it holds that $q_n z_k + (1 - q_n) \sum_{i=0}^m b_{n,i} g_i(z_i)$ is semi-contractive for y^* .

Remark 2.1 If $g(z_0, z_1, \dots, z_m) > 0$ for any $z_i > 0$, $0 \leq i \leq m$, then there are cases that we may restrict our attention only to $z_i > 0$, $0 \leq i \leq m$ and the unique positive solution $y^* > 0$ of $g(y^*, y^*, \dots, y^*) = y^*$, whether or not $g(y, y, \dots, y) = y$ has other solutions $y \leq 0$.

Example 2.1 Examples of semi-contractive function $g(z_0, z_1, \dots, z_m)$ for y^* .

- (i) $g(z_0, z_1, \dots, z_m) = z_m e^{c(1-z_m)}$, $y^* = 1$ and $c \leq 2$ (see [1]).
- (ii) $g(z_0, z_1, \dots, z_m) = z_0 \exp\{c(1 - \sum_{i=0}^m a_i z_i)\}$, $y^* = 1/(\sum_{i=0}^m a_i)$ and $c \leq 2$, where $a_0 > 0$, $a_i \geq 0$, $1 \leq i \leq m$ and $(\sum_{i=1}^m a_i)/a_0 \leq 2/e$.
This is equivalent that $h(u_0, u_1, \dots, u_m) = u_0 - c \sum_{i=0}^m b_i (e^{u_i} - 1)$ is semi-contractive for $u^* = 0$ and $c \leq 2$, where $z_i = y^* e^{u_i}$, $b_0 = y^* a_0 > 0$, $b_i = y^* a_i \geq 0$, $1 \leq i \leq m$, $\sum_{i=0}^m b_i = 1$, and $(\sum_{i=1}^m b_i)/b_0 \leq 2/e$ (see [8]).
- (iii) $g(z_0, z_1, \dots, z_m) = c(1 - e^{z_m})$, $y^* = 0$ and $c \leq 1$ (see [3]).
- (iv) $g(z_0, z_1, \dots, z_m) = \frac{cz_m}{1+bz_m^p}$, $x^* = ((c-1)/b)^{1/p}$ and $c \leq \frac{p}{p-2}$, where $p > 2$ and $b > 0$ (see [1]).

We consider the following difference equation

$$y_{n+1} = q_n y_{n-k} + (1 - q_n) g_n(y_n, y_{n-1}, \dots, y_{n-m}), \quad n = 0, 1, \dots, \quad (2.3)$$

where we assume (2.1) and

$$\begin{cases} 0 \leq q_n < 1, & \lim_{n \rightarrow \infty} q_n = q < 1, & k \in \{0, 1, \dots, m\}, & \text{and} \\ \lim_{n \rightarrow \infty} g_n(z_0, z_1, \dots, z_m) = g(z_0, z_1, \dots, z_m) & \text{for any } z_0, z_1, \dots, z_m \in (-\infty, +\infty). \end{cases} \quad (2.4)$$

Theorem 2.1 *If $g(z_0, z_1, \dots, z_m)$ is semi-contractive for y^* , then y^* of (2.3) is globally asymptotically stable for any $0 \leq q < 1$.*

Collorary 2.3 *Assume that there exists a constant $0 \leq q_0 < 1$ and some $0 \leq k \leq m$ such that $q_0 z_k + (1 - q_0)g(z_0, z_1, \dots, z_m)$, is semi-contractive for y^* . Then, for any $q_0 \leq q_n < 1$ and $g_n(z_0, z_1, \dots, z_m)$ which satisfy (2.4), the solution y^* of (2.3) is globally asymptotically stable.*

Remark 2.2 (i) The corresponding continuous case (2.3) is the following differential equation

$$\begin{cases} y'(t) = -p(t)\{y(t) - \frac{1}{1-q_n}g_n(y(n), y(n-1), \dots, y(n-m))\}, & n \leq t < n+1, \quad n = 0, 1, 2, \dots, \\ p(t) > 0, & q_n = e^{-\int_n^{n+1} p(t)dt} < 1. \end{cases}$$

(ii) In Theorem 2.1, a semi-contractivity condition is a delays and q_n -independent condition for the solution y^* of (2.3) to be globally asymptotically stable.

