

Generating the full mapping class group by involutions

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August 20, 2008

Abstract

Let $\Sigma_{g,b}$ denote a closed orientable surface of genus g with b punctures and let $\text{Mod}(\Sigma_{g,b})$ denote its mapping class group. In [Luo] Luo proved that if the genus is at least 3, $\text{Mod}(\Sigma_{g,b})$ is generated by involutions. He also asked if there exists a universal upper bound, independent of genus and the number of punctures, for the number of torsion elements/involutions needed to generate $\text{Mod}(\Sigma_{g,b})$. Brendle and Farb [BF] gave an answer in the case of $g \geq 3, b = 0$ and $g \geq 4, b = 1$, by describing a generating set consisting of 6 involutions. Kassabov showed that for every b $\text{Mod}(\Sigma_{g,b})$ can be generated by 4 involutions if $g \geq 8$, 5 involutions if $g \geq 6$ and 6 involutions if $g \geq 4$. We proved that for every b $\text{Mod}(\Sigma_{g,b})$ can be generated by 4 involutions if $g \geq 7$ and 5 involutions if $g \geq 5$.

1 Introduction

Let $\Sigma_{g,b}$ be an closed orientable surface of genus $g \geq 1$ with arbitrarily chosen b points (which we call punctures). Let $\text{Mod}(\Sigma_{g,b})$ be the mapping class group of $\Sigma_{g,b}$, which is the group of homotopy classes of orientation-preserving homeomorphisms preserving the set of punctures. Let $\text{Mod}^\pm(\Sigma_{g,b})$ be the extended mapping class group of $\Sigma_{g,b}$, which is the group of homotopy class of all (including orientation-reversing) homeomorphisms preserving the set of punctures. By $\text{Mod}_{g,b}^0$ we will denote the subgroup of $\text{Mod}_{g,b}$ which fixes the punctures pointwise.

In [MP], McCarthy and Papadopoulos proved that $\text{Mod}(\Sigma_{g,0})$ is generated by infinitely many conjugates of a single involution for $g \geq 3$. Luo, see [Luo], described the finite set of involutions which generate $\text{Mod}(\Sigma_{g,b})$ for $g \geq 3$. He also proved that $\text{Mod}(\Sigma_{g,b})$ is generated by torsion elements in all cases except $g = 2$ and $b = 5k + 4$, but this group is not generated by involutions if $g \leq 2$. Brendle and Farb proved that $\text{Mod}(\Sigma_{g,b})$ can be generated by 6 involutions for $g \geq 3, b = 0$ and $g \geq 4, b \leq 1$ (see [BF]). In [Ka], Kassabov proved that for every b $\text{Mod}(\Sigma_{g,b})$ can be generated by 4 involutions if $g \geq 8$, 5 involutions if $g \geq 6$ and 6 involutions if $g \geq 4$. He also proved in the case of $\text{Mod}^\pm(\Sigma_{g,b})$. Our main result is stronger than [Ka].

Main Theorem. *For all $g \geq 3$ and $b \geq 0$, the mapping class group $\text{Mod}(\Sigma_{g,b})$ can be generated by:*

- (a) 4 involutions if $g \geq 7$;
- (b) 5 involutions if $g \geq 5$.

2 Preliminaries

Let c be a simple closed curve on $\Sigma_{g,b}$. Then the (right hand) Dehn twist T_c about c is the homotopy class of the homeomorphism obtained by cutting $\Sigma_{g,p}$ along c , twisting one of the side by 360° to the right and gluing two sides of a back to each other. Figure 1 shows the Dehn twist about the curve c . We will

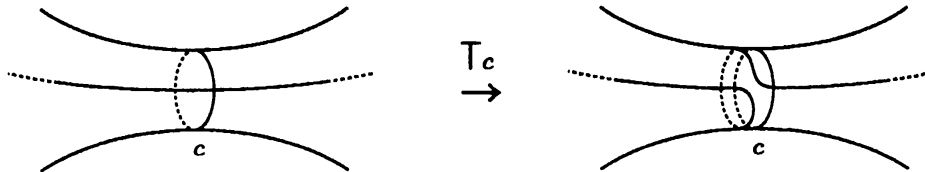


Figure 1: The Dehn twist

denote by T_c the Dehn twist around the curve c .

