

転移点と特異点を持つ 2 階線形常微分方程式について

On asymptotics of a second order linear O.D.E with
a turning-regular singular point

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§1. Introduction.

1.1. The differential equation studied is

$$(1.1) \quad \varepsilon^2 \frac{d^2 y}{dx^2} - \left(x^m - \frac{\varepsilon}{x} \right) y = 0,$$

$$x, y \in \mathbb{C}; \quad 0 < |x| \leq x_0, \quad 0 < \varepsilon \leq \varepsilon_0; \quad m \in \mathbb{N},$$

where x_0 and ε_0 are constants. This differential equation has a turning point and a regular singular point, both of which are situated at the origin. We do not have a one-step-method to obtain an asymptotic approximation to the solution as $\varepsilon \rightarrow 0$ in the whole domain $D = \{x : 0 < |x| \leq x_0\}$, so we split (1.1) into two different types of the differential equation whose solutions are obtained separately (§2) and then we connect them by a so-called matching matrix in a common domain as shown in §4.

1.2. The differential equation (1.1) is represented in the matrix form:

$$(1.2) \quad \varepsilon \frac{dY}{dx} = \begin{bmatrix} 0 & 1 \\ x^m - \varepsilon/x & 0 \end{bmatrix} Y,$$

where Y is a 2-by-2 matrix. (1.2) has the first two terms of

$$(1.3) \quad \varepsilon \frac{dY}{dx} = \left\{ \begin{bmatrix} 0 & 1 \\ x^m & 0 \end{bmatrix} + \varepsilon \begin{bmatrix} 0 & 0 \\ -1/x & 0 \end{bmatrix} + O(\varepsilon^2) \right\} Y.$$

If $O(\varepsilon^2)$ is small for $x \in D$ and ε , then a solution of (1.3) is a regular perturbation of one of (1.2) with respect to a small ε . In this sense (1.2) is dominant to (1.3)

Our aim is to get two types of the formal solution of (1.1) and match them as $\varepsilon \rightarrow 0$. In order to do it, analyzing Stokes curve configuration is important (§3). The case of $m = 1$

has been studied in Nakano [5].

Remark: We do not show any proofs or illustrations as they would take many pages.

§2. The reduced equations.

2.1. The differential equation (1.2) is written in the form

$$(2.1) \quad x^{(m+1)/2}(x^{-m-1}\varepsilon)\frac{dZ}{dx} = \left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + (x^{-m-1}\varepsilon) \begin{bmatrix} 0 & 0 \\ -1 & -mx^{m/2}/2 \end{bmatrix} \right) Z,$$

where $Y := \text{diag}[1, x^{m/2}] Z$. This differential equation is called an *outer equation* of (1.2) and it should be analyzed when $x^{-m-1}\varepsilon \rightarrow 0$, that is, for x in a sub-domain $S := \{x : K\varepsilon^{1/(m+1)} \leq |x| \leq x_0\}$ ($K = \text{large constant}$) of the whole domain D . A solution of (2.1) is called an *outer solution* of (1.2).

Theorem 2.1. *The formal outer solution $\tilde{Y}_{out}$ of (1.2) is given by*

$$(2.2) \quad \tilde{Y}_{out} := x^{m/4} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} e^{\alpha} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

$$\alpha := \frac{2}{m+2} \frac{1}{\varepsilon} x^{(m+2)/2} + \frac{1}{m} \frac{1}{x^{m/2}},$$

or

$$(2.3) \quad \tilde{Y}_{out} := \begin{bmatrix} x^{-m/4} & 0 \\ 0 & x^{m/4} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{\alpha} & 0 \\ 0 & e^{-\alpha} \end{bmatrix},$$

which is the leading term of an asymptotic expansion of a true outer solution of (1.2), namely, there exists a true outer solution Y_{out} such that

$$(2.4) \quad Y_{out} \sim \tilde{Y}_{out} \quad (x^{-m-1}\varepsilon \rightarrow 0)$$

in an outer domain, i.e., in a sector

$$(2.5) \quad S_m := \left\{ x : K\varepsilon^{1/(m+1)} \leq |x| \leq x_0, \quad -\frac{\pi}{m+2} < \arg x < \frac{3\pi}{m+2} \right\}.$$

Notice that the arguments of x in the above sector S_m correspond to the arguments of the boundaries of a canonical domain C_m^∞ (cf. (2.10)). $\tilde{Y}_{out}$ is an outer *WKB approximation*

to the solution of (1.2) of a matrix form.