By Theorem 2.1 and Example 2.1, we obtain the following result:

Example 2.2 Examples of delays and q -independent stability conditions.

(i) Ricker model $y_{n+1} = qy_n + (1-q)y_{n-m}e^{c(1-y_{n-m})}$, $n = 0, 1, 2, \dots$. The positive equilibrium $y^* = 1$ is globally asymptotically stable, if $c \leq 2$ (see [1]).

(ii) Ricker model with delayed-density dependence $y_{n+1} = qy_n + (1-q)y_n \exp\{c(1 - \sum_{i=0}^m a_i y_{n-i})\}$. The positive equilibrium $y^* = 1/(\sum_{i=0}^m a_i)$ is globally asymptotically stable, if $c \leq 2$, where $a_0 > 0$, $a_i \geq 0$, $1 \leq i \leq m$ and $(\sum_{i=1}^m a_i)/a_0 \leq 2/e$ (see [8]).

(iii) Wazewska-Czyżewska and Lasota model $y_{n+1} = qy_n + (1-q)c \sum_{i=0}^m b_i e^{-\gamma y_{n-i}}$, $n = 0, 1, 2, \dots$,

where $\gamma > 0$, $b_i \geq 0$, $0 \leq i \leq m$, and $\sum_{i=0}^m b_i = 1$.

The positive equilibrium y^* is the positive solution of the equation $y^* = ce^{-\gamma y^*}$. Put $x_n = \gamma(y^* - y_n)$. Then, this equation is equivalent to

$$x_{n+1} = qx_n - (1-q)\gamma y^* \sum_{i=0}^m b_i (e^{x_{n-i}} - 1), \quad \text{where } b_i \geq 0, \quad 0 \leq i \leq m, \quad \sum_{i=0}^m b_i = 1. \quad (2.5)$$

Thus, the positive equilibrium y^* is globally asymptotically stable, if $c \leq e/\gamma$ which is equivalent that the zero solution of (2.5) is globally asymptotically stable if $\gamma y^* \leq 1$ (see [3]).

(iv) Bobwhite quail population model $y_{n+1} = qy_n + (1-q)\frac{cy_{n-m}}{1+by_{n-m}^p}$, $n = 0, 1, 2, \dots$, where $c > 1$, $b > 0$. The positive equilibrium $y^* = ((c-1)/b)^{1/p}$ is globally asymptotically stable, if $c \leq \frac{p}{p-2}$ for $p > 2$ (see [1]).

We have the following counter example:

Example 2.3 Examples of q -dependent and delay-dependent stability conditions.

(i) A model in hematopoiesis $y_{n+1} = qy_n + (1-q)e^{2(1-y_n)}$, $n = 0, 1, 2, \dots$.

The equilibrium $y^* = 1$ is globally asymptotically stable if $q \in [1/3, 1)$, and 2-cycle if $q \in [0, 1/3)$ (see [2]).

(ii) A delayed model in hematopoiesis $y_{n+1} = qy_n + (1-q)e^{2(1-y_{n-2})}$, $n = 0, 1, 2, \dots$.

The characteristic equation takes the form $\lambda^3 - q\lambda^2 = -2(1-q)$. Then for $q = q_2 = \frac{3-\sqrt{3}}{2} =$

$0.633975 \dots > 1/3$, the roots are $-1 < \lambda_1 < 0$, $|\lambda_2| = |\lambda_3| = 1$. For $q_2 < q < 1$, the equilibrium $y^* = 1$ is locally attractive but it becomes unstable for $q = q_2$, and *Hopf bifurcation occurs* (see [2]).

(iii) Ricker's equation with delayed-density dependence $y_{n+1} = y_n \exp\{c_n(1 - \sum_{i=0}^m b_{n,i} y_{n-i})\}$, $n = 0, 1, \dots$, which is equivalent to $x_{n+1} = x_n - c_n \sum_{i=0}^m b_{n,i} (e^{x_{n-i}} - 1)$, $n = 0, 1, \dots$, where $c_n, b_{n,i} > 0$, $\sum_{i=0}^m b_{n,i} = 1$ and $y_n = e^{x_n}$.