We record the following lemmas.

Lemma 1. *For any homeomorphism h of the surface $\Sigma_{g,b}$ the twists around the curves c and $h(c)$ are conjugate in the mapping class group $\text{Mod}(\Sigma_{g,b})$,*

$$T_{h(c)} = hT_ch^{-1}.$$

Lemma 2. *Let c and d be two simple closed curves on $\Sigma_{g,b}$. If c is disjoint from d , then*

$$T_cT_d = T_dT_c$$

3 Proof of main theorem

In this section we proof maintheorem. The keypoints of proof are to generate T_α in 4 involutions by using lantern relation.

3.1 The policy of proof

We give the policy of proof of maintheorem.

Lemma 3. *Let G, Q denote the groups and let N, H denote the subgroups of G . Suppose that the group G has the following exact sequence;*

$$1 \rightarrow N \xrightarrow{i} G \xrightarrow{\pi} Q \rightarrow 1.$$

If H contains $i(N)$ and has a surjection to Q then we have that $H = G$.

Proof. We suppose that there exists some $g \in G - H$. By the existence of surjection from H to Q , we can see that there exists some $h \in H$ such that $\pi(h) = \pi(g)$. Therefore, since $\pi(g^{-1}h) = \pi(g)^{-1}\pi(h) = 1$, we can see that

$g^{-1}h \in \text{Ker } \pi = \text{Im } i$. Then there exists some $n \in N$ such that $i(n) = g^{-1}h$. By $i(N) \subset H$, since $i(n) \in H$ and $h \in H$, we have

$$g = h \cdot i(n)^{-1} \in H.$$

This is contradiction in $g \notin H$. Therefore, we can prove that $H = G$. $\square$

It is clear that we have the exact sequence:

$$1 \rightarrow \text{Mod}_{g,b}^0 \rightarrow \text{Mod}_{g,b} \rightarrow \text{Sym}_b \rightarrow 1.$$

Therefore, we can see the following corollary;

Corollary 4. *Let H denote the subgroup of $\text{Mod}(\Sigma_{g,b})$, which contains $\text{Mod}^0(\Sigma_{g,b})$ and has a surjection to Sym_b . Then H is equal to $\text{Mod}(\Sigma_{g,b})$.*

We generate the subgroup H which has the condition of corollary 4 by involutions.

Let us embed our surface $\Sigma_{g,b}$ in the Euclidian space in two different ways as shown on Figure 2. (In these pictures we will assume that genus $g = 2k + 1$ is odd and the number of punctures $b = 2l + 1$ is odd. In the case of even genus we only have to swap the top parts of the pictures, and in the case of even number of punctures we have to remove the last point.)

In Figure 2 we have also marked the puncture points as $x_1, \dots, x_b$ and we have the curves $\alpha_i, \beta_i, \gamma_i$ and δ . The curve $\alpha_i, \beta_i, \gamma_i$ are non separating curve and δ is separating curve.

Each embedding gives a natural involution of the surface—the half turn rotation around its axis of symmetry. Let us call these involutions ρ_1 and ρ_2 .

Then we can get following lemma;

Lemma 5 ([Mo]). *The subgroup of the mapping class group be generated by ρ_1, ρ_2 and 3 Dehn twists T_α, T_β and T_γ around one of the curve in each family contains the subgroup $\text{Mod}^0(\Sigma_{g,b})$.*

The existence a surjection from the subgroup H of $\text{Mod}(\Sigma_{g,b})$ to Sym_b is equivalent to showing taht the Sym_b can be generated by involutions;

$$\begin{aligned} r_1 &= (1, b-1)(2, b-2) \cdots (l, l+1)(b) \\ r_2 &= (2, b-1)(3, b-2) \cdots (l, l+2)(1)(l+1)(b) \\ r_3 &= (1, b)(2, b-1)(3, b-2) \cdots (l, l+2)(l+1) \end{aligned}$$

corresponding to 3 involutions in H .

