

MINIMUM-TIME PROBLEM OF NONLINEAR CONTROL SYSTEM ON SEPARABLE REFLEXIVE BANACH SPACES

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1. INTRODUCTION

We consider the nonlinear control system N_0 given by

$$\begin{cases} \dot{x}(t) = A_0x(t) + Fx(t) + B_0u(t), & t \geq 0 \\ x(0) = x_0 \end{cases}$$

in the separable reflexive Banach space X_0 . Along with N_0 , the sequence $\{N_n\}, n = 1, 2, \dots$, of perturbed equations

$$\begin{cases} \dot{x}_n(t) = A_nx_n(t) + Fx_n(t) + B_nu_n(t), & t \geq 0 \\ x_n(0) = x_{0,n} \end{cases}$$

in the separable reflexive Banach spaces X_n is considered with the mild solution

$$x_n(t) = S_n(t)x_{0,n} + \int_0^t S_n(t-s)\{Fx_n(s) + B_nu_n(s)\}ds$$

for every $u_n(\cdot) \in Y_n$, where Y_n is a control space and $B_n \in \mathcal{L}(Y_n, X_n)$. The operator A_n and the nonlinear operator F are assumed to satisfy that A_n generates a strongly continuous semigroup of bounded linear operator $S_n(t), t \geq 0$, on X_n and $A_n + F$ is strongly dissipative. $u_n(\cdot)$ is a locally summable function.

Linear case ($F = 0$) of above systems in Hilbert space have been treated by Carija ([3]).

In this paper, we consider the case where $B_n = I_n$ (the identity operator in X_n) and we are to prove the existence of minimal time for the nonlinear system N_0 which steers initial value x_0 to the target x_1 and to give conditions for the convergence of the sequence of minimal times for the nonlinear approximate system N_n on $X_n, n = 1, 2, \dots$, to the minimal time for the original system N_0 on X_0 .

2. MINIMUM-TIME PROBLEM

We consider the nonlinear control systems

$$(1) \quad \dot{x}_n(t) = A_n x_n(t) + F x_n(t) + u_n(t) \quad t \geq 0$$

in the separable reflexive Banach spaces X_n , $n = 0, 1, 2, \dots$, with the mild solutions

$$(2) \quad x_n(t) = S_n(t)x_{0,n} + \int_0^t S_n(t-s)\{F x_n(s) + u_n(s)\}ds.$$

For each $n \geq 0$, the set U_{ad}^n of admissible controls is defined by

$$U_{ad}^n = \{\text{strongly measurable function } u_n(\cdot); u_n(t) \in Y_n, \\ \|u_n(t)\| \leq 1, \text{ a.e.}\}.$$

For $n \geq 0$, define

$$R_n(t) = \{(x_{0,n}, x_{1,n}) \in X_n \times X_n; x_n(0) = x_{0,n}, x_n(t) = x_{1,n}, \\ \text{for some } u_n \in U_{ad}^n\}$$

where $x_n(t)$ is given by (2). Define also

$$R_n = \cup_{t \geq 0} R_n(t)$$

and the minimal-time function $T_n : R_n \rightarrow R^1$,

$$T_n(x_{0,n}, x_{1,n}) = \inf\{t : (x_{0,n}, x_{1,n}) \in R_n(t)\}.$$

We now list the assumptions which will be in effect throughout this paper:

(A1) there exist $M > 0$ and $\omega \geq 0$, such that, for $n = 0, 1, 2, \dots$ and $t \geq 0$,

$$\|S_n(t)\| \leq M e^{\omega t}$$

where M and ω are independent of n ,

(A2) $S_n(t)$ is compact,

(A3) $S_n(t) \rightarrow S_0(t)$, uniformly for t in bounded intervals,

(A4) $S_n^*(t) \rightarrow S_0^*(t)$, uniformly for t in bounded intervals.