2.2. We reduce (1.2) to another form in the complement $C := \{x : 0 < |x| < K\varepsilon^{1/(m+1)}\}$ of the sub-domain S , i.e., $D = C \cup S$. Let $x := \varepsilon^{1/(m+1)}t$ (a *stretching transform*) and $Y := \text{diag} [1, \varepsilon^{m/2(m+1)}] U$, then (1.2) becomes a form such as

$$(2.6) \quad \varepsilon^{m/2(m+1)} \frac{dU}{dt} = \begin{bmatrix} 0 & 1 \\ p(t) & 0 \end{bmatrix} U \quad \left(p(t) := t^m - \frac{1}{t}\right),$$

which has a very similar form to (1.2) but lacks a term of ε and is called an *inner equation* of (1.2). The origin $t = 0$ is a regular singular point and zeros of $p(t)$ are turning points of (2.6), which are called *secondary turning points* of (1.2). A solution of (2.6) is called an *inner solution* of (1.2).

Theorem 2.2. *The formal inner solution $\tilde{Y}_{in}$ of (1.2) is given by*

$$(2.7) \quad \tilde{Y}_{in} := \begin{bmatrix} 1 & 0 \\ 0 & \varepsilon^{m/2(m+1)} \end{bmatrix} p^{1/4} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} e^{\beta} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

$$\beta := \frac{1}{\varepsilon^{m/2(m+1)}} \int^t \sqrt{p} dt,$$

or

$$(2.8) \quad \tilde{Y}_{in} := \begin{bmatrix} 1 & 0 \\ 0 & \varepsilon^{m/2(m+1)} \end{bmatrix} \begin{bmatrix} p^{-1/4} & 0 \\ 0 & p^{1/4} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{\beta} & 0 \\ 0 & e^{-\beta} \end{bmatrix},$$

which is the leading term of the asymptotic expansion of a true inner solution of (1.2), namely, there exists a true inner solution Y_{in} of (1.2) such that

$$(2.9) \quad Y_{in} \sim \tilde{Y}_{in} \quad \text{as} \quad \begin{cases} \varepsilon \rightarrow 0 \\ t \rightarrow \infty \end{cases}$$

in a canonical domain

$$(2.10) \quad C_m^\infty := \left\{ t : 0 < |t| < \infty, -\frac{\pi}{m+2} < \arg t < \frac{3\pi}{m+2} \text{ near } t = \infty \right\}.$$

$\tilde{Y}_{in}$ is an inner *WKB approximation* to the solution of (1.2) of a matrix form. The property (2.9) is called the *double asymptotic property* (Fedoryuk [2]).

§3. Stokes curves and the canonical domains.

3.1. A Stokes curve for (2.6) is, by definition, a set of points t 's given by

$$(3.1) \quad \{t : \Re \xi(a, t) = 0\},$$

where

$$(3.2) \quad \xi(a, t) := \int_a^t \sqrt{p} dt \quad (p(a) = 0).$$

An anti-Stokes curve of (2.6) is defined by an equation

$$(3.3) \quad \Im \xi(a, t) = 0 \quad (p(a) = 0).$$

These curves are particular level curves defined by $\Re \xi(a, t) = \text{const.}$ and $\Im \xi(a, t) = \text{const.}$, namely, they are the curves of level zero.

The global property of Stokes curve configuration for a general rational function $p(t)$ is well known in Evgrafov-Fedoryuk [1], Fedoryuk [2] and Nakano [6]-[7], and Fukuhara [3], Hukuhara [4] and Paris-Wood [8] for a local property of Stokes curves. The outline of the Stokes curve configuration for (2.6) is as follows:

Theorem 3.1. *The Stokes and anti-Stokes curves for (2.6) possess the following properties:*

(i) *The origin $t = 0$ is a regular singular point from which one Stokes curve and one anti-Stokes curve emerge.*

When $m = \text{odd}$, two lines $t < -1$, $0 < t < 1$ on the real axis are Stokes curves, and two lines $-1 < t < 0$, $1 < t$ are anti-Stokes curves.