Lemma 6. *The symmetric group Sym_b is generated by r_1, r_2 and r_3 .*

Proof. The group generated by r_i contains the long cycle $r_3 r_1 = (1, 2, \dots, b)$ and transposition $r_3 r_2 = (1, b)$. These two elements generate the whole symmetric group, therefore the involutions r_i generate Sym_b . $\square$

We note that the images of ρ_1 and ρ_2 to Sym_b are r_1 and r_2 .

Therefore, by Lemma 1, Corollary 4, Lemma 5 and Lemma 6 we sufficient to generate H by ρ_1, ρ_2 and involutions which have the following conditions;

- ⟨1⟩ involutions which genarate the Dehn twist around γ ,
- ⟨2⟩ two of each involutions which exchange α and β , β and γ , γ and α ,
- ⟨3⟩ involution whose image is r_3 .

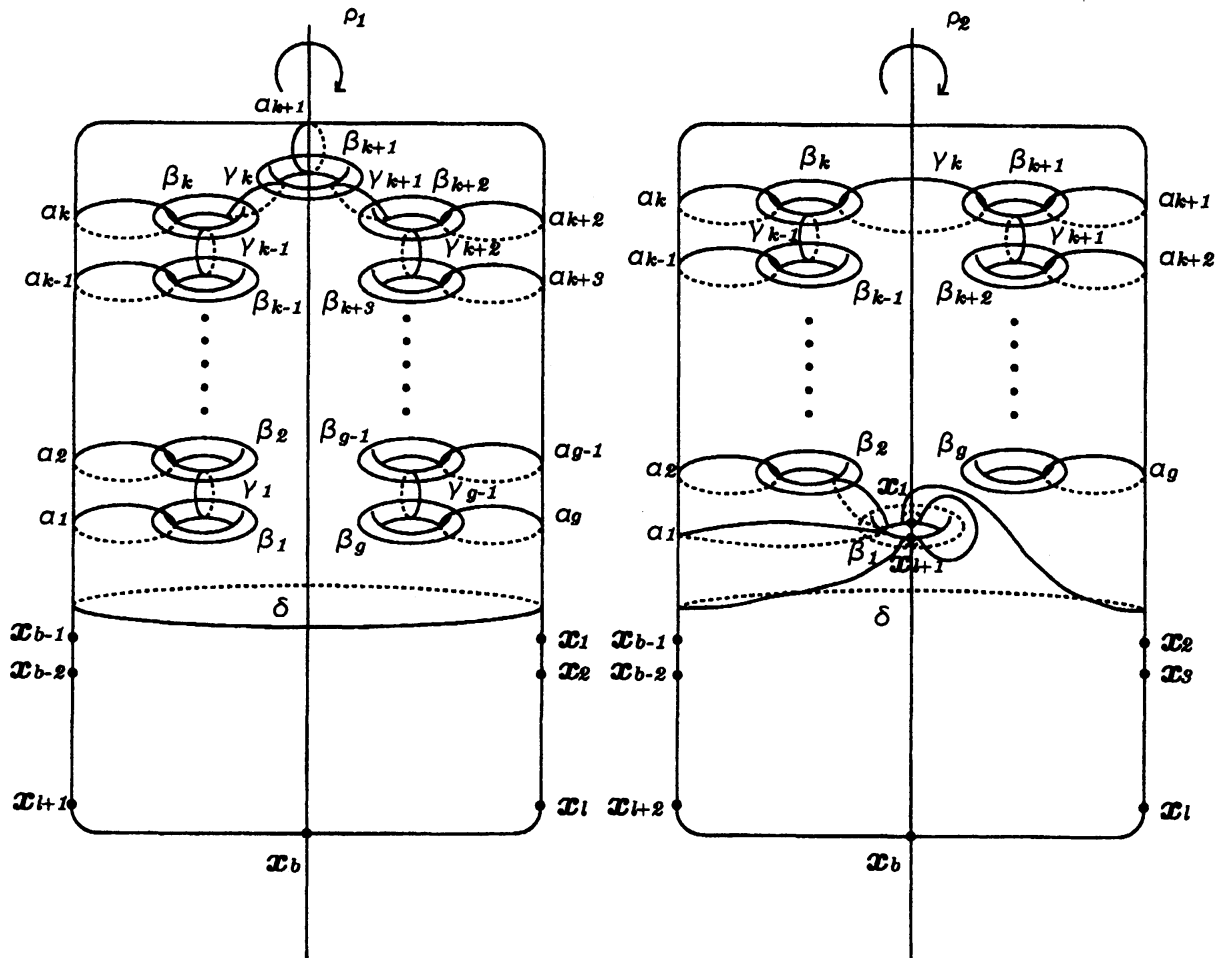


Figure 2: The embeddings of the surface $\Sigma_{g,b}$ in Euclidian space used to define the involutions ρ_1 and ρ_2 .