(F1) the nonlinear function F is Lipschitz continuous:

there exists a constant c , such that

$$\|F x_n - F y_n\| \leq c \|x_n - y_n\|, \quad x_n, y_n \in X_n,$$

(F2) F has a linear growth rate on X_n ; there exists a constant $k > 0$, such that

$$\|F x_n\| \leq k(1 + \|x_n\|).$$

THEOREM 1. *If $x_0 \in X_0$ and $x_1 \in D(A_0)$, such that*

$$(3) \quad \|(A_0 + F)x_1\| + \omega\|x_0 - x_1\| < 1 \quad \text{for } \omega > 0$$

holds, then there exists $u \in U_{ad}^0$ which steers x_0 to x_1 in a time T_0 satisfying

$$(4) \quad T_0 \leq \omega^{-1} \log \left\{ \frac{1 - \|(A_0 + F)x_1\|}{1 - \|(A_0 + F)x_1\| - \omega\|x_0 - x_1\|} \right\}$$

Proof. Consider the nonlinear equation

$$(5) \quad \begin{cases} \dot{x}(t) = A_0 x(t) + Fx(t) - \text{sign}(x(t) - x_1) \\ x(0) = x_0 \in D(A_0) \end{cases}$$

where

$$\begin{aligned} \text{sign}(y) &= y/\|y\|, \quad y \neq 0, \\ \text{sign}(0) &= \{z \in X : \|z\| \leq 1\}. \end{aligned}$$

Thus, multiplying equation (5) with $x(t) - x_1 \neq 0$, using the dissipative of $A_0 + F$, multiplying by $e^{-2\omega t}$ and then integrating 0 to t (see [4]). We have

$$\begin{aligned} & e^{-2\omega t} \|x(t) - x_1\|^2 \\ & \leq \|x_0 - x_1\|^2 - 2 \int_0^t e^{-2\omega s} (1 - \|(A_0 + F)x_1\|) \|x(s) - x_1\| ds. \end{aligned}$$

By Gronwall's inequality,

$$\begin{aligned} & e^{-\omega t} \|x(t) - x_1\| \\ & \leq \|x_0 - x_1\| - \int_0^t e^{-\omega s} (1 - \|(A_0 + F)x_1\|) ds \end{aligned}$$

and then,

$$\begin{aligned} & e^{-\omega t} \|x(t) - x_1\| \\ & \leq \|x_0 - x_1\| + \omega^{-1} (1 - \|(A_0 + F)x_1\|) e^{-\omega t} - \omega^{-1} (1 - \|(A_0 + F)x_1\|). \end{aligned}$$

Thus

$$\begin{aligned} & \|x(t) - x_1\| \\ & \leq e^{\omega t} \|x_0 - x_1\| - \omega^{-1} (1 - \|(A_0 + F)x_1\|) e^{\omega t} + \omega^{-1} (1 - \|(A_0 + F)x_1\|). \end{aligned}$$

Let $x(t) \rightarrow x_1$, then

$$e^{\omega T_0} \{ (1 - \|(A_0 + F)x_1\|) - \omega \|x_0 - x_1\| \} \leq 1 - \|(A_0 + F)x_1\|.$$

We also

$$e^{\omega T_0} \leq \frac{1 - \|(A_0 + F)x_1\|}{1 - \|(A_0 + F)x_1\| - \omega \|x_0 - x_1\|}.$$

Hence

$$T_0 \leq \omega^{-1} \log \left\{ \frac{1 - \|(A_0 + F)x_1\|}{1 - \|(A_0 + F)x_1\| - \omega \|x_0 - x_1\|} \right\}.$$

We will assume that a mild solution exists for every $u_n(\cdot) \in L^p_{Y_n}$ and clearly, because of (F1), is unique.

LEMMA 1. *Let conditions (A1)-(A4), (F1)-(F2) and*

$$(B1) \quad x_{0,n} \rightarrow x_0, \quad x_{1,n} \rightarrow x_1$$

be satisfied. If

$$(6) \quad (x_{0,n}, x_{1,n}) \in R_n(t_n)$$

$$(7) \quad t_n \rightarrow T, \quad u_n \rightarrow u \quad \text{as } n \rightarrow \infty$$

then $(x_0, x_1) \in R_0(T)$.

