

Center Manifold Theorem for Integral Equations

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1 Introduction

In this paper we are concerned with the integral equation (with infinite delay)

$$x(t) = \int_{-\infty}^t K(t-s)x(s)ds + f(x_t), \quad (E)$$

where K is a measurable $m \times m$ matrix valued function with complex components satisfying the condition $\int_0^\infty \|K(t)\|e^{\rho t}dt < \infty$ and $\text{ess sup}\{\|K(t)\|e^{\rho t} : t \geq 0\} < \infty$, and f is a nonlinear term belonging to the space $C^1(X; \mathbb{C}^m)$, the set of all continuously (Fréchet) differentiable functions mapping X into $\mathbb{C}^m$, with the property that $f(0) = 0$ and $Df(0) = 0$; here, ρ is a positive constant which is fixed throughout the paper, and $X := L_\rho^1(\mathbb{R}^-; \mathbb{C}^m)$, $\mathbb{R}^- := (-\infty, 0]$, is a Banach space (employed throughout the paper as the phase space for Eq. (E)) equipped with norm $\|\phi\|_X := \int_{-\infty}^0 |\phi(\theta)|e^{\rho\theta}d\theta$ ($\forall \phi \in X$), and x_t is an element in X defined as $x_t(\theta) = x(t+\theta)$ for $\theta \in \mathbb{R}^-$. The linearized equation of Eq. (E) (around the equilibrium point 0) is given by

$$x(t) = \int_{-\infty}^t K(t-s)x(s)ds, \quad (1)$$

which possesses the characteristic matrix $\Delta(\lambda) := E_m - \int_0^\infty K(t)e^{-\lambda t}dt$ ($\text{Re } \lambda > -\rho$); here E_m is the $m \times m$ unit matrix. Recently, Diekmann and Gyllenberg [3] have treated Eq. (E), and established the principle of linearized stability for integral equations. In the paper, as a further development in the stability problem of Eq. (E), we treat the case that the equilibrium point zero is nonhyperbolic (that is, the set $\{\lambda \in \mathbb{C} : \det \Delta(\lambda) = 0 \text{ \& } \text{Re } \lambda = 0\}$ is nonempty), and establish center manifold theorem for Eq. (E); and then we will investigate stability properties of the zero solution of Eq. (E) in the critical case.

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2 Several preparatory results for integral equations

In this section, following [6] we summarize several preliminary results necessary for our later arguments. Eq.(E) can be formulated as an abstract equation on the space X of the form

$$x(t) = L(x_t) + f(x_t),$$

where $L : X \rightarrow \mathbb{C}^m$ is a bounded linear operator defined by $L(\phi) := \int_{-\infty}^0 K(-\theta)\phi(\theta)d\theta$ for $\phi \in X$. Let us consider Eq.(E) with the initial condition

$$x_\sigma = \phi, \quad \text{that is,} \quad x(\sigma + \theta) = \phi(\theta) \quad \text{for } \theta \in \mathbb{R}^-, \quad (2)$$

where $(\sigma, \phi) \in \mathbb{R} \times X$ is given arbitrarily. A function $x : (-\infty, a) \rightarrow \mathbb{C}^m$ is said to be a solution of the initial value problem (E)-(2) on the interval (σ, a) if x satisfies the following conditions: (i) $x_\sigma = \phi$, that is, $x(\sigma + \theta) = \phi(\theta)$ for $\theta \in \mathbb{R}^-$; (ii) $x \in L^1_{\text{loc}}[\sigma, a)$, x is locally integrable on $[\sigma, a)$; (iii) $x(t) = L(x_t) + f(x_t)$ for $t \in (\sigma, a)$.

By virtue of [6, Proposition 1], the initial value problem (E)-(2) has a unique (local) solution which is denoted by $x(t; \sigma, \phi, f)$; in fact, $x(t; \sigma, \phi, f)$ is defined globally if, in particular, $f(\phi)$ is globally Lipschitz continuous in ϕ . Moreover we remark that if $x(t)$ is a solution of Eq.(E) on (σ, a) , then x_t is an X -valued continuous function on $[\sigma, a)$. Now suppose that $\phi = \psi$ in X , that is, $\phi(\theta) = \psi(\theta)$ a.e. $\theta \in \mathbb{R}^-$. Then by the uniqueness of solutions of (E)-(2) it follows that $x(t; \sigma, \phi, f) = x(t; \sigma, \psi, f)$ for $t \in (\sigma, a)$, so that $x_t(\sigma, \phi, f) = x_t(\sigma, \psi, f)$ in X for $t \in [\sigma, a)$. In particular, given $\sigma \in \mathbb{R}$, $x_t(\sigma, \cdot, f)$ induces a transformation on X for each $t \in [\sigma, a)$ provided that $x(t; \sigma, \phi, f)$ is the solution of (E)-(2) on (σ, a) .

For any $t \geq 0$ and $\phi \in X$, we define $T(t)\phi \in X$ by

$$[T(t)\phi](\theta) := x_t(\theta; 0, \phi, 0) = \begin{cases} x(t + \theta; 0, \phi, 0), & -t < \theta \leq 0, \\ \phi(t + \theta), & \theta \leq -t. \end{cases}$$

Then $T(t)$ defines a bounded linear operator on X . In fact, $\{T(t)\}_{t \geq 0}$ is a strongly continuous semigroup of bounded linear operators on X , called the solution semigroup for Eq.(1). Denote by A the generator of $\{T(t)\}_{t \geq 0}$, and let $\sigma(A)$ and $P_\sigma(A)$ be the spectrum and the point spectrum of the generator A , respectively. Between the spectrum of A and the characteristic roots of Eq. (1), the relation $\sigma(A) \cap \mathbb{C}_{-\rho} = P_\sigma(A) \cap \mathbb{C}_{-\rho} = \{\lambda \in \mathbb{C}_{-\rho} : \det \Delta(\lambda) = 0\} (=:\Sigma)$ holds, where $\mathbb{C}_{-\rho} := \{z \in \mathbb{C} : \operatorname{Re} z > -\rho\}$. Moreover, for $\operatorname{ess}(A)$, the essential spectrum of A , we have the estimate $\sup_{\lambda \in \operatorname{ess}(A)} \operatorname{Re} \lambda \leq -\rho$. Now set $\Sigma^u := \{\lambda \in \sigma(A) : \operatorname{Re} \lambda > 0\}$, $\Sigma^c := \{\lambda \in \sigma(A) : \operatorname{Re} \lambda = 0\}$, and $\Sigma^s := \sigma(A) \setminus (\Sigma^c \cup \Sigma^u)$. Then these observations, combined with the analyticity of $\det \Delta(\lambda)$ on the domain $\mathbb{C}_{-\rho}$, yield the following result ([6, Theorem 2]):

Proposition 1. *Let $\{T(t)\}_{t \geq 0}$ be the solution semigroup of Eq.(1). Then X is decomposed as a direct sum of closed subspaces E^u , E^c , and E^s*

$$X = E^u \oplus E^c \oplus E^s$$

with the following properties:

- (i) $\dim(E^u \oplus E^c) < \infty$,
- (ii) $T(t)E^u \subset E^u$, $T(t)E^c \subset E^c$, and $T(t)E^s \subset E^s$ for $t \in \mathbb{R}^+ := [0, \infty)$,
- (iii) $\sigma(A|_{E^u}) = \Sigma^u$, $\sigma(A|_{E^c}) = \Sigma^c$ and $\sigma(A|_{E^s \cap \mathcal{D}(A)}) = \Sigma^s$,
- (iv) $T^u(t) := T(t)|_{E^u}$ and $T^c(t) := T(t)|_{E^c}$ are extendable for $t \in \mathbb{R} := (-\infty, \infty)$ as groups of bounded linear operators on E^u and E^c , respectively,
- (v) $T^s(t) := T(t)|_{E^s}$ is a strongly continuous semigroup of bounded linear operators on E^s , and its generator is identical with $A|_{E^s \cap \mathcal{D}(A)}$,
- (vi) there exist positive constants α, ε with $\alpha > \varepsilon$ and a constant $C \geq 1$ such that

$$\begin{aligned} \|T^s(t)\|_{\mathcal{L}(X)} &\leq Ce^{-\alpha t}, \quad t \in \mathbb{R}^+, \\ \|T^u(t)\|_{\mathcal{L}(X)} &\leq Ce^{\alpha t}, \quad t \in \mathbb{R}^-, \\ \|T^c(t)\|_{\mathcal{L}(X)} &\leq Ce^{\varepsilon|t|}, \quad t \in \mathbb{R}. \end{aligned}$$

In (vi) we note that C is a constant depending only on α and ε , and that the value of $\varepsilon > 0$ can be taken arbitrarily small. Also, we will use the notations $E^{cu} = E^c \oplus E^u$, $E^{su} = E^s \oplus E^u$ etc, and denote by Π^s the projection from X onto E^s along E^{cu} , and similarly for Π^u , Π^{cu} etc.

We now introduce a continuous function $\Gamma^n : \mathbb{R}^- \rightarrow \mathbb{R}^+$ for each natural number n which is of compact support with *support* $\Gamma^n \subset [-1/n, 0]$ and satisfies $\int_{-\infty}^0 \Gamma^n(\theta) d\theta = 1$. Notice that $\Gamma^n \beta \in X$ for any $\beta \in \mathbb{C}^m$. Let us recall that $x(\cdot; \sigma, \varphi, p)$ is the (unique) solution of the integral equation

$$x(t) = \int_{-\infty}^t K(t-s)x(s)ds + p(t), \quad t > \sigma \quad (3)$$

through (σ, φ) ; here $\varphi \in X$. The following result ([6, Theorem 3]), which will often be referred to as *VCF* for short, gives a representation formula for $x_t(\sigma, \varphi, p)$ in the space X by using $T(t)$, φ and p .

Proposition 2. *Let $p \in C([\sigma, \infty); \mathbb{C}^m)$. Then*

$$x_t(\sigma, \varphi, p) = T(t-\sigma)\varphi + \lim_{n \rightarrow \infty} \int_{\sigma}^t T(t-s)(\Gamma^n p(s))ds, \quad \forall t \geq \sigma \quad (4)$$

in X .

Let us consider a subset $\bar{X}$ consisting of all elements $\phi \in X$ which are continuous on $[-\varepsilon_\phi, 0]$ for some $\varepsilon_\phi > 0$, and set

$$X_0 = \{\varphi \in X \mid \varphi = \phi \text{ a.e. on } \mathbb{R}^- \text{ for some } \phi \in \bar{X}\}.$$

For any $\varphi \in X_0$, we define the value of φ at zero by $\varphi[0] = \phi(0)$, where ϕ is an element belonging to $\bar{X}$ satisfying $\phi = \varphi$ a.e. on $\mathbb{R}^-$. We note that the value $\varphi[0]$ is well-defined; that is, it does not depend on the particular choice of ϕ since $\phi(0) = \psi(0)$ for any other $\psi \in \bar{X}$ such that $\phi = \psi$ a.e. on $\mathbb{R}^-$. It is clear that X_0 is a normed space equipped with norm

$$\|\varphi\|_{X_0} := \|\varphi\|_X + |\varphi[0]|, \quad \forall \varphi \in X_0.$$

We note that the solution $x(\cdot; \sigma, \psi, p)$ of Eq. (3) through $(\sigma, \psi) \in \mathbb{R} \times X$ satisfies the relation $x_t(\sigma, \psi, p) \in X_0$ with $(x_t(\sigma, \psi, p))[0] = x(t; \sigma, \psi, p)$ whenever $t > \sigma$.