3.2 Generating Dehn twists by 4 involutions

In this subsection, we argue about $\langle 1 \rangle$. Moreover, we generate Dehn twists by 4 involutions. The basic idea is to use the lantern relation.

We begin by recalling the lantern relation in the mapping class group. This relation was first discovered by Dehn and later rediscovered by Johnson.

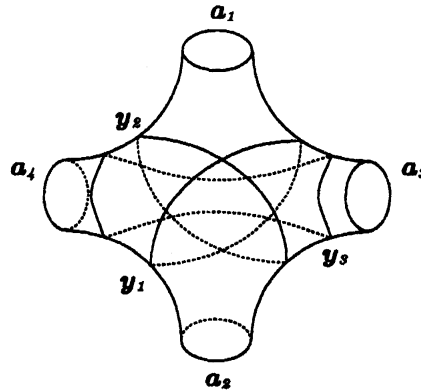


Figure 3: Lantern

From now on we will assume that the genus g of the surface is at least 5.

Let the $S_{0,4}$ be a surface of genus 0 with 4 boundary components. Denote by a_1, a_2, a_3 and a_4 the four boundary curves of the surface $S_{0,4}$ and let the interior curves y_1, y_2 and y_3 be as shown in Figure 3.

The following relation:

$$T_{y_1} T_{y_2} T_{y_3} = T_{a_1} T_{a_2} T_{a_3} T_{a_4}. \quad (1)$$

among the Dehn twists around the curves a_i and y_i is known as the lantern relation. Notice that the curves a_i do not intersect any other curve and that the Dehn twists T_{a_i} commute with every twists in this relation. This allows us to rewrite the lantern relation as follows

$$T_{a_4} = (T_{y_1} T_{a_1}^{-1}) (T_{y_2} T_{a_2}^{-1}) (T_{y_3} T_{a_3}^{-1}). \quad (2)$$

Let R denote the product $\rho_2 \rho_1$. By Figure 2 we can see that $R = \rho_2 \rho_1$ acts as follows:

$$\begin{aligned} R\alpha_i &= \alpha_{i+1}, \quad (1 \leq i < g) \\ R\beta_i &= \beta_{i+1}, \quad (1 \leq i < g) \\ R\gamma_i &= \gamma_{i+1}, \quad (1 \leq i < g-1). \end{aligned} \quad (3)$$

The lanterns S and $R^{-2}S$ have a common boundary component $a_1 = R^{-2}a_2$ and their union is a surface S_2 homeomorphic to a sphere with 6 boundary components. By Figure 4 we can see that there exists an involution $\bar{J}$ of S_2 which takes S to $R^{-2}S$.

Let us embed the surface S_2 in $\Sigma_{g,b}$ as shown on Figure 5. The boundary components of S_2 are $a_1 = \alpha_k$, $a_2 = \alpha_{k+2}$, $a_3 = \gamma_{k+1}$, $a_4 = \gamma_k$, $R^{-2}a_1 = \alpha_{k-2}$, $R^{-2}a_2 = \alpha_k$, $R^{-2}a_3 = \gamma_{k-1}$ and $R^{-2}a_4 = \gamma_{k-2}$; and the middle curve $y_1 =$

α_{k+1} . The Figure 5 shows the existence of the involution $\tilde{J}$ on the complement of S_2 which is a surface of genus $g - 5$ with 6 boundary components. Gluing together $\bar{J}$ and $\tilde{J}$ gives us the involution J of the surface $\Sigma_{g,b}$. By Figure 4 J acts as follows

$$J(a_1) = R^{-2}a_2, \quad J(a_3) = R^{-2}a_1, \quad J(y_1) = R^{-2}y_2, \quad J(y_3) = R^{-2}y_1.$$

Therefore, we have

$$\begin{aligned} R^2 J(a_1) &= a_2, \quad R^2 J(y_1) = y_2 \\ JR^{-2}(a_1) &= a_3, \quad JR^{-2}(y_1) = y_3. \end{aligned} \quad (4)$$

Let ρ_3 denote $T_{a_1}\rho_2T_{a_1}^{-1}$.