Proof. Condition (6) implies that there exists $u_n \in U^n_{ad}$ such that

$$x_{1,n} = S_n(t_n)x_{0,n} + \int_0^{t_n} S_n(t_n - s) \{ Fx_{1,n}(s) + u_n(s) \} ds$$

By (7), there exists T_0 such that

$$t_n \leq T_0, \quad n \geq 1.$$

For every $n \geq 1$ and every $t \in [0, T_0]$, we have

$$\begin{aligned} & \|x_{1,n} - x_1\| \\ &= \|S_n(t_n)x_{0,n} + \int_0^{t_n} S_n(t_n - s) \{ Fx_{1,n}(s) + u_n(s) \} ds \\ &\quad - S_0(T)x_0 - \int_0^T S_0(T - s) \{ Fx_1(s) + u(s) \} ds\| \\ &\leq \|S_n(t_n)x_{0,n} - S_0(T)x_0\| \\ &\quad + \left\| \int_0^{t_n} S_n(t_n - s)u_n(s)ds - \int_0^T S_0(T - s)u(s)ds \right\| \end{aligned}$$

$$\begin{aligned}
& + \left\| \int_0^{t_n} S_n(t_n - s) F x_{1,n}(s) ds - \int_0^T S_0(T - s) F x_1(s) ds \right\| \\
& = I + II + III.
\end{aligned}$$

First, it is not hard to show that

$$S_n(t_n)x_{0,n} \rightarrow S_0(T)x_0.$$

$$\begin{aligned}
II & \leq \left\| \int_T^{t_n} S_n(t_n - s) u_n(s) ds \right\| \\
& + \left\| \int_0^T (S_n(t_n - s) u_n(s) - S_0(T - s) u_n(s)) ds \right\| \\
& + \left\| \int_0^T (S_0(T - s) u_n(s) - S_0(T - s) u(s)) ds \right\|.
\end{aligned}$$

The first term converges to zero by (A1). The second term converges to zero by (A4). For the moment, let us concentrate on the third term. From the Hahn-Banach theorem, we know that we can find $x_n^* \in B_1^* =$ dual unit ball such that

$$\begin{aligned}
& \left| \left(\int_0^T S_0(T - s) (u_n(s) - u(s)) ds, x_n^* \right) \right| \\
& = \left\| \int_0^T (S_0(T - s) u_n(s) - S_0(T - s) u(s)) ds \right\| \\
\Rightarrow & \left| \int_0^T (u_n(s) - u(s), S_0^*(T - s) x_n^*) ds \right| \\
& = \left\| \int_0^T (S_0(T - s) u_n(s) - S_0(T - s) u(s)) ds \right\|.
\end{aligned}$$

From Schauder's theorem, we know that, for $T > s$, $S_0^*(T - s)$ is compact. By Alaoglu's theorem, we know that B_1^* is w-compact. So by passing to subsequence if necessary, we may assume that $x_n^* \rightarrow x^* \in B_1^*$. Hence, $S_0^*(T - s) x_n^* \rightarrow z^*(t)$. Since $u_n \rightharpoonup u$,

$$\begin{aligned}
& \left| \int_0^t (u_n(s) - u(s), S_0^*(T - s) x_n^*) ds \right| \rightarrow 0 \\
\Rightarrow & \left\| \int_0^t S_0(T - s) (u_n(s) - u(s)) ds \right\| \rightarrow 0,
\end{aligned}$$

as $n \rightarrow \infty$.

$$\begin{aligned}
III & \leq \left\| \int_T^{t_n} S_n(t_n - s) F x_{1,n}(s) ds \right\| \\
& + \left\| \int_0^T (S_n(t_n - s) F x_{1,n}(s) - S_n(t_n - s) F x_1(s)) ds \right\| \\
& + \left\| \int_0^T (S_n(t_n - s) F x_1(s) - S_0(T - s) F x_1(s)) ds \right\|.
\end{aligned}$$