The positive equilibrium $y^* = 1$ is *globally asymptotically stable* if $\limsup_{n \rightarrow \infty} \sum_{i=n}^{n+m} r_i < \frac{3}{2} + \frac{1}{2(m+1)}$ (see [7]).

(iv) A model of the growth of bobwhite quail populations $y_{n+1} = qy_n + (1-q)\frac{cy_n}{1+y_{n-m}^p}$, $n = 0, 1, \dots$,

where $c, p > 0$. If $c \leq 1$, then for any $0 < q < 1$, $\lim_{n \rightarrow \infty} y_n = 0$. If $c > 1$, then the positive equilibrium $y^* = (c-1)^{1/p}$ of the model exists. Moreover, if $p \leq \frac{2c}{(c-1)(1-q)}$ for $m = 0$, or $p < \frac{c}{(c-1)(1-q)} \frac{3m+4}{2(m+1)^2}$ for $m \geq 1$, then the positive equilibrium y^* is globally asymptotically stable (see [4]).

3 Delays-independent stability conditions for (1.1)

After setting

$$r_1 = a_0, r_2 = \sum_{i=1}^m a_i, r = r_1 + r_2, \varphi(x) = qx - r_1 f(x), \hat{z}(q) = (-1 + \sqrt{1+4q})/(2q), \quad (3.1)$$

we have the following result.

Theorem 3.1 Assume that $f(x) = f_0(x) = e^x - 1$ and $0 < q < 1$, and suppose that

$$r_1 < q, \quad r \leq q + (1-q) \ln(q/r_1) \quad \text{and} \quad (q/r_1)^q e^{r-q} (r_1 - r_2) + (1-q) \geq 0, \quad (3.2)$$

or

$$\begin{cases} r_1 \leq q, & r > q + (1-q) \ln(q/r_1), & qr_2 \leq r_1, \\ r - r_2(q/r_1)^q e^{r-q} - (1-q)(\bar{L} - 1) \geq 0 & \text{and} & \bar{L} = \ln \frac{r-q-(1-q)\ln(q/r_1)}{r_2} \leq 0, \end{cases} \quad (3.3)$$

or

$$\begin{cases} r_1 > q, & r \leq 1+q, & r - r_2(q/r_1)^q e^{r-q} - (1-q)(\ln(q/r_1) - 1) \geq 0, \\ \text{and} & \frac{r}{q}(q/r_1)^q e^{r-q} \leq \frac{e^{\hat{z}(q)}}{1-\hat{z}(q)}. \end{cases} \quad (3.4)$$

Then, the zero solution of (1.1) is globally asymptotically stable.

Numerical result 3.1 Assume that $f(x) = f_0(x) = e^x - 1$ and $0 < q < 1$.

(i) The last inequality in (3.4) can be eliminated from (3.4).

(ii) Under the condition $\frac{r_2}{r_1} \leq \frac{2}{e}$ and $r \leq 1+q$, the third inequality of (3.4) is satisfied, and hence the zero solution of (1.1) is globally asymptotically stable.

Example 3.1 Wazewska-Czyżewska and Lasota model (see [9]).

$$y_{n+1} = qy_n + (1-q)c \sum_{i=0}^m b_i e^{-\gamma y_{n-i}}, \quad \text{where } c, \gamma > 0, b_i \geq 0 \text{ and } \sum_{i=0}^m b_i = 1. \quad (3.5)$$

(3.5) is equivalent to (2.5). For equation (3.5), the positive equilibrium of (3.5), say y^* , is globally asymptotically stable, if $\gamma y^* \leq 1$ (see [3] and Example 2.2 iii). For the case $\gamma y^* > 1$, by using