The following lemma can be established by applying Proposition 2 and [6, Theorem 4]. We omit the proof.

Lemma 1. *Let $f_* \in C(X; \mathbb{C}^m)$, and consider the equation*

$$x(t) = \int_{-\infty}^t K(t-s)x(s)ds + f_*(x_t). \quad (E_*)$$

Moreover, let $\psi \in E^c$, and η be a constant such that $\varepsilon < \eta < \alpha$. Then we have:

- (i) *If $x(t)$ is a solution of Eq. (E_*) defined on $\mathbb{R}$ with the properties that $\Pi^c x_0 = \psi$, $\sup_{t \in \mathbb{R}} \|x_t\|_X e^{-\eta|t|} < \infty$ and $\sup_{t \in \mathbb{R}} |f_*(x_t)| < \infty$, then the X -valued function $u(t) := x_t$ satisfies*

$$\begin{aligned} u(t) = & T^c(t)\psi + \lim_{n \rightarrow \infty} \int_0^t T^c(t-s)\Pi^c \Gamma^n f_*(u(s))ds \\ & - \lim_{n \rightarrow \infty} \int_t^\infty T^u(t-s)\Pi^u \Gamma^n f_*(u(s))ds + \lim_{n \rightarrow \infty} \int_{-\infty}^t T^s(t-s)\Pi^s \Gamma^n f_*(u(s))ds \end{aligned}$$

for $t \in \mathbb{R}$, and moreover u belongs to $C(\mathbb{R}; X_0)$.

- (ii) *Conversely, if $y \in C(\mathbb{R}; X)$ with $\sup_{t \in \mathbb{R}} \|y(t)\|_X e^{-\eta|t|} < \infty$ and $\sup_{t \in \mathbb{R}} |f_*(y(t))| < \infty$ satisfies*

$$\begin{aligned} y(t) = & T^c(t)\psi + \lim_{n \rightarrow \infty} \int_0^t T^c(t-s)\Pi^c \Gamma^n f_*(y(s))d\tau \\ & - \lim_{n \rightarrow \infty} \int_t^\infty T^u(t-s)\Pi^u \Gamma^n f_*(y(s))ds + \lim_{n \rightarrow \infty} \int_{-\infty}^t T^s(t-s)\Pi^s \Gamma^n f_*(y(s))ds \end{aligned}$$

for $t \in \mathbb{R}$, then y belongs to $C(\mathbb{R}; X_0)$ and the function $\xi(t)$ defined by

$$\xi(t) := (y(t))[0], \quad t \in \mathbb{R}$$

is a solution of Eq. (E_*) on $\mathbb{R}$ satisfying $\Pi^c \xi_0 = \psi$, $\sup_{t \in \mathbb{R}} \|\xi_t\|_X e^{-\eta|t|} < \infty$ and $\xi_t = y(t)$ for $t \in \mathbb{R}$.

3 Center manifold and its exponential attractivity

In what follows we assume that $f \in C^1(X; \mathbb{C}^m)$ satisfies $f(0) = 0$ and $Df(0) = 0$. In this section we will establish the existence of local center manifolds of the equilibrium point 0 of $\text{Eq.}(E)$ and study their properties. To do so, in parallel with $\text{Eq.}(E)$, we will consider a modified equation of (E) of the form

$$x(t) = \int_{-\infty}^t K(t-s)x(s)ds + f_\delta(x_t), \quad (E_\delta)$$

where f_δ with $\delta > 0$ is a modification of the original nonlinear term f ; more precisely let $\chi : \mathbb{R} \rightarrow [0, 1]$ be a C^∞ -function such that $\chi(t) = 1$ ($|t| \leq 2$) and $\chi(t) = 0$ ($|t| \geq 3$), and define

$$f_\delta(\phi) := \chi(\|\Pi^{su}\phi\|_X/\delta)\chi(\|\Pi^c\phi\|_X/\delta)f(\phi), \quad \phi \in X.$$

The function $f_\delta : X \rightarrow \mathbb{C}^m$ is continuous on X , and is of class C^1 when restricted to the open set $S_\delta := \{\phi \in X : \|\Pi^{su}\phi\|_X < \delta\}$ since we may assume that $\|\Pi^c\phi\|_X$ is of class C^1 for $\phi \neq 0$ because of $\dim E^c < \infty$. Moreover, by the assumption $f(0) = Df(0) = 0$, there exist a $\delta_1 > 0$ and a nondecreasing continuous function $\zeta_* : (0, \delta_1] \rightarrow \mathbb{R}^+$ such that $\zeta_*(+0) = 0$,

$$\|f_\delta(\phi)\|_X \leq \delta\zeta_*(\delta) \quad \text{and} \quad \|f_\delta(\phi) - f_\delta(\psi)\|_X \leq \zeta_*(\delta)\|\phi - \psi\|_X \quad (5)$$

for $\phi, \psi \in X$ and $\delta \in (0, \delta_1]$. Indeed, we may put

$$\zeta_*(\delta) = \left(\sup_{\|\phi\|_X \leq 3\delta} \|Df(\phi)\|_{\mathcal{L}(X; \mathbb{C}^m)} \right) \cdot \left(1 + 3 \sup_{0 \leq t \leq 3} |\chi'(t)| \right)$$

(cf. [2, Lemma 4.1]). Taking $\delta_1 > 0$ small, we may also assume that there exists a positive number $M_1(\delta_1) =: M_1$ such that

$$\|Df_\delta(\phi)\|_{\mathcal{L}(X; \mathbb{C}^m)} \leq M_1, \quad \phi \in S_\delta \quad (6)$$

for any $\delta \in (0, \delta_1]$. Fix a positive number η such that

$$\varepsilon < \eta < \alpha,$$

where ε and α are the constants in Proposition 1.

For the existence of center manifold for $\text{Eq.}(E_\delta)$ and its exponential attractivity, we have the following:

Theorem 1. *There exist a positive number δ and a C^1 -map $F_{*,\delta} : E^c \rightarrow E^{su}$ with $F_{*,\delta}(0) = 0$ such that the following properties hold:*

- (i) $W_\delta^c := \text{graph } F_{*,\delta}$ is tangent to E^c at zero,

- (ii) W_δ^c is invariant for Eq. (E_δ) , that is, if $\xi \in W_\delta^c$, then $x_t(0, \xi, f) \in W_\delta^c$ for $t \in \mathbb{R}$.
- (iii) Assume moreover that $\Sigma^u = \emptyset$. Then there exists a positive constant β_0 with the property that if x is a solution of Eq. (E_δ) on an interval $J = [t_0, t_1]$, then the inequality

$$\|\Pi^s x_t - F_{*,\delta}(\Pi^c x_t)\|_X \leq C \|\Pi^s x_{t_0} - F_{*,\delta}(\Pi^c x_{t_0})\|_X e^{-\beta_0(t-t_0)}, \quad t \in J$$

holds true. In particular, if x is a solution on an interval $[t_0, \infty)$, x_t tends to W_δ^c exponentially as $t \rightarrow \infty$.

As will be shown in Proposition 4 given later, the map $F_{*,\delta} : E^c \rightarrow E^{su}$ in Theorem 1 is globally Lipschitz continuous with the Lipschitz constant $L(\delta) = 4C^2 C_1 \zeta_*(\delta) / (\alpha - \eta)$. Noticing that $L(\delta) \rightarrow 0$ as $\delta \rightarrow 0$, one can assume that the number δ satisfies $\delta \in (0, \delta_1]$ together with $L(\delta) \leq 1$. Let us take a small $r \in (0, \delta)$ so that $\|F_{*,\delta}(\psi)\|_X < \delta$ for any $\psi \in B_{E^c}(r) := \{\phi \in E^c : \|\phi\|_X < r\}$. Such a choice of r is possible by the continuity of $F_{*,\delta}$. Set $F_* := F_{*,\delta}|_{B_{E^c}(r)}$ and consider an open neighborhood Ω_0 of 0 in X defined by

$$\Omega_0 := \{\phi \in X : \|\Pi^{su}\phi\|_X < \delta, \|\Pi^c\phi\|_X < r\}.$$

Observe that $f \equiv f_\delta$ on Ω_0 . Then the following theorem which yields a local center manifold for Eq. (E) as the graph of F_* immediately follows from Theorem 1.

Theorem 2. Assume that $f \in C^1(X; \mathbb{C}^m)$ with $f(0) = Df(0) = 0$. Then there exist positive numbers r, δ , and a C^1 -map $F_* : B_{E^c}(r) \rightarrow E^{su}$ with $F_*(0) = 0$, together with an open neighborhood Ω_0 of 0 in X , such that the following properties hold:

- (i) $W_{\text{loc}}^c(r, \delta) := \text{graph } F_*$ is tangent to E^c at zero,
- (ii) $W_{\text{loc}}^c(r, \delta)$ is locally invariant for Eq. (E) , that is,
 - (a) for any $\xi \in W_{\text{loc}}^c(r, \delta)$ there exists a $t_\xi > 0$ such that $x_t(0, \xi, f) \in W_{\text{loc}}^c(r, \delta)$ for $|t| \leq t_\xi$,
 - (b) if $\xi \in W_{\text{loc}}^c(r, \delta)$ and $x_t(0, \xi, f) \in \Omega_0$ for $0 \leq t \leq T$, then $x_t(0, \xi, f) \in W_{\text{loc}}^c(r, \delta)$ for $0 \leq t \leq T$.
- (iii) Assume moreover that $\Sigma^u = \emptyset$. Then there exists a positive constant β_0 with the property that if x is a solution of Eq. (E) on an interval $J = [t_0, t_1]$ satisfying $x_t \in \Omega_0$ on J , then the inequality

$$\|\Pi^s x_t - F_*(\Pi^c x_t)\|_X \leq C \|\Pi^s x_{t_0} - F_*(\Pi^c x_{t_0})\|_X e^{-\beta_0(t-t_0)}, \quad t \in J$$

holds true. In particular, if the solution $x(t)$ is defined on $[t_0, \infty)$ satisfying $x_t \in \Omega_0$ on $[t_0, \infty)$, then x_t tends to $W_{\text{loc}}^c(r, \delta)$ exponentially as $t \rightarrow \infty$.