By Lemma 1, (4) and that ρ_2 sends $a_1 = \alpha_k$ to $y_1 = \alpha_{k+1}$, we have

$$\begin{aligned} T_{y_1}T_{a_1}^{-1} &= \rho_2T_{a_1}\rho_2T_{a_1}^{-1} = \rho_2\rho_3, \\ T_{y_2}T_{a_2}^{-1} &= R^2J\rho_2\rho_3JR^{-2}, \quad T_{y_3}T_{a_3}^{-1} = JR^{-2}\rho_2\rho_3R^2J. \end{aligned} \quad (5)$$

By (2) and (5) we have

$$T_{\gamma_k} = (\rho_2\rho_3)(R^2J\rho_2\rho_3JR^{-2})(JR^{-2}\rho_2\rho_3R^2J). \quad (6)$$

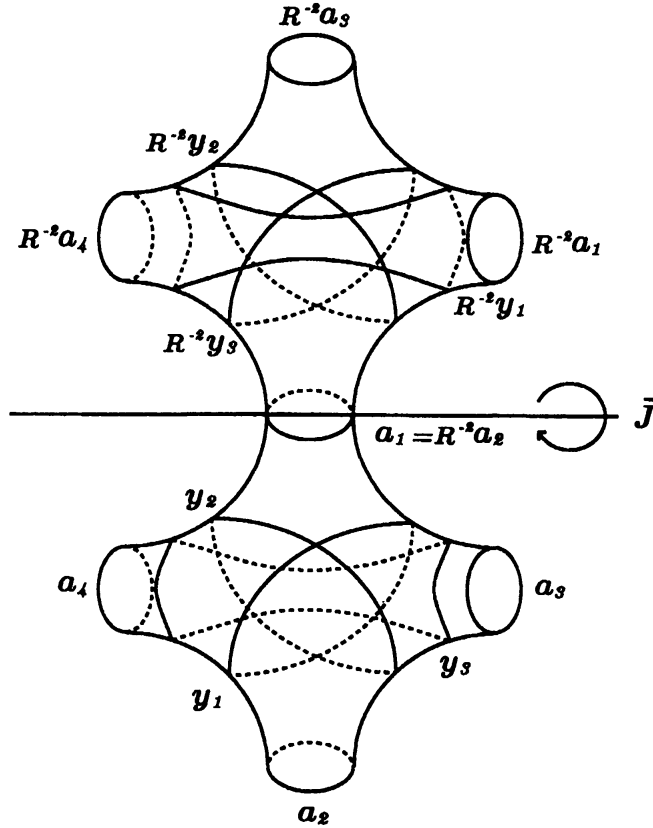


Figure 4: S_2 and the involution $\bar{J}$

Figure 5: The involution J on $\Sigma_{g,b}$

3.3 Genus at least 5

We proof that the mapping class group is generated by 5 involutions.

The five involutions are $\rho_1, \rho_2, \rho_3, J$ and another involution I . We construct involution I in the same way as involution J like Figure 6.