the generalized Yorke condition, [6, Theorem 8] extended these to $\gamma y^* \leq (1 + q^{m+1})/(1 - q^{m+1})$ with some restricted conditions " $V_k(q) < 0$, $W_k(q) < 0$ ". Note that the last condition contains the restriction $(q + q^2 + \dots + q^m)q^m \leq 1$ for $0 < q < 1$. On the other hand, by applying Theorem 3.1 and Numerical result 3.1 to (2.5) for $a_i = (1 - q)\gamma y^* b_i$, $0 \leq i \leq m$, we obtain another sufficient condition, for example, $\sum_{i=1}^m b_i \leq \frac{2}{e} b_0$ and $\gamma y^* \leq (1 + q)/(1 - q)$ for the solution y^* of (3.5) to be globally asymptotically stable. Note that $e^x - 1 < x/(1 - x)$ for $0 < x < 1$ and $\frac{1+q^{m+1}}{1-q^{m+1}} < \frac{1+q}{1-q}$ for $0 < q < 1$. Thus, compared with [6, Proof of Theorem 2] (and [1]-[9] and references therein), one can see that our results offer new stability conditions to (3.5).

4 Semi-contractivity with a sign condition

For $0 \leq q < 1$, consider the following nonautonomous equation

$$x_{n+1} = qx_n - \sum_{j=0}^m a_{n,j} f_j(x_{n-j}), \quad n = 0, 1, \dots, \quad (4.1)$$

where $0 < q \leq 1$, $a_{n,j} \geq 0$, $0 \leq j \leq m$, $n = 0, 1, \dots$, and $\sum_{j=0}^m a_{n,j} > 0$, and we assume that there is a function $f(x)$ such that (1.2) holds.

For (4.1) and any $0 \leq l_n \leq m$, we can derive the following equation.

$$\begin{cases} x_{n+1} = \{q^{l_n+1} x_{n-l_n} + (1-q) \sum_{k=0}^{l_n} q^k \sum_{j=0}^{m-k} a_{n-k,j} f_j(x_{n-k-j})\} \\ \quad - \sum_{k=1}^{l_n} q^k \sum_{j=m-k+1}^m a_{n-k,j} f_j(x_{n-k-j}), \quad n = 2m, 2m+1, \dots \end{cases} \quad (4.2)$$

Similar to the proofs of [5, Lemmas 2.3 and 2.4], we have the following two lemmas for (4.1).

Lemma 4.1 *Let $\{x_n\}_{n=0}^\infty$ be the solution of (4.1). If there exists an integer $n \geq m$ such that $x_{n+1} \geq 0$ and $x_{n+1} > x_n$, then there exists an integer $\underline{g}_n \in [n-m, n]$ such that*

$$x_{\underline{g}_n} = \min_{0 \leq j \leq m} x_{n-j} < 0. \quad (4.3)$$

If there exists an integer $n \geq m$ such that $x_{n+1} \leq 0$ and $x_{n+1} < x_n$, then there exists an integer $\bar{g}_n \in [n-m, n]$ such that

$$x_{\bar{g}_n} = \max_{0 \leq j \leq m} x_{n-j} > 0. \quad (4.4)$$

After setting

$$\begin{cases} \bar{r}_1 = \sup_{n \geq m} \sum_{k=0}^m q^k \sum_{j=0}^{m-k} a_{n-k,j}, & \bar{r}_2 = \sup_{n \geq m} \sum_{k=1}^m q^k \sum_{j=m-k+1}^m a_{n-k,j}, \\ \bar{r} = \bar{r}_1 + \bar{r}_2, & \bar{\varphi}(x) = \bar{q}x - \bar{r}_1 f(x), \quad \bar{q} = q^{m+1}, \quad \bar{z} = (-1 + \sqrt{1 + 4\bar{q}})/(2\bar{q}), \end{cases} \quad (4.5)$$

and

$$\bar{g}(z_0, z_1, \dots, z_m; \bar{q}) = \bar{\varphi}(z_0) + \sum_{k=1}^m q^k \sum_{j=m-k+1}^m a_{n-k,j} g(z_j), \quad (4.6)$$

we are able to prove the following results.