In what follows, we will prove Theorem 1 by establishing several propositions. We now take a $\delta_1 > 0$ sufficiently small so that

$$\zeta_*(\delta_1)CC_1 \left(\frac{1}{\eta - \varepsilon} + \frac{2}{\alpha + \eta} + \frac{2}{\alpha - \eta} \right) < \frac{1}{2} \quad (7)$$

holds, and let $\delta \in (0, \delta_1]$. Also, let us consider the Banach space Y_η defined by

$$Y_\eta := \{y \in C(\mathbb{R}; X) : \sup_{t \in \mathbb{R}} \|y(t)\|_X e^{-\eta|t|} < \infty\}$$

with norm $\|y\|_{Y_\eta} := \sup_{t \in \mathbb{R}} \|y(t)\|_X e^{-\eta|t|}$, $y \in Y_\eta$. For any $(\psi, y) \in E^c \times Y_\eta$, we set

$$\begin{aligned} \mathcal{F}_\delta(\psi, y)(t) &:= T^c(t)\psi + \lim_{n \rightarrow \infty} \int_0^t T^c(t-s)\Pi^c\Gamma^n f_\delta(y(s))ds \\ &\quad - \lim_{n \rightarrow \infty} \int_t^\infty T^u(t-s)\Pi^u\Gamma^n f_\delta(y(s))ds \\ &\quad + \lim_{n \rightarrow \infty} \int_{-\infty}^t T^s(t-s)\Pi^s\Gamma^n f_\delta(y(s))ds \end{aligned} \quad (8)$$

for $t \in \mathbb{R}$. Notice that the right-hand side is well-defined and that $\mathcal{F}_\delta(\psi, y)$ is an X -valued function on $\mathbb{R}$ for each $(\psi, y) \in E^c \times Y_\eta$. It is straightforward to certify that $\mathcal{F}_\delta(\psi, y) \in Y_\eta$ by virtue of Proposition 1 and (5); in other words, $\mathcal{F}_\delta$ defines a map from $E^c \times Y_\eta$ to Y_η . In fact, for each $\psi \in E^c$, $\mathcal{F}_\delta(\psi, \cdot)$ is a contraction map from Y_η into itself with Lipschitz constant $1/2$, because of the inequality

$$\begin{aligned} \|\mathcal{F}_\delta(\psi, y_1) - \mathcal{F}_\delta(\psi, y_2)\|_{Y_\eta} &\leq \sup_{t \in \mathbb{R}} e^{-\eta|t|} \left| \int_0^t CC_1\zeta_*(\delta)e^{-\varepsilon(t-s)}\|y_1 - y_2\|_{Y_\eta}e^{\eta|s|}ds \right| \\ &\quad + \sup_{t \in \mathbb{R}} e^{-\eta|t|} \int_t^\infty CC_1\zeta_*(\delta)e^{\alpha(t-s)}\|y_1 - y_2\|_{Y_\eta}e^{\eta|s|}ds \\ &\quad + \sup_{t \in \mathbb{R}} e^{-\eta|t|} \int_{-\infty}^t CC_1\zeta_*(\delta)e^{-\alpha(t-s)}\|y_1 - y_2\|_{Y_\eta}e^{\eta|s|}ds \\ &\leq \zeta_*(\delta_1)CC_1 \left(\frac{1}{\eta - \varepsilon} + \frac{2}{\alpha + \eta} + \frac{2}{\alpha - \eta} \right) \|y_1 - y_2\|_{Y_\eta} \\ &\leq (1/2)\|y_1 - y_2\|_{Y_\eta} \end{aligned}$$

for $y_1, y_2 \in Y_\eta$. Thus, the map $\mathcal{F}_\delta(\psi, \cdot)$ has a unique fixed point for each $\psi \in E^c$, say $\Lambda_{*,\delta}(\psi) \in Y_\eta$, i.e., we have

$$\begin{aligned} \Lambda_{*,\delta}(\psi)(t) &= T^c(t)\psi + \lim_{n \rightarrow \infty} \int_0^t T^c(t-s)\Pi^c\Gamma^n f_\delta(\Lambda_{*,\delta}(\psi)(s))ds \\ &\quad - \lim_{n \rightarrow \infty} \int_t^\infty T^u(t-s)\Pi^u\Gamma^n f_\delta(\Lambda_{*,\delta}(\psi)(s))ds \\ &\quad + \lim_{n \rightarrow \infty} \int_{-\infty}^t T^s(t-s)\Pi^s\Gamma^n f_\delta(\Lambda_{*,\delta}(\psi)(s))ds \end{aligned} \quad (9)$$

for $t \in \mathbb{R}$, whenever $0 < \delta \leq \delta_1$.

Proposition 3. $\Lambda_{*,\delta}(\psi)$ satisfies the following:

- (i) $\|\Lambda_{*,\delta}(\psi_1) - \Lambda_{*,\delta}(\psi_2)\|_{Y_\eta} \leq 2C\|\psi_1 - \psi_2\|_X$ for $\psi_1, \psi_2 \in E^c$.
- (ii) $\Lambda_{*,\delta}(\psi)(t + \tau) = \Lambda_{*,\delta}(\Pi^c(\Lambda_{*,\delta}(\psi)(\tau)))(t)$ holds for $t, \tau \in \mathbb{R}$.

Proof. Since $\varepsilon < \eta$, (i) immediately follows from the estimate

$$\begin{aligned} \|\Lambda_{*,\delta}(\psi_1) - \Lambda_{*,\delta}(\psi_2)\|_{Y_\eta} &= \|\mathcal{F}_\delta(\psi_1, \Lambda_{*,\delta}(\psi_1)) - \mathcal{F}_\delta(\psi_2, \Lambda_{*,\delta}(\psi_2))\|_{Y_\eta} \\ &\leq \|\mathcal{F}_\delta(\psi_1, \Lambda_{*,\delta}(\psi_1)) - \mathcal{F}_\delta(\psi_1, \Lambda_{*,\delta}(\psi_2))\|_{Y_\eta} \\ &\quad + \|\mathcal{F}_\delta(\psi_1, \Lambda_{*,\delta}(\psi_2)) - \mathcal{F}_\delta(\psi_2, \Lambda_{*,\delta}(\psi_2))\|_{Y_\eta} \\ &\leq (1/2)\|\Lambda_{*,\delta}(\psi_1) - \Lambda_{*,\delta}(\psi_2)\|_{Y_\eta} + \|T^c(\cdot)(\psi_1 - \psi_2)\|_{Y_\eta} \\ &\leq (1/2)\|\Lambda_{*,\delta}(\psi_1) - \Lambda_{*,\delta}(\psi_2)\|_{Y_\eta} + \sup_{t \in \mathbb{R}} (Ce^{\varepsilon|t|}\|\psi_1 - \psi_2\|_X e^{-\eta|t|}). \end{aligned}$$

Next, given $\tau \in \mathbb{R}$, let us consider the function $\tilde{\Lambda}(t)$ defined by $\tilde{\Lambda}(t) := \Lambda_{*,\delta}(\psi)(t + \tau)$, $t \in \mathbb{R}$. Obviously, $\tilde{\Lambda}(\cdot) \in Y_\eta$. Also, it is easy to check that $\tilde{\Lambda}(t) = \mathcal{F}_\delta(\Pi^c(\Lambda_{*,\delta}(\psi)(\tau)), \tilde{\Lambda})(t)$ for all $t \in \mathbb{R}$; that is, $\tilde{\Lambda}$ is a fixed point of $\mathcal{F}_\delta(\Pi^c(\Lambda_{*,\delta}(\psi)(\tau)), \cdot)$. The uniqueness of the fixed points yields $\tilde{\Lambda} = \Lambda_{*,\delta}(\Pi^c(\Lambda_{*,\delta}(\psi)(\tau)))$, and hence

$$\Lambda_{*,\delta}(\psi)(t + \tau) = \tilde{\Lambda}(t) = \Lambda_{*,\delta}(\Pi^c(\Lambda_{*,\delta}(\psi)(\tau)))(t), \quad t \in \mathbb{R},$$

which shows (ii). □

For $\delta \in (0, \delta_1]$ let $F_{*,\delta} : E^c \rightarrow E^{su}$ be the map defined by $F_{*,\delta}(\psi) := \Pi^{su} \circ \text{ev}_0 \circ \Lambda_{*,\delta}(\psi)$ for $\psi \in E^c$, where ev_0 is the evaluation map: $\text{ev}_0(y) := y(0)$ for $y \in C(\mathbb{R}; X)$. Then

$$\begin{aligned} F_{*,\delta}(\psi) &= - \lim_{n \rightarrow \infty} \int_0^\infty T^u(-s) \Pi^u \Gamma^n f_\delta(\Lambda_{*,\delta}(\psi)(s)) ds \\ &\quad + \lim_{n \rightarrow \infty} \int_{-\infty}^0 T^s(-s) \Pi^s \Gamma^n f_\delta(\Lambda_{*,\delta}(\psi)(s)) ds, \quad \psi \in E^c; \end{aligned} \quad (10)$$

and in particular $\Lambda_{*,\delta}(\psi)(0) = \psi + F_{*,\delta}(\psi)$ for $\psi \in E^c$. Let us set

$$W_\delta^c := \text{graph } F_{*,\delta} = \{\psi + F_{*,\delta}(\psi) : \psi \in E^c\}.$$

Proposition 4. The map $F_{*,\delta}$ and its graph W_δ^c have the following properties:

- (i) $F_{*,\delta}$ is (globally) Lipschitz continuous, i.e.,

$$\|F_{*,\delta}(\psi_1) - F_{*,\delta}(\psi_2)\|_X \leq L(\delta)\|\psi_1 - \psi_2\|_X, \quad \psi_1, \psi_2 \in E^c,$$

where $L(\delta) := 4C^2 C_1 \zeta_*(\delta) / (\alpha - \eta)$.

- (ii) Let $\hat{\phi} \in W_\delta^c$ and $\tau \in \mathbb{R}$. Then the solution of (E_δ) through $(\tau, \hat{\phi})$, $x(t; \tau, \hat{\phi}, f_\delta)$, exists on $\mathbb{R}$ and

$$x_t(\tau, \hat{\phi}, f_\delta) = \Lambda_{*,\delta}(\hat{\psi})(t - \tau), \quad t \in \mathbb{R},$$

where $\hat{\psi} = \Pi^c \hat{\phi}$.

- (iii) Moreover for $\hat{\phi} \in W_\delta^c$ and $\tau \in \mathbb{R}$,

$$\Pi^{su} x_t(\tau, \hat{\phi}, f_\delta) = F_{*,\delta}(\Pi^c x_t(\tau, \hat{\phi}, f_\delta)), \quad t \in \mathbb{R}.$$

In particular W_δ^c is invariant for (E_δ) , that is, $x_t(\tau, \hat{\phi}, f_\delta) \in W_\delta^c$ for $t \in \mathbb{R}$, provided that $\hat{\phi} \in W_\delta^c$.

Proof. (i) By (10) and Proposition 3 (i), we get

$$\begin{aligned} \|F_{*,\delta}(\psi_1) - F_{*,\delta}(\psi_2)\|_X &\leq \int_0^\infty CC_1 e^{-\alpha s} \zeta_*(\delta) \|\Lambda_{*,\delta}(\psi_1)(s) - \Lambda_{*,\delta}(\psi_2)(s)\|_X ds \\ &\quad + \int_{-\infty}^0 CC_1 e^{\alpha s} \zeta_*(\delta) \|\Lambda_{*,\delta}(\psi_1)(s) - \Lambda_{*,\delta}(\psi_2)(s)\|_X ds \\ &\leq \frac{2CC_1 \zeta_*(\delta)}{\alpha - \eta} \times 2C \|\psi_1 - \psi_2\|_X = L(\delta) \|\psi_1 - \psi_2\|_X, \end{aligned}$$

as required.