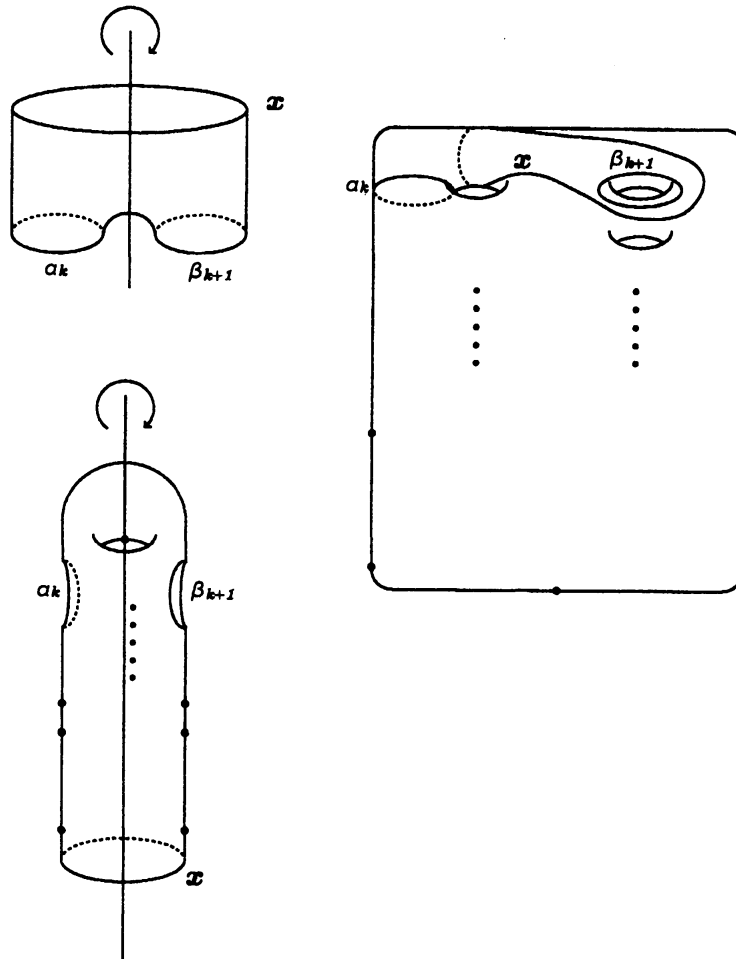


Figure 6: The involution I on $\Sigma_{g,b}$

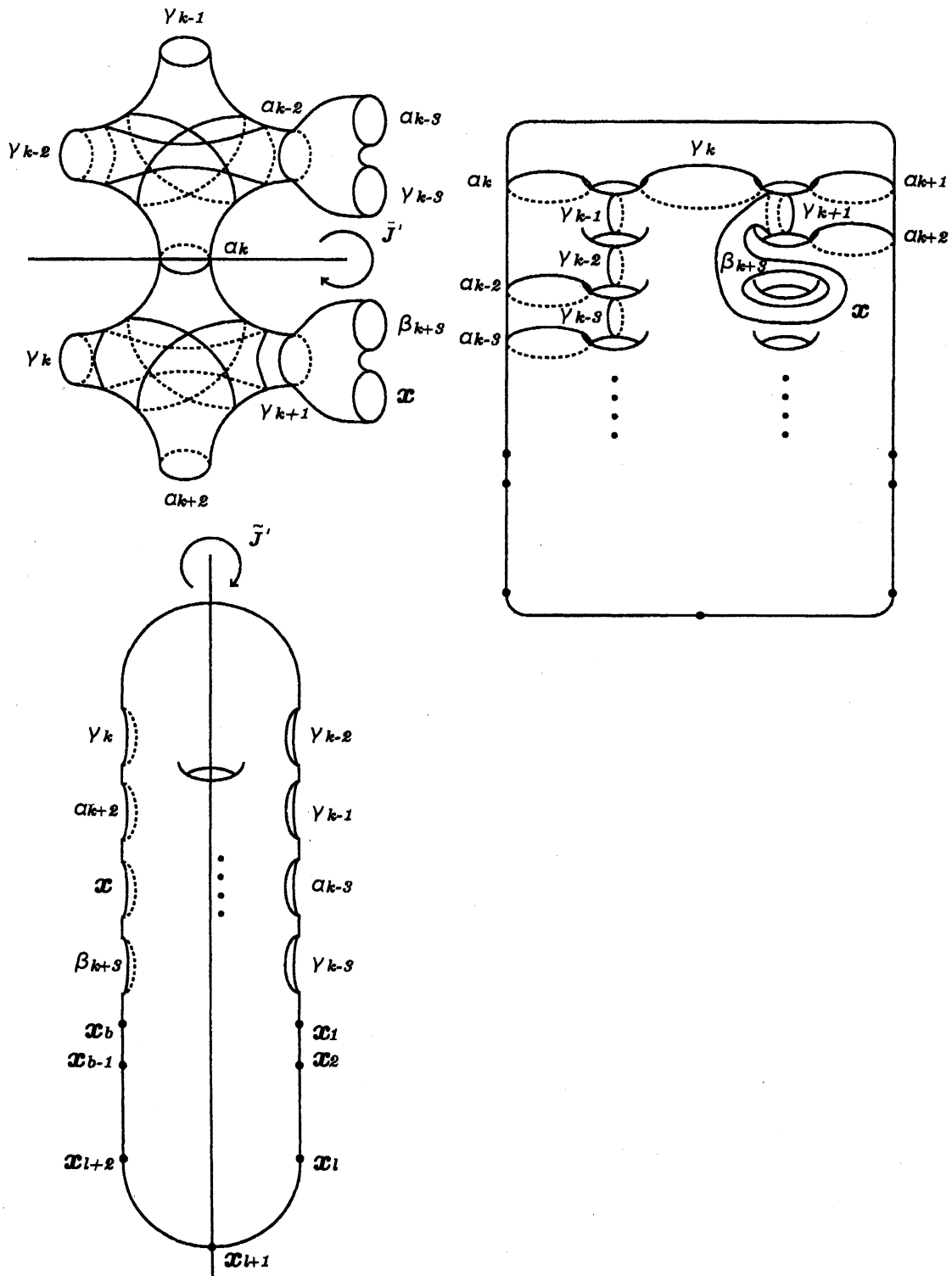
Theorem 7. *If $g \geq 5$, the group G_3 generated by $\rho_1, \rho_2, \rho_3, I$ and J is the whole mapping class group $\text{Mod}(\Sigma_{g,b})$.*