If there exists an integer $n \geq m$ such that $x_{n+1} \geq 0$ and $x_{n+1} > x_n$, then by (4.3) and (4.2) with $l_n = n - g_n$, we have that

$$x_{n+1} \leq \bar{\varphi}(x_{g_n}) - \bar{r}_2 f(L_n), \quad L_n = \min_{0 \leq j \leq 2m} x_{n-j}. \quad (4.7)$$

If there exists an integer $n \geq m$ such that $x_{n+1} \leq 0$ and $x_{n+1} < x_n$, then by (4.4) and (4.2) with $l_n = n - \bar{g}_n$, we have that

$$x_{n+1} \geq \bar{\varphi}(x_{\bar{g}_n}) - \bar{r}_2 f(R_n), \quad R_n = \max_{0 \leq j \leq 2m} x_{n-j}. \quad (4.8)$$

Lemma 4.2 Suppose that the solution x_n of (4.1) is oscillatory about 0. If for some real number $L < 0$, there exists a positive integer $n_L \geq 2m$ such that $x_n \geq L$ for $n \geq n_L$, then for any integer $n \geq n_L + 2m$,

$$x_{n+1} \leq R_L \text{ for } n \geq n_L + 2m, \quad \text{and} \quad x_{n+1} \geq S_L \text{ for } n \geq n_L + 4m, \quad (4.9)$$

where $R_L = \max_{L \leq x \leq 0} \varphi(x) - r_2 f(L) > 0$ and $S_L = \min_{0 \leq x \leq R_L} \varphi(x) - r_2 f(R_L) < 0$. Moreover, if $S_L > L$ for any $L < 0$, then $\lim_{n \rightarrow \infty} x_n = 0$.

Assume that $g(z_0, z_1, \dots, z_m)$ is continuous for $(z_0, z_1, \dots, z_m) \in R^{m+1}$ and $g(y^*, y^*, \dots, y^*) = y^*$ has a unique solution y^* .

Definition 4.1 The function $g(z_0, z_1, \dots, z_m)$ is said to be semi-contractive with a sign condition z_0 for y^* , if

(i) for any constants $\underline{z} < y^*$ and $z_i \geq \underline{z}$, $0 \leq i \leq m$ with $z_0 \leq y^*$, there exists a constant $y^* < \bar{z} < +\infty$ such that $g(z_0, z_1, \dots, z_m) \leq \bar{z}$ and for any $\underline{z} \leq z_i \leq \bar{z}$, $0 \leq i \leq m$ with $z_0 \geq y^*$, there exists a constant $\bar{z} > \underline{z}$ such that $\bar{z} \leq g(z_0, z_1, \dots, z_m)$,

or

(ii) for any constants $\bar{z} > y^*$ and $z_i \leq \bar{z}$, $0 \leq i \leq m$ with $z_0 \geq y^*$, there exists a constant $y^* > \underline{z} > -\infty$ such that $g(z_0, z_1, \dots, z_m) \geq \underline{z}$ and for any $\underline{z} \leq z_i \leq \bar{z}$, $0 \leq i \leq m$ with $z_0 \leq y^*$, there exists a constant $\bar{z} < \underline{z}$ such that $\bar{z} \geq g(z_0, z_1, \dots, z_m)$.

Then by (4.7), (4.8) and Lemma 4.2, we can obtain the following result.

Theorem 4.1 If $\bar{g}(z_0, z_1; \bar{q}) = \bar{\varphi}(z_0) - \bar{r}_2 f(z_1)$ is semi-contractive with a sign condition z_0 for $x^* = 0$, then the zero solution of (4.1) is globally asymptotically stable.

Note that if $\bar{g}(z_0, z_1; \bar{q}) = \bar{\varphi}(z_0) - \bar{r}_2 f(z_1)$ is semi-contractive with a sign condition z_0 for $x^* = 0$, then the zero solution $x^* = 0$ of (4.1) is uniformly stable and hence $x^* = 0$ is globally asymptotically stable.

For the special case $f(x) = e^x - 1$, we establish the following sufficient conditions for $0 < q < 1$ which are some extensions of the result in [5] for $q = 1$.