First and third term converges to zero. Let

$$r_n(t) = I + II + [\text{First and third term of } III] \rightarrow 0,$$

as $n \rightarrow \infty$. We have

$$\begin{aligned} & \|x_{1,n}(t) - x(t)\| \\ & \leq r_n(t) + \left\| \int_0^T (S_n(t_n - s)F x_{1,n}(s) - S_n(t_n - s)F x_1(s)) ds \right\| \\ & \leq r_n(t) + MK \int_0^T e^{\omega(t_n - s)} \|x_{1,n}(s) - x_1(s)\| ds. \end{aligned}$$

Using Gronwall's inequality, we get that

$$\begin{aligned} & \|x_{1,n}(t) - x_1(t)\| \\ & \leq r_n(t) + MK \int_0^T r_n(s) e^{\omega(t_n - s)} \exp\left(\int_0^T e^{\omega(t_n - \tau)} d\tau\right) ds. \end{aligned}$$

But note that $\int_0^T e^{\omega(t_n - \tau)} d\tau \leq R$. So we have

$$\|x_{1,n}(t) - x_1(t)\| \leq r_n(t) + MK \exp(R) \int_0^T e^{\omega(t_n - s)} r_n(s) ds.$$

Recall that, for all $t \in [0, T]$, $r_n(t) \rightarrow 0$. So using the dominated convergence theorem, we get that $r_n(\cdot) \rightarrow 0$. Since

$$\begin{aligned} & \int_0^T e^{\omega(t_n - s)} r_n(s) ds \leq M' \|r_n\|, \\ & \lim_{n \rightarrow \infty} \int_0^T e^{\omega(t_n - s)} r_n(s) ds \rightarrow 0. \end{aligned}$$

Therefore

$$\|x_{1,n}(t) - x_1(t)\| \rightarrow 0$$

as $n \rightarrow \infty$. Hence $x_{1,n}(t) \rightarrow x_1(t)$, as $n \rightarrow \infty$, for all $t \in [0, T]$.

LEMMA 2. Assume (A1)-(A4), (B1),

$$(B2) \quad x_1 \in D(A_0), \quad \|(A_n + F)x_1\| < 1;$$

$$(B3) \quad x_{1,n} \in D(A_n), \quad A_n x_{1,n} \rightarrow A_0 x_1.$$

If $(x_0, x_1) \in R_0(t)$, then there exists a sequence $\{\gamma_n\}$, convergent to zero, such that

$$(8) \quad (x_{0,n}, x_{1,n}) \in R_n(t + \gamma_n)$$

for n sufficiently large.

Proof. First of all, we prove the following assertion: if $y_n \rightarrow x_1$, then there exists a sequence $\{\gamma_n\}$ convergent to zero such that

$$(9) \quad (y_n, x_{1,n}) \in R_n(\gamma_n)$$

for n sufficiently large. Indeed, since

$$\|(A_0 + F)x_1\| < 1,$$

there exists a positive integer n_1 such that, for $n \geq n_1$, we have

$$\|(A_n + F)x_{1,n}\| \leq c_1 < 1.$$

Furthermore, since $y_n \rightarrow x_1$, we may conclude that

$$\|(A_n + F)x_{1,n}\| + \omega \|y_n - x_{1,n}\| < 1, \quad n \geq n_1.$$

So, by Theorem 1, $x_{1,n}$ can be reached from y_n in a time T_n which satisfies

$$T_n \leq \omega^{-1} \log \left\{ \frac{1 - \|(A_n + F)x_{1,n}\|}{1 - \|(A_n + F)x_{1,n}\| - \omega \|y_n - x_{1,n}\|} \right\}.$$

Taking $\gamma_n = T_n$, we obtain (9) and $\gamma_n \rightarrow 0$ as $n \rightarrow \infty$ as claimed. Since $(x_0, x_1) \in R_0(t)$, there exists $u \in U_{ad}^0$ such that

