

ラプラス変換の実逆変換への再生核空間の応用 II

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1 Introduction

The present paper contains new results on the modified Laplace transform

$$\mathcal{L}f(p) = p Lf(p) = \int_0^\infty p e^{-pt} f(t) dt.$$

The results are intended to publish elsewhere.

2 Preliminaries

Theorem 2.1. *The following estimates hold.*

$$\begin{aligned} \sup_{p>0} |\mathcal{L}f(p)| &\leq \sup_{t>0} |f(t)| \\ \int_0^\infty |\mathcal{L}f(p)| dp &\leq \int_0^\infty \frac{|f(t)|}{t^2} dt. \end{aligned}$$

If we interpolate the results above, we obtain the following inequality.

Theorem 2.2.

$$\int_0^\infty |\mathcal{L}f(p)|^2 dp \leq 4 \int_0^\infty |f(t)|^2 \frac{dt}{t^2}.$$

Proof. By using the distribution function, we have

$$\int_0^\infty |\mathcal{L}f(p)|^2 dp = 4 \int_0^\infty \lambda |\{p > 0 : |\mathcal{L}f(p)| > 2\lambda\}| d\lambda.$$

By the L^∞ -estimate, we obtain

$$\int_0^\infty |\mathcal{L}f(p)|^2 dp \leq 4 \int_0^\infty \lambda |\{p > 0 : |\mathcal{L}[\chi_{\{|f| \leq \lambda\}}f](p)| > \lambda\}| d\lambda.$$

Next, we invoke the L^1 -estimate and the Chebychev inequality. The result is

$$\int_0^\infty |\mathcal{L}f(p)|^2 dp \leq 4 \int_0^\infty \left(\int_0^\infty |\mathcal{L}[\chi_{\{|f| \leq \lambda\}}f](p)| dp \right) d\lambda.$$

Having used up our estimates which were already proved, we have only to calculate the integral elaborately.

$$\int_0^\infty |\mathcal{L}f(p)|^2 dp \leq 4 \int_0^\infty \left(\int_0^\infty \chi_{\{|f(t)| \leq \lambda\}} |f(t)| \frac{dt}{t^2} \right) d\lambda \leq 4 \int_0^\infty |f(t)|^2 \frac{dt}{t^2}.$$

□

The power 2 is best possible in the following sense.

Example 2.3. Let us establish that

$$\int_{\mathbf{R}} \mathcal{L}f(p)^2 dp \lesssim \int_{\mathbf{R}} |f(t)|^2 \frac{dt}{t^{1+\beta}}$$

fails for $0 < \beta < 1$. Take $\alpha \in \mathbf{R}$ so that $\frac{\beta}{2} < \alpha < \frac{1}{2}$. Then

$$f_\alpha(x) = (\chi_{[0,1]}(x)x)^\alpha$$

satisfies

$$\begin{aligned} \mathcal{L}f_\alpha(p) &= p \int_0^1 t^\alpha e^{-tp} dt \\ &= p \int_0^p (p^{-1}s)^\alpha e^{-s} d(p^{-1}s) \\ &\simeq p^{-\alpha} \end{aligned}$$

as $p \rightarrow \infty$. As a result, we have $f_\alpha \in L^2\left((0, \infty), \frac{dt}{t^{1+\beta}}\right)$, $0 < \beta < \alpha$, while $\mathcal{L}f \notin L^2(0, \infty)$.

In the rest of this paper, we consider

$$H_K = \{f : [0, \infty) \rightarrow [0, \infty) : f(0) = 0, f \text{ is absolutely continuous and } \|f\|_{H_K} < \infty\},$$

where the norm is given by

$$\|f\|_{H_K} = \left(\int_0^\infty |f'(t)|^2 \frac{e^t dt}{t} \right)^{\frac{1}{2}}.$$

To prove that $\mathcal{L}$ is compact, we have only to establish the following.

Theorem 2.4. $H_K \subset L^2\left((0, \infty), \frac{dt}{t^2}\right)$ in the sense of compact embedding.

Proof. This is because

$$H_K \subset L^\infty((0, \infty), \max(|t|^{-1}, 1))$$

is a continuous embedding and

$$L^\infty((0, \infty), \max(|t|^{-1}, 1)) \subset L^2\left((0, \infty), \frac{dt}{t^2}\right)$$

is a compact embedding. □