- (ii) Applying Lemma 1 (i), we deduce that $\Lambda_{*,\delta}(\hat{\psi}) \in C(\mathbb{R}; X_0)$ and that the X -valued function $\xi(t) := (\Lambda_{*,\delta}(\hat{\psi})(t))[0]$ ($t \in \mathbb{R}$) satisfies $\xi_t = \Lambda_{*,\delta}(\hat{\psi})(t)$ for $t \in \mathbb{R}$ and is a solution of (E_δ) on $\mathbb{R}$ with $\xi_0 = \Lambda_{*,\delta}(\hat{\psi})(0) = \hat{\psi} + F_{*,\delta}(\hat{\psi}) = \hat{\phi}$. Let $x(t) := \xi(t - \tau)$. Then $x(t)$ is a solution of (E_δ) on $\mathbb{R}$ with $x_\tau = \hat{\phi}$, so that $x(t) = x(t; \tau, \hat{\phi}, f_\delta)$ for $t \in \mathbb{R}$. Consequently,

$$x_t(\tau, \hat{\phi}, f_\delta) = \xi_{t-\tau} = \Lambda_{*,\delta}(\hat{\psi})(t - \tau), \quad t \in \mathbb{R}.$$

- (iii) Notice from Proposition 3 (ii) that $\Lambda_{*,\delta}(\hat{\psi})(t - \tau) = \Lambda_{*,\delta}(\Pi^c(\Lambda_{*,\delta}(\hat{\psi})(t - \tau)))(0)$ for $\hat{\psi} := \Pi^c \hat{\phi}$, which, combined with (ii), yields that

$$\begin{aligned} \Pi^{su} x_t(\tau, \hat{\phi}, f_\delta) &= \Pi^{su}(\Lambda_{*,\delta}(\Pi^c(\Lambda_{*,\delta}(\hat{\psi})(t - \tau)))(0)) \\ &= \Pi^{su}(\Lambda_{*,\delta}(\Pi^c x_t(\tau, \hat{\phi}, f_\delta))(0)) = F_{*,\delta}(\Pi^c x_t(\tau, \hat{\phi}, f_\delta)); \end{aligned}$$

which is the desired one. The latter part of (iii) is obvious. $\square$

Now assume that $\Sigma^u = \emptyset$, i.e., $E^u = \{0\}$. Fix a $\delta \in (0, \delta_1]$ and let

$$K := CC_1 \zeta_*(\delta), \quad \mu := K + \varepsilon.$$

Proposition 5. Let $x(t)$ be a solution of (E_δ) on an interval $J := [t_0, t_1]$. Given $\tau \in J$, put $\hat{\phi} := \Pi^c x_\tau + F_{*,\delta}(\Pi^c x_\tau)$. Then the following inequalities hold:

(i) For $t_0 \leq t \leq \tau$

$$\|\Pi^c x_t - \Pi^c x_t(\tau, \hat{\phi}, f_\delta)\|_X \leq K \int_t^\tau e^{\mu(s-t)} \|\Pi^s x_s - \Pi^s x_s(\tau, \hat{\phi}, f_\delta)\|_X ds.$$

(ii) Moreover for $t_0 \leq t \leq \tau$

$$\|\Pi^c x_t - \Pi^c x_t(\tau, \hat{\phi}, f_\delta)\|_X \leq K \int_t^\tau e^{\mu'(s-t)} \|\xi(s)\|_X ds,$$

where $\mu' := \mu + KL(\delta)$ and $\xi(t) := \Pi^s x_t - F_{*,\delta}(\Pi^c x_t)$ for $t \in \mathbb{R}$.

Proof. By virtue of Proposition 4 (ii) and (iii), the solution $x(t; \tau, \hat{\phi}, f_\delta)$ exists on $\mathbb{R}$ and $\Pi^s x_t(\tau, \hat{\phi}, f_\delta) = F_{*,\delta}(\Pi^c x_t(\tau, \hat{\phi}, f_\delta))$ for $t \in \mathbb{R}$. Let $t_0 \leq t \leq \tau$. VCF gives

$$x_\tau(\tau, \hat{\phi}, f_\delta) = T(\tau - t)x_t(\tau, \hat{\phi}, f_\delta) + \lim_{n \rightarrow \infty} \int_t^\tau T(\tau - s)\Gamma^n f_\delta(x_s(\tau, \hat{\phi}, f_\delta))ds,$$

in particular

$$\Pi^c x_\tau(\tau, \hat{\phi}, f_\delta) = T^c(\tau - t)\Pi^c x_t(\tau, \hat{\phi}, f_\delta) + \lim_{n \rightarrow \infty} \int_t^\tau T^c(\tau - s)\Pi^c \Gamma^n f_\delta(x_s(\tau, \hat{\phi}, f_\delta))ds.$$

By the group property of $\{T^c(t)\}_{t \in \mathbb{R}}$, we get

$$\Pi^c x_t(\tau, \hat{\phi}, f_\delta) = T^c(t - \tau)\Pi^c x_\tau(\tau, \hat{\phi}, f_\delta) - \lim_{n \rightarrow \infty} \int_t^\tau T^c(t - s)\Pi^c \Gamma^n f_\delta(x_s(\tau, \hat{\phi}, f_\delta))ds. \quad (11)$$

Similarly for the solution $x(t)$

$$\Pi^c x_t = T^c(t - \tau)\Pi^c x_\tau - \lim_{n \rightarrow \infty} \int_t^\tau T^c(t - s)\Pi^c \Gamma^n f_\delta(x_s)ds.$$

Then, since $\Pi^c x_\tau(\tau, \hat{\phi}, f_\delta) = \Pi^c \hat{\phi} = \Pi^c x_\tau$, it follows that

$$\begin{aligned} e^{\varepsilon t} \|\Pi^c x_t - \Pi^c x_t(\tau, \hat{\phi}, f_\delta)\|_X &\leq \int_t^\tau K e^{\varepsilon s} \|\Pi^s x_s - \Pi^s x_s(\tau, \hat{\phi}, f_\delta)\|_X ds \\ &\quad + \int_t^\tau K e^{\varepsilon s} \|\Pi^c x_s - \Pi^c x_s(\tau, \hat{\phi}, f_\delta)\|_X ds \end{aligned}$$

for $t_0 \leq t \leq \tau$. Hence we get

$$e^{\varepsilon t} \|\Pi^c x_t - \Pi^c x_t(\tau, \hat{\phi}, f_\delta)\|_X \leq \int_t^\tau K e^{K(s-t)} e^{\varepsilon s} \|\Pi^s x_s - \Pi^s x_s(\tau, \hat{\phi}, f_\delta)\|_X ds,$$

which implies (i).

Next we will verify (ii). By Proposition 4 (iii) and (i), we get $\|\Pi^s x_s - \Pi^s x_s(\tau, \hat{\phi}, f_\delta)\|_X \leq \|\xi(s)\|_X + L(\delta)\|\Pi^c x_s - \Pi^c x_s(\tau, \hat{\phi}, f_\delta)\|_X$ for $s \in J$. Hence it follows from (i) that

$$\begin{aligned} e^{\mu t} \|\Pi^c x_t - \Pi^c x_t(\tau, \hat{\phi}, f_\delta)\|_X &\leq \int_t^\tau K e^{\mu s} \|\xi(s)\|_X ds \\ &\quad + \int_t^\tau KL(\delta) e^{\mu s} \|\Pi^c x_s - \Pi^c x_s(\tau, \hat{\phi}, f_\delta)\|_X ds; \end{aligned}$$

then

$$e^{\mu t} \|\Pi^c x_t - \Pi^c x_t(\tau, \hat{\phi}, f_\delta)\|_X \leq \int_t^\tau K e^{KL(\delta)(s-t)} e^{\mu s} \|\xi(s)\|_X ds,$$

which implies (ii). $\square$

Recall that

$$K := CC_1 \zeta_*(\delta), \quad \mu := K + \varepsilon, \quad \mu' := \mu + KL(\delta) = K(1 + L(\delta)) + \varepsilon. \quad (12)$$

Proposition 6. Assume that $\Sigma^u = \emptyset$ and $x(t)$ is a solution of (E_δ) on $J = [t_0, t_1]$. Define $\hat{x}_t \in W_\delta^c$ by $\hat{x}_t := \Pi^c x_t + F_{*,\delta}(\Pi^c x_t)$ for $t \in J$, and set $y(s; t) := \Pi^c x_s(t, \hat{x}_t, f_\delta)$ for $t \in J$ and $s \leq t$. Then the following inequality holds:

$$\|y(s; t) - y(s; t_0)\|_X \leq K \int_{t_0}^t e^{\mu'(\theta-s)} \|\xi(\theta)\|_X d\theta, \quad s \leq t_0,$$

where $\xi(\theta) := \Pi^s x_\theta - F_{*,\delta}(\Pi^c x_\theta)$ for $\theta \in [t_0, t]$.

Proof. Suppose that $s \leq t_0$. By the same reasoning as (11)

$$\Pi^c x_s(t, \hat{x}_t, f_\delta) = T^c(s-t) \Pi^c \hat{x}_t - \lim_{n \rightarrow \infty} \int_s^t T^c(s-\sigma) \Pi^c \Gamma^n f_\delta(x_\sigma(t, \hat{x}_t, f_\delta)) d\sigma. \quad (13)$$

Applying VCF to x_t and using $\Pi^c \hat{x}_\tau = \Pi^c x_\tau$ ($\tau \in J$), we deduce that

$$\Pi^c \hat{x}_t = T^c(t-t_0) \Pi^c \hat{x}_{t_0} + \lim_{n \rightarrow \infty} \int_{t_0}^t T^c(t-\sigma) \Pi^c \Gamma^n f_\delta(x_\sigma) d\sigma,$$

and thus, (13) becomes

$$\begin{aligned} \Pi^c x_s(t, \hat{x}_t, f_\delta) &= T^c(s-t_0) \Pi^c \hat{x}_{t_0} + \lim_{n \rightarrow \infty} \int_{t_0}^t T^c(s-\sigma) \Pi^c \Gamma^n f_\delta(x_\sigma) d\sigma \\ &\quad - \lim_{n \rightarrow \infty} \int_s^t T^c(s-\sigma) \Pi^c \Gamma^n f_\delta(x_\sigma(t, \hat{x}_t, f_\delta)) d\sigma, \quad t \in J. \end{aligned}$$

Therefore

$$\begin{aligned} \|y(s; t) - y(s; t_0)\|_X &= \|\Pi^c x_s(t, \hat{x}_t, f_\delta) - \Pi^c x_s(t_0, \hat{x}_{t_0}, f_\delta)\|_X \\ &= \left\| \lim_{n \rightarrow \infty} \int_{t_0}^t T^c(s-\sigma) \Pi^c \Gamma^n f_\delta(x_\sigma) d\sigma \right. \\ &\quad \left. - \lim_{n \rightarrow \infty} \int_s^t T^c(s-\sigma) \Pi^c \Gamma^n f_\delta(x_\sigma(t, \hat{x}_t, f_\delta)) d\sigma \right. \\ &\quad \left. + \lim_{n \rightarrow \infty} \int_s^{t_0} T^c(s-\sigma) \Pi^c \Gamma^n f_\delta(x_\sigma(t_0, \hat{x}_{t_0}, f_\delta)) d\sigma \right\|_X \\ &\leq \int_{t_0}^t CC_1 e^{\varepsilon|s-\sigma|} \zeta_*(\delta) \|x_\sigma - x_\sigma(t, \hat{x}_t, f_\delta)\|_X d\sigma \\ &\quad + \int_s^{t_0} CC_1 e^{\varepsilon|s-\sigma|} \zeta_*(\delta) \|x_\sigma(t_0, \hat{x}_{t_0}, f_\delta) - x_\sigma(t, \hat{x}_t, f_\delta)\|_X d\sigma. \quad (14) \end{aligned}$$