$$(10) \quad x_1 = S_0(t)x_0 + \int_0^t S_0(t-s)\{Fx_1(s) + u(s)\}ds.$$

Denoting

$$(11) \quad y_n = S_n(t)x_{0,n} + \int_0^t S_n(t-s)\{Fy_n(s) + u_n(s)\}ds.$$

$$\begin{aligned}
& \|y_n(t) - x_1(t)\| \\
& \leq \|S_n(t)x_{0,n} - S_0(t)x_0\| \\
& \quad + \left\| \int_0^t S_n(t-s)(Fy_n(s) + u(s))ds - \int_0^t S_0(t-s)(Fx_1(s) + u(s))ds \right\| \\
& \leq \|S_n(t)x_{0,n} - S_0(t)x_0\| \\
& \quad + \int_0^t \|S_n(t-s) - S_0(t-s)\| \|u_n(s)\| ds \\
& \quad + \int_0^t \|S_0(t-s)(u_n(s) - u(s))\| ds \\
& \quad + \int_0^t \|S_n(t-s) - S_0(t-s)\| \|Fx_1(s)\| ds \\
& \quad + \int_0^t \|S_n(t-s)\| \|Fy_n(s) - Fx_1(s)\| ds \\
& = J1 + J2 + J3 + J4 + J5.
\end{aligned}$$

$J1, J2$ and $J4$ are converge to zero as $n \rightarrow \infty$. By same method of Lemma 2, $J3$ is converge to zero as $n \rightarrow \infty$. Let $k_n(t) = J1 + J2 + J3 + J4 \rightarrow 0$ as $n \rightarrow \infty$. We have

$$\begin{aligned}
& \|y_n(t) - x_1(t)\| \\
& \leq k_n(t) + \int_0^t \|S_n(t-s)\| \|Fy_n(s) - Fx_1(s)\| ds \\
& \leq k_n(t) + MK \int_0^t e^{\omega(t-s)} \|y_n(s) - x_1(s)\| ds.
\end{aligned}$$

Using Gronwall's inequality, we get that

$$\begin{aligned}
& \|y_n(t) - x_1(t)\| \\
& \leq k_n(t) + MK \int_0^t k_n(s) e^{\omega(t-s)} \exp\left(\int_0^t e^{\omega(t-\tau)} d\tau\right) ds.
\end{aligned}$$

But note that $\int_0^t e^{\omega(t-\tau)} d\tau \leq R'$. So, we have

$$\|y_n(t) - x_1(t)\| \leq k_n(t) + MK \exp(R') \int_0^t e^{\omega(t-s)} k_n(s) ds.$$

Recall that, for any $t > 0$, $k_n(t) \rightarrow 0$. So using the dominated convergence theorem, we get that $k_n(\cdot) \rightarrow 0$. Since

$$\int_0^t e^{\omega(t-s)} k_n(s) ds \leq M'' \|k_n\|,$$

where M'' is constant,

$$\lim_{n \rightarrow \infty} \int_0^t e^{\omega(t-s)} k_n(s) ds \rightarrow 0.$$

Therefore $\|y_n(t) - x_1(t)\| \rightarrow 0$ as $n \rightarrow \infty$. Hence $y_n \rightarrow x_1$, as $n \rightarrow \infty$, for any $t > 0$. Therefore, (9) holds for y_n defined by (11).

Finally, by (9) and (11), we obtain (8), thereby completing our proofs.

THEOREM 2. Under conditions (A1)-(A4), (F1)-(F2), assume that (B1)-(B3) and $(x_0, x_1) \in R_0$. Then, the following results hold:

(a) $(x_{0,n}, x_{1,n}) \in R_n$, for n sufficiently large.

(b) $\lim_{n \rightarrow \infty} T_n(x_{0,n}, x_{1,n}) = T_0(x_0, x_1)$.

Proof. By Lemma 2, there exists a subsequence of $\{T_n(x_{0,n}, x_{1,n})\}$, denoted by $\{T_{n'}\}$, which converges, say to T' . Using once again Lemma 2, with $t = T_0(x_0, x_1)$, we obtain

$$T_{n'} \leq T_0(x_0, x_1) + \gamma_{n'}.$$

Hence, we obtain

$$T' \leq T_0(x_0, x_1).$$

Finally, using Lemma 1, we may infer that

$$T_0(x_0, x_1) \leq T',$$

and thus we obtain

$$T_0(x_0, x_1) = T'.$$

Since the last equality can be obtained for all convergent subsequence, the proof is complete.

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