Theorem 4.2 Suppose that $f(x) = e^x - 1$ and that one of the following condition is fulfilled:

$$\begin{cases} \bar{r}_2 \leq 1 \quad \text{and} \quad \frac{\bar{r}}{\bar{q}} e^{\bar{r}_2} \leq \frac{e^{\bar{z}}}{1-\bar{z}} & \text{if } \bar{r}_1 \leq \bar{q}, \\ \bar{r} \leq 1 + \bar{q} \quad \text{and} \quad \frac{\bar{r}}{\bar{q}} (\bar{q}/\bar{r}_1)^{\bar{q}} e^{\bar{r}-\bar{q}} \leq \frac{e^{\bar{z}}}{1-\bar{z}} & \text{if } \bar{r}_1 > \bar{q}, \end{cases} \quad (4.10)$$

$$\text{or} \quad \begin{cases} \bar{r}_2 \leq 1, \quad \frac{\bar{r}}{\bar{q}} e^{\bar{r}_2} > \frac{e^{\bar{z}}}{1-\bar{z}} \quad \text{and} \quad G_3(\delta) > 0 & \text{if } \bar{r}_1 \leq \bar{q}, \\ \bar{r} \leq 1 + \bar{q}, \quad \frac{\bar{r}}{\bar{q}} (\bar{q}/\bar{r}_1)^{\bar{q}} e^{\bar{r}-\bar{q}} > \frac{e^{\bar{z}}}{1-\bar{z}} \quad \text{and} \quad G_1(\alpha) > 0 & \text{if } \bar{r}_1 > \bar{q}, \end{cases} \quad (4.11)$$

$$\text{with } \begin{cases} G_1(x) = \bar{q} \left(\bar{q} \ln(\bar{q}/\bar{r}_1) + \bar{r} - \bar{q} - \bar{r}_2 e^x \right) + \bar{r} - \bar{r}(\bar{q}/\bar{r}_1)^{\bar{q}} e^{\bar{r}-\bar{q}-\bar{r}_2 e^x} - x, \\ G_3(x) = (\bar{r}_1 + (1 + \bar{q})\bar{r}_2) - \bar{q}\bar{r}_2 e^x - \bar{r}e^{\bar{r}_2 - \bar{r}_2 e^x} - x, \end{cases} \quad (4.12)$$

where α and δ are the lowest solutions of $G_1(x) = 0$ and $G_3(x) = 0$, respectively, and $\bar{z}$ is a positive solution of $\bar{q}z^2 + z - 1 = 0$. Then, the solution $x^* = 0$ of (4.1) is globally asymptotically stable.

As an immediate consequence we have the following corollary.

Corollary 4.1 Assume that $f(x) = e^x - 1$ and that

$$\bar{r} \leq 1 + \bar{q} \quad \text{and} \quad \bar{r}_1 \geq \bar{q}\bar{r}_2. \quad (4.13)$$

If

$$(i) \frac{\bar{r}}{\bar{q}} (\bar{q}/\bar{r}_1)^{\bar{q}} e^{\bar{r}-\bar{q}} \leq \frac{e^{\bar{z}}}{1-\bar{z}}, \quad \text{or} \quad (ii) \frac{\bar{r}}{\bar{q}} (\bar{q}/\bar{r}_1)^{\bar{q}} e^{\bar{r}-\bar{q}} > \frac{e^{\bar{z}}}{1-\bar{z}} \quad \text{and} \quad G_1(\alpha) > 0, \quad (4.14)$$

then, the zero solution of (4.1) is globally asymptotically stable.

Example 4.1 Consider a model $x_{n+1} = qx_n - \sum_{i=0}^m a_i(e^{-x_{n-i}} - 1)$, $n = 0, 1, 2, \dots$, where $a_i \geq 0$, $0 \leq i \leq m$, and $\sum_{i=0}^m a_i > 0$. This equation is equivalent to (2.5), if $\sum_{i=0}^m a_i = (1-q)\gamma y^*$ and $0 < q < 1$. By Corollary 4.1, the zero solution $x^* = 0$ is globally asymptotically stable for $\bar{r} \leq 1 + \bar{q}$, if for the setting (4.5) and $\hat{r}_1 = \bar{q}(\frac{1+\bar{q}}{\bar{q}}(1-\bar{z})e^{1-\bar{z}})^{1/\bar{q}}$, it holds that $\frac{\bar{r}_2}{\bar{r}_1} \leq \frac{1+\bar{q}}{\bar{r}_1} - 1$. Since $e^x - 1 < x/(1-x)$ for $0 < x < 1$ and we do not need the restriction $(q + q^2 + \dots + q^m)q^m \leq 1$ for $0 < q < 1$ in [6, Theorem 2], our results improve some of [6, Theorem 8] (see [5]).

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