Observe that

$$\begin{aligned} \|x_\sigma - x_\sigma(t, \hat{x}_t, f_\delta)\|_X &\leq \|\Pi^s x_\sigma - F_{*,\delta}(\Pi^c x_\sigma)\|_X + \|F_{*,\delta}(\Pi^c x_\sigma) - F_{*,\delta}(\Pi^c x_\sigma(t, \hat{x}_t, f_\delta))\|_X \\ &\quad + \|\Pi^c x_\sigma - \Pi^c x_\sigma(t, \hat{x}_t, f_\delta)\|_X \\ &\leq \|\xi(\sigma)\|_X + (1 + L(\delta))\|\Pi^c x_\sigma - \Pi^c x_\sigma(t, \hat{x}_t, f_\delta)\|_X, \end{aligned} \quad (15)$$

where we used Proposition 4 (i) and (iii). Note also that

$$\begin{aligned} \|x_\sigma(t_0, \hat{x}_{t_0}, f_\delta) - x_\sigma(t, \hat{x}_t, f_\delta)\|_X &\leq \|F_{*,\delta}(\Pi^c x_\sigma(t_0, \hat{x}_{t_0}, f_\delta)) - F_{*,\delta}(\Pi^c x_\sigma(t, \hat{x}_t, f_\delta))\|_X \\ &\quad + \|\Pi^c x_\sigma(t_0, \hat{x}_{t_0}, f_\delta) - \Pi^c x_\sigma(t, \hat{x}_t, f_\delta)\|_X \\ &\leq (1 + L(\delta))\|y(\sigma; t) - y(\sigma; t_0)\|_X. \end{aligned} \quad (16)$$

In view of (14), (15) and (16), combined with Proposition 5 (ii), we deduce

$$\begin{aligned} \|y(s; t) - y(s; t_0)\|_X &\leq \int_{t_0}^t K e^{\varepsilon(\sigma-s)} (\|\xi(\sigma)\|_X + (1 + L(\delta))\|\Pi^c x_\sigma - \Pi^c x_\sigma(t, \hat{x}_t, f_\delta)\|_X) d\sigma \\ &\quad + \int_s^{t_0} K e^{\varepsilon(\sigma-s)} (1 + L(\delta)) \|y(\sigma; t) - y(\sigma; t_0)\|_X d\sigma \\ &\leq \int_{t_0}^t K e^{\varepsilon(\sigma-s)} \|\xi(\sigma)\|_X d\sigma \\ &\quad + \int_{t_0}^t K e^{\varepsilon(\sigma-s)} (1 + L(\delta)) \left(K \int_\sigma^t e^{\mu'(\tau-\sigma)} \|\xi(\tau)\|_X d\tau \right) d\sigma \\ &\quad + \int_s^{t_0} K e^{\varepsilon(\sigma-s)} (1 + L(\delta)) \|y(\sigma; t) - y(\sigma; t_0)\|_X d\sigma. \end{aligned} \quad (17)$$

Notice that the second term of the right-hand side becomes

$$K \int_{t_0}^t (e^{\varepsilon(t_0-s)+\mu'(\sigma-t_0)} - e^{\varepsilon(\sigma-s)}) \|\xi(\sigma)\|_X d\sigma$$

because of (12). So we see from (17) that for $s \leq t_0$

$$\begin{aligned} e^{\varepsilon s} \|y(s; t) - y(s; t_0)\|_X &\leq K \int_{t_0}^t e^{(\varepsilon-\mu')t_0+\mu'\sigma} \|\xi(\sigma)\|_X d\sigma \\ &\quad + K(1 + L(\delta)) \int_s^{t_0} e^{\varepsilon\sigma} \|y(\sigma; t) - y(\sigma; t_0)\|_X d\sigma. \end{aligned}$$

By Gronwall's inequality and (12)

$$\begin{aligned} e^{\varepsilon s} \|y(s; t) - y(s; t_0)\|_X &\leq \left(K \int_{t_0}^t e^{(\varepsilon-\mu')t_0+\mu'\sigma} \|\xi(\sigma)\|_X d\sigma \right) e^{K(1+L(\delta))(t_0-s)} \\ &= K e^{-(\mu'-\varepsilon)s} \int_{t_0}^t e^{\mu'\sigma} \|\xi(\sigma)\|_X d\sigma, \end{aligned}$$

which yields the the desired one. $\square$

Proposition 7. Assume that $\Sigma^u = \emptyset$, and let $\delta \in (0, \delta_1]$ be a sufficiently small number satisfying

$$\max \left(\mu', \frac{K(\alpha - \varepsilon)}{\alpha - \mu'} \right) < \alpha. \quad (18)$$

If $x(t)$ is a solution of (E_δ) on $J = [t_0, t_1]$, then the function $\xi(t) := \Pi^s x_t - F_{*,\delta}(\Pi^c x_t)$ satisfies the inequality

$$\|\xi(t)\|_X \leq C \|\xi(t_0)\|_X e^{-\beta_0(t-t_0)}, \quad t \in J,$$

where $\beta_0 := \alpha - K(\alpha - \varepsilon)/(\alpha - \mu') > 0$. If in particular $J = [t_0, \infty)$, $\text{dist}(x_t, W_\delta^c)$ tends to 0 exponentially as $t \rightarrow \infty$.

Proof. By applying VCF, one can easily deduce the relation

$$\begin{aligned} \xi(t) - T^s(t - t_0)\xi(t_0) &= \lim_{n \rightarrow \infty} \int_{t_0-t}^0 T^s(-s) \Pi^s \Gamma^n (f_\delta(x_{s+t}) - f_\delta(\Lambda_{*,\delta}(\Pi^c x_t)(s))) ds \\ &\quad + \lim_{n \rightarrow \infty} \int_{-\infty}^{t_0-t} T^s(-s) \Pi^s \Gamma^n (f_\delta(\Lambda_{*,\delta}(\Pi^c x_{t_0})(t - t_0 + s)) \\ &\quad - f_\delta(\Lambda_{*,\delta}(\Pi^c x_t)(s))) ds, \quad t \in J. \end{aligned}$$

If we set $\hat{x}_t := \Pi^c x_t + F_{*,\delta}(\Pi^c x_t)$ for $t \in J$, by Proposition 4 (ii)

$$\Lambda_{*,\delta}(\Pi^c x_t)(s) = x_s(0, \hat{x}_t, f_\delta) = x_{s+t}(t, \hat{x}_t, f_\delta)$$

and

$$\Lambda_{*,\delta}(\Pi^c x_{t_0})(t - t_0 + s) = x_{t-t_0+s}(0, \hat{x}_{t_0}, f_\delta) = x_{s+t}(t_0, \hat{x}_{t_0}, f_\delta)$$

in particular for $s \in \mathbb{R}^-$. So

$$\begin{aligned} \xi(t) &= T^s(t - t_0)\xi(t_0) + \lim_{n \rightarrow \infty} \int_{t_0-t}^0 T^s(-s) \Pi^s \Gamma^n (f_\delta(x_{s+t}) - f_\delta(x_{s+t}(t, \hat{x}_t, f_\delta))) ds \\ &\quad + \lim_{n \rightarrow \infty} \int_{-\infty}^{t_0-t} T^s(-s) \Pi^s \Gamma^n (f_\delta(x_{s+t}(t_0, \hat{x}_{t_0}, f_\delta)) - f_\delta(x_{s+t}(t, \hat{x}_t, f_\delta))) ds, \end{aligned}$$

and thus

$$\begin{aligned} \|\xi(t)\|_X &\leq C e^{-\alpha(t-t_0)} \|\xi(t_0)\|_X + \int_{t_0}^t K e^{\alpha(\theta-t)} \|x_\theta - x_\theta(t, \hat{x}_t, f_\delta)\|_X d\theta \\ &\quad + \int_{-\infty}^{t_0} K e^{\alpha(\theta-t)} \|x_\theta(t_0, \hat{x}_{t_0}, f_\delta) - x_\theta(t, \hat{x}_t, f_\delta)\|_X d\theta. \end{aligned}$$

Since $x_\theta(t, \hat{x}_t, f_\delta)$ ($t \in J$, $\theta \in \mathbb{R}$) can be written as

$$\begin{aligned} x_\theta(t, \hat{x}_t, f_\delta) &= \Pi^c x_\theta(t, \hat{x}_t, f_\delta) + \Pi^s x_\theta(t, \hat{x}_t, f_\delta) \\ &= \Pi^c x_\theta(t, \hat{x}_t, f_\delta) + F_{*,\delta}(\Pi^c x_\theta(t, \hat{x}_t, f_\delta)) \end{aligned}$$

by Proposition 4 (iii), it follows from Proposition 4 (i) and Proposition 6 that for $\theta \leq t_0$

$$\begin{aligned} \|x_\theta(t_0, \hat{x}_{t_0}, f_\delta) - x_\theta(t, \hat{x}_t, f_\delta)\|_X &\leq \|\Pi^c x_\theta(t_0, \hat{x}_{t_0}, f_\delta) - \Pi^c x_\theta(t, \hat{x}_t, f_\delta)\|_X \\ &\quad + \|F_{*,\delta}(\Pi^c x_\theta(t_0, \hat{x}_{t_0}, f_\delta)) - F_{*,\delta}(\Pi^c x_\theta(t, \hat{x}_t, f_\delta))\|_X \\ &\leq (1 + L(\delta)) \|y(\theta; t) - y(\theta; t_0)\|_X \\ &\leq (1 + L(\delta)) K \int_{t_0}^t e^{\mu'(\tau-\theta)} \|\xi(\tau)\|_X d\tau, \end{aligned}$$

where $y(\theta; t)$ ($t \in J$) is the one in Proposition 6. On the other hand, for $t_0 \leq \theta \leq t$

$$\begin{aligned} \|x_\theta - x_\theta(t, \hat{x}_t, f_\delta)\|_X &\leq \|\Pi^s x_\theta - F_{*,\delta}(\Pi^c x_\theta)\|_X + \|F_{*,\delta}(\Pi^c x_\theta) - F_{*,\delta}(\Pi^c x_\theta(t, \hat{x}_t, f_\delta))\|_X \\ &\quad + \|\Pi^c x_\theta - \Pi^c x_\theta(t, \hat{x}_t, f_\delta)\|_X \\ &\leq \|\xi(\theta)\|_X + (1 + L(\delta)) \|\Pi^c x_\theta - \Pi^c x_\theta(t, \hat{x}_t, f_\delta)\|_X \\ &\leq \|\xi(\theta)\|_X + (1 + L(\delta)) K \int_\theta^t e^{\mu'(\sigma-\theta)} \|\xi(\sigma)\|_X d\sigma, \end{aligned}$$

where we used Proposition 4 (i), (iii) and Proposition 5 (ii). Thus we have

$$\begin{aligned} \|\xi(t)\|_X &\leq C e^{-\alpha(t-t_0)} \|\xi(t_0)\|_X \\ &\quad + \int_{t_0}^t K e^{\alpha(\theta-t)} \left(\|\xi(\theta)\|_X + (1 + L(\delta)) K \int_\theta^t e^{\mu'(\sigma-\theta)} \|\xi(\sigma)\|_X d\sigma \right) d\theta \\ &\quad + \int_{-\infty}^{t_0} K e^{\alpha(\theta-t)} (1 + L(\delta)) K \left(\int_{t_0}^t e^{\mu'(\tau-\theta)} \|\xi(\tau)\|_X d\tau \right) d\theta \\ &= C e^{-\alpha(t-t_0)} \|\xi(t_0)\|_X + \left(K + \frac{K^2(1 + L(\delta))}{\alpha - \mu'} \right) \int_{t_0}^t e^{\alpha(\sigma-t)} \|\xi(\sigma)\|_X d\sigma, \end{aligned}$$

so that

$$e^{\alpha t} \|\xi(t)\|_X \leq C e^{\alpha t_0} \|\xi(t_0)\|_X + \hat{K} \int_{t_0}^t e^{\alpha \sigma} \|\xi(\sigma)\|_X d\sigma,$$

where $\hat{K} := K + K^2(1 + L(\delta))/(\alpha - \mu')$. An application of Gronwall's inequality gives $e^{\alpha t} \|\xi(t)\|_X \leq C e^{\alpha t_0} \|\xi(t_0)\|_X e^{\hat{K}(t-t_0)}$, and hence

$$\|\xi(t)\|_X \leq C \|\xi(t_0)\|_X e^{-(\alpha - \hat{K})(t-t_0)}, \quad t \in J,$$

which is the desired one because of $\hat{K} = K(\alpha - \varepsilon)/(\alpha - \mu') = \alpha - \beta_0$.