Proof. By the relation (6) we satisfy the condition (1). Since J sends α_{k-2} to γ_{k+1} and I sends α_k to β_{k+1} , we consist the condition (2). We can also see that we satisfy the condition (3) from a way to the construction of the involution J .

Therefore, we can finish the proof of the theorem because we can satisfy the conditions in 3.1. $\square$

3.4 Genus at least 7

We want to improve the above argument and show that for the genus $g \geq 7$ we do not need the involution I in order to generate the mapping class group. Assume that the genus of the surface is at least 7.

Figure 7: The involution J' on $\Sigma_{g,b}$

The S_2 and two pairs of pants have common boundary components $R^{-2}a_1$ and a_3 and their union is a surface S_3 homeomorphic to a sphere with 8 boundary components. Figure 7 shows the existence of the involution $\bar{J}'$ on S_3 which extends the involution $\bar{J}$ on S_2 .

Let us embed S_3 in the $\Sigma_{g,b}$ as shown on Figure 7. From Figure 7 we can find the involution $\bar{J}'$ of the complement of S_3 . Let J' be the involution obtained by gluing together $\bar{J}'$ and $\bar{J}$. Moreover, from Figure 7 we can construct J' which acts on the punctures as the involution r_3 .

Theorem 8. *If $g \geq 7$, the group G_4 generated by ρ_1, ρ_2, ρ_3 and J' is the whole mapping class group $\text{Mod}(\Sigma_{g,b})$.*

Proof. From the construction of J' we have

$$T_{\gamma_k} = (\rho_2\rho_3)(R^2J'\rho_2\rho_3J'R^{-2})(J'R^{-2}\rho_2\rho_3R^2J') \in G_4.$$

Therefore, we can see that we satisfy the condition $\langle 1 \rangle$. Since J' can send α_{k-2} to γ_{k+1} and β_{k+3} to γ_{k-3} , we can satisfy the condition $\langle 2 \rangle$ only in J' . Moreover, By that J' acts as r_3 , we consist the condition $\langle 3 \rangle$. Therefore, the group G_4 is the whole mapping class group. $\square$

4 Acknowledgement

I would like to thank Professor Hisaaki Endo for careful readings and for many helpful suggestions and comments. And I would like to thank Hitomi Fukushima, Yeonhee Jang and Kouki Masumoto for many advices.

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