When $m = \text{even}$, a line $0 < t < 1$ on the real axis is a Stokes curve and two lines $t < 0$, $1 < t$ on the real axis are anti-Stokes curves.

(ii) *The point at infinity $t = \infty$ is an irregular singular point and $m+3$ Stokes curves emerge from (or tend to) $t = \infty$ at angles $\pm \frac{\pi}{m+2}$, $\pm \frac{3\pi}{m+2}$, $\pm \frac{5\pi}{m+2}$, $\dots$.*

Also, $m+3$ anti-Stokes curves emerge from (or tend to) $t = \infty$ at middle angles between neighboring two Stokes curves.

(iii) *All the zero $t = e^{2k\pi i/(m+1)}$ ($k = 0, 1, 2, 3, \dots$) of $p(t)$ are situated on the unit circle $|t| = 1$ symmetrically with respect to the real axis and they are simple secondary*

turning points. From a turning point $t = e^{2k\pi i/(m+1)}$ three Stokes curves emerge at angles $\pm \frac{\pi}{3} + \frac{4k\pi}{3(m+1)}, \pi + \frac{4k\pi}{3(m+1)}$.

Three anti-Stokes curves emerge from every zero at middle angles between neighboring two Stokes curves.

(iv) There is a Stokes curve connecting $\alpha := e^{2k\pi i/(m+1)}$ and $\alpha^* := e^{2\pi i - 2k\pi i/(m+1)}$. This Stokes curve crosses the anti-Stokes curve $-1 < t < 0$ and can not cross lines $t < -1$ or $0 < t < 1$.

(v) There is an anti-Stokes curve connecting $\alpha := e^{2k\pi i/(m+1)}$ and $\bar{\alpha} := e^{-2k\pi i/(m+1)}$. This anti-Stokes curve crosses only the Stokes curve $0 < t < 1$.

(vi) Any Stokes curve (resp., any anti-Stokes curve) can not cross other Stokes curves (resp., anti-Stokes curves) except for at turning points or at $t = \infty$.

(vii) A Stokes curve and an anti-Stokes curve emerging from a turning point tend to another turning point or to $t = \infty$.

(viii) Any Stokes curve or any anti-Stokes curve can not cross itself.

(ix) When a point $t = \alpha$ is a turning point or a simple pole, there are no (sums of) Stokes or anti-Stokes curves homotopic to a circle surrounding α . Therefor there are no circle-like Stokes or anti-Stokes curves for (6.1).

3.2. A canonical domain on the t -plane (or the Riemann surface) is, by definition, a simply connected domain bounded by Stokes curves which is mapped by $\xi = \xi(a, t)$ onto the whole ξ -plane except several slits. Referring Theorem 3.1 we can get several canonical domains whose illustration is omitted here.

§4. A matching matrix.

Existence domains S_m and C_m^∞ of the outer and the inner solutions have a common part where two solutions relate linearly. This linear relation is represented by a so-called *matching matrix*. The matching matrix $M := [m_{ij}]$ between Y_{out} and Y_{in} is defined by the equality $Y_{out}M = Y_{in}$, i.e.,

$$(4.1) \quad \tilde{Y}_{out}M \sim \tilde{Y}_{in} \quad (\varepsilon \rightarrow 0).$$

Theorem 4.1. *The matching matrix defined by (4.1) is given by*

$$(4.2) \quad M \sim \varepsilon^{m/4(m+1)} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (\varepsilon \rightarrow 0).$$

§5. The main theorem.

Summing up the results so far, we can get

The main theorem. *The differential equation (1.1) (or (1.2)) possesses a formal outer solution (an outer WKB approximation) (2.2) (or (2.3)) which is an asymptotic expansion of the true outer solution in a sector (i.e., an outer domain) (2.5) as $x^{-m-1}\varepsilon \rightarrow 0$. The differential equation (1.1) possesses a formal inner solution (an inner WKB approximation) (2.7) (or (2.8)) which is an asymptotic expansion of the true inner solution in a canonical domain (i.e., an inner domain) as $\varepsilon \rightarrow 0$ or $t \rightarrow \infty$. The arguments of the outer domain's boundaries are $-\pi/(m+2)$ and $3\pi/(m+2)$, and those of the inner domain's boundaries are identical for a large t , and two domains have a common part in which the outer and the inner solutions are related by the matching matrix (4.2).*

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