The latter part of the proposition is evident. This completes the proof. $\square$

Proof of Theorem 1. The properties (ii) and (iii) of Theorem 1 are now immediate consequences of Propositions 4 and 7, respectively. We verify the property (i). Observe that Y_η is a subspace of $Y_{\eta'}$ if $\eta < \eta' < \alpha$, and denote the inclusion map by $\mathcal{J} : Y_\eta \rightarrow Y_{\eta'}$. By

almost the same reasoning as in [8], we see that $\mathcal{J}\Lambda_{*,\delta}$ is C^1 smooth as a map from E^c to $Y_{\eta'}$; and hence $F_{*,\delta} = \Pi^{su} \circ \text{ev}_0 \circ \mathcal{J}\Lambda_{*,\delta}$ is also C^1 smooth. Moreover, since

$$[[D(\mathcal{J}\Lambda_{*,\delta})(0)](t)]\psi = T^c(t)\psi, \quad \psi \in E^c, \quad t \in \mathbb{R}$$

holds by virtue of $Df_\delta(0) = Df(0) = 0$, it follows that

$$DF_{*,\delta}(0)\psi = D(\Pi^{su} \circ \text{ev}_0 \circ \mathcal{J}\Lambda_{*,\delta})(0)\psi = \Pi^{su}T^c(0)\psi = \Pi^{su}\psi = 0, \quad \psi \in E^c;$$

hence $DF_{*,\delta}(0) = 0$, which implies (i). $\square$

4 Stability analysis of integral equations via central equations

Center manifolds play a crucial role in the stability analysis of systems around non-hyperbolic equilibria. Indeed, center manifolds for several kinds of equations allow us to reduce the stability analysis of an original system to that of its restriction to a center manifold; see e.g., [1, 4, 5, 9]. In this section, introducing an ordinary differential equation (called the "central equation" of Eq. (E)) which is expressed by using the explicit formula of the projection Π^c , we will establish the reduction principle for integral equations that the stability properties for the central equation imply those of Eq. (E) in the neighborhood of its zero solution.

Assume that $\Sigma^c \neq \emptyset$. Let $\{\phi_1, \dots, \phi_{d_c}\}$ be a basis for E^c , where d_c is the dimension of E^c . Then based on the formal adjoint theory for Eq. (1) developed in [7], one can consider its dual basis as elements in the Banach space

$$X^\sharp := L^1_\rho(\mathbb{R}^+; (\mathbb{C}^*)^m) = \{\psi : \mathbb{R}^+ \rightarrow (\mathbb{C}^*)^m : \psi(\tau)e^{-\rho\tau} \text{ is integrable on } \mathbb{R}^+\}$$

with norm

$$\|\psi\|_{X^\sharp} := \int_0^\infty |\psi(\tau)|e^{-\rho\tau} d\tau, \quad \psi \in X^\sharp,$$

where $(\mathbb{C}^*)^m$ is the space of m -dimensional row vectors with complex components equipped with the norm which is compatible with the one in $\mathbb{C}^m$, that is, $|z^*z| \leq |z^*||z|$ for $z^* \in (\mathbb{C}^*)^m$ and $z \in \mathbb{C}^m$. To be more precise, if we set

$$\langle \psi, \phi \rangle := \int_{-\infty}^0 \left(\int_\theta^0 \psi(\xi - \theta)K(-\theta)\phi(\xi)d\xi \right) d\theta, \quad (\psi, \phi) \in X^\sharp \times X,$$

then this pairing defines a bounded bilinear form on $X^\sharp \times X$ with the property

$$|\langle \psi, \phi \rangle| \leq \|K\|_{\infty, \rho} \|\psi\|_{X^\sharp} \|\phi\|_X, \quad (\psi, \phi) \in X^\sharp \times X;$$

here we recall that $\|K\|_{\infty, \rho} = \text{ess sup}\{\|K(t)\|e^{\rho t} : t \geq 0\}$. Then there exist $\{\psi_1, \dots, \psi_{d_c}\}$, elements of $X^\#$, such that $\langle \psi_i, \phi_j \rangle = 1$ if $i = j$ and 0 otherwise, and $\langle \psi_i, \phi \rangle = 0$ for $\phi \in E^s$ and $i = 1, 2, \dots, d_c$; we call $\{\psi_1, \dots, \psi_{d_c}\}$ the dual basis of $\{\phi_1, \dots, \phi_{d_c}\}$; see [7] for details. Denote by Φ_c and Ψ_c , $(\phi_1, \dots, \phi_{d_c})$ and ${}^t(\psi_1, \dots, \psi_{d_c})$, the transpose of $(\psi_1, \dots, \psi_{d_c})$, respectively. Then, for any $\phi \in X$ the coordinate of its E^c -component with respect to the basis $\{\phi_1, \dots, \phi_{d_c}\}$, or Φ_c for short, is given by $\langle \Psi_c, \phi \rangle := {}^t(\langle \psi_1, \phi \rangle, \dots, \langle \psi_{d_c}, \phi \rangle) \in \mathbb{C}^{d_c}$, and therefore the projection Π^c is expressed, in terms of the basis Φ_c and its dual basis Ψ_c , by

$$\Pi^c \phi = \Phi_c \langle \Psi_c, \phi \rangle, \quad \phi \in X. \quad (19)$$

Since $\{T^c(t)\}_{t \geq 0}$ is a strongly continuous semigroup on the finite dimensional space E^c , there exists a $d_c \times d_c$ matrix G_c such that

$$T^c(t)\Phi_c = \Phi_c e^{tG_c}, \quad t \geq 0, \quad (20)$$

and $\sigma(G_c)$, the spectrum of G_c , is identical with Σ^c . The E^c -components of solutions of Eq.(E_δ) can be described by a certain ordinary differential equation in $\mathbb{C}^{d_c}$. More precisely, let $x(t)$ be a solution of Eq.(E_δ) through (σ, ϕ) , that is, $x(t) = x(t; \sigma, \phi, f)$. If we denote by $z_c(t)$ the component of $\Pi^c x_t$ with respect to the basis Φ_c , that is, $\Phi_c z_c(t) := \Pi^c x_t$, or $z_c(t) := \langle \Psi_c, x_t \rangle$, then by virtue of [6, Theorem 7] $z_c(t)$ satisfies the ordinary differential equation

$$\dot{z}_c(t) = G_c z_c(t) + H_c f_\delta(\Phi_c z_c(t) + \Pi^{su} x_t), \quad (21)$$

where H_c is the $d_c \times m$ matrix such that $H_c x := \lim_{n \rightarrow \infty} \langle \Psi_c, \Gamma^n x \rangle$ for $x \in \mathbb{C}^m$.

In connection with Eq. (21), let us consider the ordinary differential equations on $\mathbb{C}^{d_c}$

$$\dot{z}(t) = G_c z(t) + H_c f_\delta(\Phi_c z(t) + F_{*, \delta}(\Phi_c z(t))) \quad (CE_\delta)$$

and

$$\dot{z}(t) = G_c z(t) + H_c f(\Phi_c z(t) + F_*(\Phi_c z(t))). \quad (CE)$$

We call Eq. (CE) (resp. Eq. (CE_δ)) the central equation of (E) (resp. (E_δ)). Applying Proposition 4 (iii), one can easily derive the following result on relationships among solutions of Eq.(E_δ) (resp. Eq.(E)) and (CE_δ) (resp. (CE)).

Proposition 8. *The following statements hold true:*

- (i) *Let x be a solution of Eq.(E_δ) on an interval J such that $x_t \in W_\delta^c$ ($t \in J$). Then the function $z_c(t) := \langle \Psi_c, x_t \rangle$ satisfies the equation (CE) on J . Conversely, if $z(t)$ satisfies the equation (CE_δ) on an interval J , then there exists a unique solution x of Eq.(E_δ) on J such that $x_t \in W_\delta^c$ and $\Pi^c x_t = \Phi_c z(t)$ on J .*

- (ii) Let x be a solution of $\text{Eq.}(E)$ on an interval J such that $x_t \in W_{\text{loc}}^c(r, \delta)$ ($t \in J$). Then the function $z_c(t) := \langle \Psi_c, x_t \rangle$ satisfies the equation (CE) on J , together with the inequality $\sup_{t \in J} \|\Phi_c z_c(t)\|_X \leq r$.
Conversely, if $z(t)$ satisfies the equation (CE) on an interval J together with the inequality $\sup_{t \in J} \|\Phi_c z(t)\|_X \leq r$, then there exists a unique solution x of $\text{Eq.}(E)$ on J such that $x_t \in W_{\text{loc}}^c(r, \delta)$ and $\Pi^c x_t = \Phi_c z(t)$ on J .

Since $f(0) = f_\delta(0) = 0$, both equations (CE) and (CE_δ) (as well as (E) and (E_δ)) possess the zero solution. Notice that the zero solution of (CE) (resp. (E)) is uniformly asymptotically stable if and only if the zero solution of (CE_δ) (resp. (E_δ)) is uniformly asymptotically stable. Likewise, the zero solution of (CE) (resp. (E)) is unstable if and only if the zero solution of (CE_δ) (resp. (E_δ)) is unstable. Here, for the definition of several stability properties utilized in this paper, we refer readers to the books [10, 5].

Now suppose that $\Sigma^u = \emptyset$. Then the dynamics near the zero solution of (E) is determined by the dynamics near $z_c = 0$ of (CE) in the following sense.

Theorem 3. *Assume that $\Sigma^u = \emptyset$. If the zero solution of (CE) is uniformly asymptotically stable (resp. unstable), then the zero solution of (E) is also uniformly asymptotically stable (resp. unstable).*

Proof. By the fact stated in the preceding paragraph of the theorem, it is sufficient to establish that the uniform asymptotic stability (resp. instability) of the zero solution of (CE_δ) implies the uniform asymptotic stability (resp. instability) of the zero solution of (E_δ) .

If the zero solution of (CE_δ) is unstable, the instability of the zero solution of (E_δ) immediately follows from the invariance of W_δ^c (Proposition 4 (iii)). In what follows, under the assumption that the zero solution of (CE_δ) is uniformly asymptotically stable, we will establish the uniform asymptotic stability of the zero solution of (E_δ) . By virtue of [5, Theorem 4.2.1], there exist positive constants a , $\bar{K}$ and a Liapunov function V defined on $S_a := \{y \in \mathbb{C}^{d_c} : |y| \leq a\}$ satisfying the following properties:

- (i) There exists a $b \in C(\mathbb{R}^+; \mathbb{R}^+)$ which is strictly increasing with $b(0) = 0$ and

$$b(|y|) \leq V(y) \leq |y| \quad \text{for } y \in S_a.$$

- (ii) $|V(y) - V(z)| \leq \bar{K}|y - z|$ for $y, z \in S_a$.

- (iii) $\dot{V}(z) \leq -V(z)$ for $z \in S_a$, where $\dot{V}(z) := \limsup_{h \rightarrow +0} (1/h)\{V(y(h)) - V(z)\}$, and $y(h)$ is the solution of (CE_δ) with $y(0) = z$.

Choose a positive number τ_0 such that

$$e^{-\tau_0} \leq \frac{1}{2} \quad \text{and} \quad Ce^{-\beta_0 \tau_0} \leq \frac{1}{4}, \quad (22)$$

where β_0 is the one in Proposition 7, and we may assume that $\beta_0 > \mu'$, taking δ so small if necessary. Put $K_\infty := \|K\|_{\infty, \rho}$ and take a positive number P in such a way that

$$P > \max \left(1, \frac{4C}{\beta_0 - \mu'} \bar{K} K K_\infty \|\Psi_c\| \right), \quad (23)$$

and set $a_0 := ae^{-\eta \tau_0} / (4CK_\infty \|\Psi_c\|)$, where $\|\Psi_c\| := (\sum_{j=1}^{d_c} \|\psi_j\|_{X^\#}^2)^{1/2}$. Let Ω be a neighborhood of 0 in X such that

$$\langle \Psi_c, \phi \rangle \in S_a, \quad \|\Pi^c \phi\|_X \leq a_0, \quad \text{and} \quad Q \leq b(a)$$

for $\phi \in \Omega$, where

$$Q := V(\langle \Psi_c, \phi \rangle) + \left(PC + \frac{\bar{K} K_\infty \|\Psi_c\| KC}{\beta_0 - \mu'} \right) (\|\Pi^s \phi\|_X + \|F_{*, \delta}(\Pi^c \phi)\|_X),$$

and consider the function $W(\phi)$ on Ω defined by

$$W(\phi) := V(\langle \Psi_c, \phi \rangle) + P \|\Pi^s \phi - F_{*, \delta}(\Pi^c \phi)\|_X, \quad \phi \in \Omega.$$

W is continuous in Ω with $W(0) = 0$ and is positive in $\Omega \setminus \{0\}$ because of (i) and (ii).

We will first certify the following claim.

Claim 1. *There exists a positive number c_0 such that, for any $t_0 \in \mathbb{R}^+$ and $\phi \in X$ with $W(\phi) \leq c_0$, the solution $x(t; t_0, \phi, f_\delta)$ exists on $[t_0, t_0 + \tau_0]$ and satisfies $x_t(t_0, \phi, f_\delta) \in \Omega$ for $t \in [t_0, t_0 + \tau_0]$; in particular, $\|\Pi^c x_t(t_0, \phi, f_\delta)\|_X \leq a_0$ in this interval.*

Indeed, suppose that $x_t(t_0, \phi, f_\delta)$ is defined on the interval $[t_0, t_0 + t_*)$ with $t_* \leq \tau_0$. Applying VCF, we get

$$\|x_t(t_0, \phi, f_\delta)\|_X \leq M \|\phi\|_X + \int_{t_0}^t M \zeta_*(\delta) \|x_s(t_0, \phi, f_\delta)\|_X ds$$

for $t \in [t_0, t_0 + t_*)$, where $M := \sup_{0 \leq t \leq \tau_0} \|T(t)\|_{\mathcal{L}(X)}$. Then Gronwall's inequality yields that $\|x_t(t_0, \phi, f_\delta)\|_X \leq M \|\phi\|_X e^{M \zeta_*(\delta)(t-t_0)} \leq M \|\phi\|_X e^{M \zeta_*(\delta) \tau_0}$ for $t \in [t_0, t_0 + t_*)$; which means that $x_t(t_0, \phi, f_\delta)$ can be defined on the interval $[t_0, t_0 + t_*]$ and therefore on $[t_0, t_0 + \tau_0]$ (cf. [6, Corollary 1]). Thus it turns out that if $\|\phi\|_X$ is small enough, $x_t(t_0, \phi, f_\delta)$ exists on $[t_0, t_0 + \tau_0]$ and moreover belongs to Ω in this interval. The claim readily follows from the fact that $\inf\{W(\phi) : \phi \in \Omega, \|\phi\|_X \geq r\} > 0$ for small $r > 0$, together with the property of Ω .

Now given $t_0 \in \mathbb{R}^+$ and $\phi \in X$ with $W(\phi) \leq c_0$, let us consider the solution $x(t) := x(t; t_0, \phi, f_\delta)$. By Proposition 3 (i)

$$\|\Lambda_{*,\delta}(\Pi^c x_t)(s)\|_X \leq \|\Lambda_{*,\delta}(\Pi^c x_t)\|_{Y_\eta} e^{\eta|s|} \leq e^{\eta|s|} 2C \|\Pi^c x_t\|_X, \quad s \in \mathbb{R};$$

hence taking account of $\Lambda_{*,\delta}(\Pi^c x_t)(s) = x_{t+s}(t, \hat{x}_t, f_\delta)$ for $s \in \mathbb{R}$ (Proposition 4 (ii)), we get $\|x_{t+s}(t, \hat{x}_t, f_\delta)\|_X \leq e^{\eta_0} 2C \|\Pi^c x_t\|_X$ for $s \in [-\tau_0, 0]$, where $\hat{x}_t := \Pi^c x_t + F_{*,\delta}(\Pi^c x_t)$. Set $y^\circ(t+s; t) := \langle \Psi_c, x_{t+s}(t, \hat{x}_t, f_\delta) \rangle$. Then $|y^\circ(t+s; t)| \leq K_\infty \|\Psi_c\| \|x_{t+s}(t, \hat{x}_t, f_\delta)\|_X \leq 2CK_\infty \|\Psi_c\| e^{\eta_0} \|\Pi^c x_t\|_X \leq 2CK_\infty \|\Psi_c\| e^{\eta_0} a_0 = a/2$ for $s \in [-\tau_0, 0]$; hence $y^\circ(s; t) \in S_{a/2}$ and thus $V(y^\circ(s; t))$ is well-defined for $s \in [t_0, t]$ with $t \in [t_0, t_0 + \tau_0]$.

We next confirm:

Claim 2. $\sup\{W(x_t) : t \in [t_0, t_0 + \tau_0]\} \leq Q$ and $W(x_{t_0+\tau_0}(t_0, \phi, f_\delta)) \leq c_0/2$.

Indeed, fix a $t \in [t_0, t_0 + \tau_0]$ and set $z(s) := y^\circ(s; t)$ for $s \in [t_0, t]$. Since $y^\circ(s; t) = \langle \Psi_c, x_s(t, \hat{x}_t, f_\delta) \rangle = \langle \Psi_c, \Pi^c x_s(t, \hat{x}_t, f_\delta) \rangle$ for $s \in [t_0, t]$, $z(s)$ is a solution of (CE_δ) on $[t_0, t]$ with $z(t) = y^\circ(t; t) = \langle \Psi_c, \Pi^c x_t \rangle$. By the property (i), we see that $\dot{V}(z(s)) \leq -V(z(s))$ for $s \in [t_0, t]$, which implies that $(d/ds)(e^{s-t}V(z(s))) = e^{s-t}(V(z(s)) + \dot{V}(z(s))) \leq 0$, so that

$$V(\langle \Psi_c, \Pi^c x_t \rangle) - e^{t_0-t}V(y^\circ(t_0; t)) = V(z(t)) - e^{t_0-t}V(z(t_0)) \leq \int_{t_0}^t \frac{d}{ds}(e^{s-t}V(z(s)))ds \leq 0;$$

consequently,

$$\begin{aligned} V(\langle \Psi_c, \Pi^c x_t \rangle) &\leq e^{t_0-t}V(\langle \Psi_c, \Pi^c x_{t_0} \rangle) + e^{t_0-t}(V(y^\circ(t_0; t)) - V(\langle \Psi_c, \Pi^c x_{t_0} \rangle)) \\ &\leq e^{t_0-t}V(\langle \Psi_c, \Pi^c x_{t_0} \rangle) + e^{t_0-t}\bar{K}|y^\circ(t_0; t) - \langle \Psi_c, \Pi^c x_{t_0} \rangle| \\ &\leq e^{t_0-t}V(\langle \Psi_c, \Pi^c \phi \rangle) + e^{t_0-t}\bar{K}K_\infty \|\Psi_c\| \|\Pi^c x_{t_0}(t, \hat{x}_t, f_\delta) - \Pi^c x_{t_0}\|_X \\ &\leq e^{t_0-t}V(\langle \Psi_c, \Pi^c \phi \rangle) + e^{t_0-t}\bar{K}K_\infty \|\Psi_c\| K \int_{t_0}^t e^{\mu'(\theta-t_0)} \|\xi(\theta)\|_X d\theta, \end{aligned}$$

where the last inequality is due to Proposition 5 (ii). Therefore, applying Proposition 7,

$$\begin{aligned} W(x_t) &= V(\langle \Psi_c, \Pi^c x_t \rangle) + P\|\xi(t)\|_X \\ &\leq e^{t_0-t}V(\langle \Psi_c, \Pi^c \phi \rangle) + e^{t_0-t}\bar{K}K_\infty \|\Psi_c\| K \int_{t_0}^t e^{\mu'(\theta-t_0)} (C\|\xi(t_0)\|_X e^{-\beta_0(\theta-t_0)}) d\theta \\ &\quad + PC\|\xi(t_0)\|_X e^{-\beta_0(t-t_0)} \\ &\leq e^{t_0-t}V(\langle \Psi_c, \Pi^c \phi \rangle) + C\|\xi(t_0)\|_X \left(\frac{\bar{K}K_\infty K \|\Psi_c\|}{\beta_0 - \mu'} e^{t_0-t} + P e^{-\beta_0(t-t_0)} \right). \end{aligned} \quad (24)$$

In particular,

$$\begin{aligned} W(x_{t_0+\tau_0}) &\leq e^{-\tau_0}V(\langle \Psi_c, \Pi^c \phi \rangle) + C\|\xi(t_0)\|_X \left(\frac{\bar{K}K_\infty K \|\Psi_c\|}{\beta_0 - \mu'} e^{-\tau_0} + P e^{-\beta_0\tau_0} \right) \\ &\leq (1/2)V(\langle \Psi_c, \Pi^c \phi \rangle) + (1/2)P\|\xi(t_0)\|_X \\ &= (1/2)W(x_{t_0}) = (1/2)W(\phi) \leq (1/2)c_0. \end{aligned}$$

Since $\|\xi(t_0)\|_X \leq \|\Pi^s \phi\|_X + \|F_{*,\delta}(\Pi^c \phi)\|_X$, (24) implies also

$$\sup\{W(x_t) : t \in [t_0, t_0 + \tau_0]\} \leq V(\langle \Psi_c, \Pi^c \phi \rangle) + C\|\xi(t_0)\|_X \left(\frac{\bar{K}K_\infty K \|\Psi_c\|}{\beta_0 - \mu'} + P \right) \leq Q,$$

as required.

By Claim 2, combined with Claim 1, $x(t) = x(t; t_0, \phi, f_\delta)$ is defined on $[t_0, t_0 + 2\tau_0]$, and $y^\circ(s; t) \in S_{a/2}$ still holds for $s \in [t_0, t]$ with $t \in [t_0, t_0 + 2\tau_0]$. More generally, one can deduce that $x(t) = x(t; t_0, \phi, f_\delta)$ is defined on $[t_0, t_0 + n\tau_0]$, and $y^\circ(s; t) \in S_{a/2}$ holds for $s \in [t_0, t]$ with $t \in [t_0, t_0 + n\tau_0]$ for any $n \in \mathbb{N}$, together with the relations

$$\sup\{W(x_t) : t \in [t_0 + (n-1)\tau_0, t_0 + n\tau_0]\} \leq \frac{Q}{2^{n-1}} \quad \text{and} \quad W(x_{t_0+n\tau_0}) \leq \frac{c_0}{2^n}$$

for $n \in \mathbb{N}$. This means that $x(t) = x(t; t_0, \phi, f_\delta)$ is actually defined on $[t_0, \infty)$ and that

$$V(\langle \Psi_c, x_t(t_0, \phi, f_\delta) \rangle) + P\|\Pi^s x_t - F_{*,\delta}(\Pi^c x_t)\|_X \leq Q 2^{-(t-t_0)/\tau_0}, \quad t \in [t_0, \infty).$$

In view of (i) and $P > 1$, it follows that $b(|\langle \Psi_c, x_t(t_0, \phi, f_\delta) \rangle|) \leq Q 2^{-(t-t_0)/\tau_0} \leq b(a)$ and $\|\Pi^s x_t - F_{*,\delta}(\Pi^c x_t)\|_X \leq Q 2^{-(t-t_0)/\tau_0}$. Since $\|\Pi^c x_t(t_0, \phi, f_\delta)\|_X = \|\Phi_c \langle \Psi_c, x_t(t_0, \phi, f_\delta) \rangle\|_X \leq \|\Phi_c\| b^{-1}(Q 2^{-(t-t_0)/\tau_0})$ with $\|\Phi_c\| := (\sum_{j=1}^{d_c} \|\phi_j\|_X^2)^{1/2}$ and $\|\Pi^s x_t(t_0, \phi, f_\delta)\|_X \leq \|\Pi^s x_t - F_{*,\delta}(\Pi^c x_t)\|_X + \|F_{*,\delta}(\Pi^c x_t)\|_X \leq Q 2^{-(t-t_0)/\tau_0} + L(\delta)\|\Pi^c x_t\|_X$, we obtain that for any $\phi \in \Omega$ and $t \in [t_0, \infty)$

$$\begin{aligned} \|x_t(t_0, \phi, f_\delta)\|_X &\leq \|\Pi^c x_t(t_0, \phi, f_\delta)\|_X + \|\Pi^s x_t(t_0, \phi, f_\delta)\|_X \\ &\leq Q 2^{-(t-t_0)/\tau_0} + (1 + L(\delta))\|\Phi_c\| b^{-1}(Q 2^{-(t-t_0)/\tau_0}), \end{aligned}$$

which shows that the zero solution of (E_δ) is uniformly asymptotically stable. $\square$

Before concluding this section, we will provide an example to illustrate how our Theorem 3 is available for stability analysis of some concrete equations. Let us consider nonlinear (scalar) integral equation

$$x(t) = \int_{-\infty}^t P(t-s)x(s)ds + f(x_t), \quad (25)$$

where P is a nonnegative continuous function on $\mathbb{R}^+$ satisfying $\int_0^\infty P(t)dt = 1$ together with the condition $\|P\|_{1,\rho} := \int_0^\infty P(t)e^{\rho t}dt < \infty$ and $\|P\|_{\infty,\rho} := \text{ess sup}\{P(t)e^{\rho t} : t \geq 0\} < \infty$ for some positive constant ρ , and $f \in C^1(X; \mathbb{C})$, $X := L_\rho^1(\mathbb{R}^-; \mathbb{C})$, satisfies $f(0) = 0$ and $Df(0) = 0$. Eq. (25) is written as Eq. (E) with $m = 1$ and $K \equiv P$. The characteristic operator $\Delta(\lambda)$ of Eq. (25) is given by $\Delta(\lambda) = 1 - \int_0^\infty P(t)e^{-\lambda t}dt$. We thus get $\Sigma^u = \emptyset$ and $\Sigma^c = \{0\}$. Indeed, in this case, 0 is a simple root of the equation $\Delta(\lambda) = 0$, and E^c is 1-dimensional space with a basis $\{\phi_1\}$, $\phi_1 \equiv 1$, together with $\{\psi_1\}$, $\psi_1 \equiv 1/r$ (here

$r := \int_0^\infty \tau P(\tau) d\tau$, as the dual basis of $\{\phi_1\}$; see [7] for details. The projection Π^c is given by the formula $\Pi^c \phi = \Phi_c \langle \Psi_c, \phi \rangle$, $\forall \phi \in X$, and hence

$$\begin{aligned} \Pi^c \phi &= \phi_1 \langle \psi_1, \phi \rangle = \phi_1 \left(\int_{-\infty}^0 \int_{\theta}^0 \psi_1(\xi - \theta) P(-\theta) \phi(\xi) d\xi d\theta \right) \\ &= \Phi_c \left(\frac{1}{r} \int_{-\infty}^0 P(-\theta) \left(\int_{\theta}^0 \phi(\xi) d\xi \right) d\theta \right). \end{aligned}$$

Thus, for a solution $x(t)$ of Eq. (25), the component $z_c(t)$ of $\Pi^c x_t$ with respect to Φ_c is given by

$$z_c(t) = \frac{1}{r} \int_{-\infty}^t \hat{P}(t-s) x(s) ds$$

with $\hat{P}(t) := \int_t^\infty P(\tau) d\tau$, because of

$$\begin{aligned} r z_c(t) &= \int_{-\infty}^0 P(-\theta) \left(\int_{\theta}^0 x(t+\xi) d\xi \right) d\theta = \int_{-\infty}^0 P(-\theta) \left(\int_{t+\theta}^t x(s) ds \right) d\theta \\ &= \int_{-\infty}^t P(t-\tau) \left(\int_{\tau}^t x(s) ds \right) d\tau = \int_{-\infty}^t \left(\int_{t-s}^\infty P(w) dw \right) x(s) ds. \end{aligned}$$

Observe that $z_c(t)$ satisfies the ordinary equation

$$r \dot{z}_c(t) = \hat{P}(0) x(t) + \int_{-\infty}^t (-P(t-s)) x(s) ds = x(t) - \int_{-\infty}^t P(t-s) x(s) ds,$$

that is, $r \dot{z}_c(t) = f(x_t) = f(\Phi_c z_c(t) + \Pi^s x_t)$. In particular, if x is a solution of Eq. (25) satisfying $x_t \in W_{\text{loc}}^c(r, \delta)$ on an interval J , then $\Pi^s x_t = F_*(\Phi_c z_c(t))$ on J ; hence we get

$$\dot{z}_c(t) = (1/r) f(\Phi_c z_c(t) + F_*(\Phi_c z_c(t)))$$

on J . This observation leads to that $G_c = 0$ and $H_c = 1/r$ in the central equation (CE); in fact, by noticing that $\Sigma^c = \{0\}$ and $H_c x = \lim_{n \rightarrow \infty} \langle \psi_1, \Gamma^n x \rangle = (1/r)x$, $\forall x \in \mathbb{C}$, one can also certify this fact. Consequently, the central equation of Eq. (25) is identical with the scalar equation $\dot{z} = H(z)$; here

$$H(w) := (1/r) f(\Phi_c w + F_*(\Phi_c w)), \quad (w \in \mathbb{C} \text{ and } |w| \text{ is small}).$$

In what follows, we will determine the function H for some special functions f .

Let us assume that f is of the form

$$f(\phi) = \varepsilon \left(\int_{-\infty}^0 Q(-\theta) \phi(\theta) d\theta \right)^m + g(\phi), \quad \forall \phi \in X, \quad (26)$$

where m is a natural number such that $m \geq 2$, ε is a nonzero real number, Q is a function satisfying $\|Q\|_{1,\rho} < \infty$ and $\|Q\|_{\infty,\rho} < \infty$ and $c_0 := \int_{-\infty}^0 Q(-\theta) d\theta > 0$, and

$g \in C^1(X; \mathbb{C})$ satisfies $|g(\phi)| = o(\|\phi\|_X^m)$ as $\|\phi\|_X \rightarrow 0$ (here, o means Landau's notation "small oh"). One can easily see that the function f given by (26) satisfies $f \in C^1(X; \mathbb{C})$ and $f(0) = Df(0) = 0$. For any w with small $|w|$, we get

$$\begin{aligned} f(\Phi_c w) &= \varepsilon \left(\int_{-\infty}^0 Q(-\theta)(\Phi_c w)(\theta) d\theta \right)^m + g(\Phi_c w) \\ &= \varepsilon \left(w \int_{-\infty}^0 Q(-\theta) d\theta \right)^m + o(w^m) = \varepsilon(c_0 w)^m + o(w^m); \end{aligned}$$

hence,

$$\begin{aligned} rH(w) &= f(\Phi_c w + F_*(\phi_c w)) \\ &= f(\Phi_c w) + \{f(\Phi_c w + F_*(\phi_c w)) - f(\Phi_c w)\} \\ &= f(\Phi_c w) + \varepsilon \{[L_1(\Phi_c w + F_*(\phi_c w))]^m - [L_1(\Phi_c w)]^m\} + o(w^m) \\ &= \varepsilon(c_0 w)^m + o(w^m) + \varepsilon \sum_{k=0}^{m-1} \binom{m}{k} \{L_1(\Phi_c w)\}^k \{L_1(F_*(\Phi_c w))\}^{m-k}, \end{aligned}$$

here L_1 is a bounded linear functional on L_ρ^1 defined by $L_1(\phi) := \int_{-\infty}^0 Q(-\theta)\phi(\theta)d\theta$. Recall that $L_1(F_*(\Phi_c w)) = o(w)$ as $w \rightarrow 0$; hence

$$\sum_{k=0}^{m-1} \binom{m}{k} \{L_1(\Phi_c w)\}^k \{F_*(\Phi_c w)\}^{m-k} = o(w^m) \quad \text{as } w \rightarrow 0.$$

Thus $rH(w) = \varepsilon(c_0 w)^m + o(w^m)$ as $w \rightarrow 0$. Hence it follows that

$$H(w) = (\varepsilon/r)c_0^m w^m + o(w^m) \quad \text{as } w \rightarrow 0.$$

Consequently, one can easily see that the zero solution of the central equation of Eq. (25) is uniformly asymptotically stable if $\varepsilon < 0$ and if m is an odd natural number; while it is unstable if $\varepsilon > 0$ and if m is an odd natural number, or if $\varepsilon \neq 0$ and if m is an even natural number. Therefore, by virtue of Theorem 3, we get the following result:

Proposition 9. *Assume that*

$$f(\phi) = \varepsilon \left(\int_{-\infty}^0 Q(-\theta)\phi(\theta)d\theta \right)^m + g(\phi), \quad \forall \phi \in X, \quad (27)$$

here ε is a nonzero constant, m is a natural number such that $m \geq 2$, Q is a function satisfying $\|Q\|_{1,\rho} < \infty$, $\|Q\|_{\infty,\rho} < \infty$ and $\int_0^\infty Q(t)dt > 0$ and $g(\phi) = o(\|\phi\|_X^m)$ as $\|\phi\|_X \rightarrow 0$ with $g \in C^1(X; \mathbb{C})$. Then the following statements hold true;

- (i) if m is odd and $\varepsilon < 0$, then the zero solution of Eq. (25) is uniformly asymptotically stable (in L_ρ^1);
- (ii) if m is odd and $\varepsilon > 0$, then the zero solution of Eq. (25) is unstable (in L_ρ^1);
- (iii) if m is even and $\varepsilon \neq 0$, then the zero solution of Eq. (25) is unstable (in L_ρ^1).

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