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On Alexander polynomials of some reduced curves

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1 Introduction

Let C be a plane curve in $\mathbb{P}^2$. We are interested in several topological invariants. In this report, we consider *Alexander polynomials of some reduced curves* which is defined as the following. Let Q be a reduced quartic. Suppose that Q has at most A_1 singularities. Let $C_1, \ldots, C_n$ be smooth conics such that:

1. Each C_i is tangent to Q with intersection multiplicity 2 at 4 smooth points for any i.
2. For all pairs (i,j) $(i \neq j)$, C_i intersects transversely with C_j at all intersection points.
3. $C_i \cap C_j \cap C_k = \emptyset$ and $C_i \cap C_j \cap \mathcal{Q} = \emptyset$.

Let $B := Q + C_1 + \cdots + C_n$ be the reduced curve which consists of the above quartic and smooth conics and let $Q \cap C_i = \{P_{i1}, \ldots, P_{i4}\}$ be the tangent points of C_i and Q for $i = 1, \ldots, n$. Note that the configurations of singularities of B is

$$\Sigma(B) = \Sigma(Q) + \{2n(n-1)A_1, 4nA_3\}.$$

For example, in [7], Namba and Tsuchihashi considered the case $n = 2$ and Q is a union of two smooth conics which are intersects transversely each other.

In this report, we consider the case $n = 3$ and then we determine their Alexander polynomials.

2 Alexander polynomials

2.1 Definition of Alexander Polynomials

Let C be an affine curve of degree d. Suppose that the line at infinity L_∞ intersects transversely with C. Let $\phi : \pi_1(X) \to \mathbb{Z}$ be the composition of Hurewicz homomorphism and the summation homomorphism. Let t be a generator of $\mathbb{Z}$ and we put the Laurent polynomial ring $\Lambda := \mathbb{C}[t, t^{-1}]$. We consider an infinite cyclic covering $p : \tilde{X} \to X$ such that $p_*(\pi_1(\tilde{X})) = \ker \phi$. Then $H_1(\tilde{X}, \mathbb{C})$ has a structure of Λ-module. Thus we have

$$H_1(\tilde{X}, \mathbb{C}) = \Lambda/\lambda_1(t) \oplus \cdots \oplus \Lambda/\lambda_m(t)$$

where we can take $\lambda_i(t) \in \Lambda$ is a polynomial in t such that $\lambda_i(0) \neq 0$ for $i = 1, \dots, m$. The Alexander polynomial $\Delta_C(t)$ is defined by the product $\prod_{i=1}^m \lambda_i(t)$.

In this report, we use the *Loeser-Vaquié formula* ([10, 9]) for calculating Alexander polynomials. Hereafter we follow the notations and terminologies of [4, 9] for the Loeser-Vaquié formula.

2.2 Loeser-Vaquié formula

Let $[X, Y, Z]$ be homogenous coordinates of $\mathbb{P}^2$ and let us consider the affine space $\mathbb{C}^2 = \mathbb{P}^2 \setminus \{Z = 0\}$ with affine coordinates $(x, y) = (X/Z, X/Y)$. Let $f(x, y)$ be the defining polynomial of C. Let $\mathrm{Sing}(C)$ be the singular locus of C and let $P \in \mathrm{Sing}(C)$ be a singular point. Consider a resolution $\pi : \tilde{U} \to U$ of (C, P), and let $E_1, \dots, E_s$ be the exceptional divisors of π. Let (u, v) be a local coordinate system centered at P and k_i and m_i be respective order of zero of the canonical two form

$\pi^*(du \wedge dv)$ and $\pi^* f$ along the divisor E_i. *The adjunction ideal* $\mathcal{J}_{P,k,d}$ *of* $\mathcal{O}_P$ is defined by

$$\mathcal{J}_{P,k,d} = \left\{ \phi \in \mathcal{O}_P \mid (\pi^*\phi) \geq \sum_i (\lfloor km_i/d \rfloor - k_i)E_i \right\}, \quad k = 1, \ldots, d-1$$

where $\lfloor * \rfloor = \max\{n \in \mathbb{Z} \mid n \leq *\}$ and we call it the floor function.

Let $O(j)$ be the set of polynomials in x, y whose degree is less than or equal to j. We consider the canonical mapping $\sigma : \mathbb{C}[x, y] \to \bigoplus_{P \in \mathrm{Sing}(C)} \mathcal{O}_P$ and its restriction:

$$\sigma_k : O(k-3) \to \bigoplus_{P \in \mathrm{Sing}(C)} \mathcal{O}_P.$$

Put $V_k(P) := \mathcal{O}_P / \mathcal{J}_{P,k,d}$ and denote the composition of σ_k and the natural surjection $\bigoplus \mathcal{O}_P \to \bigoplus V_k(P)$ by $\bar{\sigma}_k$. Then the Alexander polynomial of C is given as follows:

Theorem 1. ([5, 6, 1, 3]) *The reduced Alexander polynomial* $\tilde{\Delta}_C(t)$ *is given by the product*

$$\tilde{\Delta}_C(t) = \prod_{k=1}^{d-1} \Delta_k(t)^{\ell_k}, \quad \ell_k := \dim \operatorname{coker} \bar{\sigma}_k \tag{1}$$

where

$$\Delta_k(t) = \left(t - \exp\left(\frac{2k\pi i}{d}\right)\right)\left(t - \exp\left(-\frac{2k\pi i}{d}\right)\right).$$

The Alexander polynomial $\Delta_C(t)$ *is given as*

$$\Delta_C(t) = (t-1)^{r-1}\tilde{\Delta}_C(t)$$

where r *is the number of irreducible components of* C.

2.3 The adjunction ideal for non-degenerate singularities

In general, the computation of the ideal $\mathcal{J}_{P,k,d}$ requires an explicit computation of the resolution of the singularity (C, P). However for the case of non-degenerate singularities, the ideal $\mathcal{J}_{P,k,d}$ can be obtained combinatorially by a toric modification. Let (u, v) be a local coordinate system centered at P such that (C, P) is defined by a function germ $f(u, v)$ and the Newton boundary $\Gamma(f; u, v)$ is non-degenerate. Let $R_1, \dots, R_s$ be the primitive weight vectors which correspond to the faces $\Delta_1, \dots, \Delta_s$ of $\Gamma(f; u, v)$. Let $\pi : \tilde{U} \to U$ be the canonical toric modification and let $\hat{E}(R_i)$ be the exceptional divisor corresponding to R_i. Recall that the order of zeros of the canonical two form $\pi^*(du \wedge dv)$ along the divisor $\hat{E}(R_i)$ is simply given by $|R_i| - 1$ where $|Q_i| = p + q$ for a weight vector $R_i = {}^t(p_i, q_i)$ (see [8]). For a function germ $g(u, v)$, let $m(g, R_i)$ be the multiplicity of the pull-back $(\pi^* g)$ on $\hat{E}(R_i)$. Then

Lemma 1 ([2, 9]). *A function germ $g \in \mathcal{O}_P$ is contained in the ideal $\mathcal{J}_{P,k,d}$ if and only if g satisfies following condition:*

$$m(g, R_i) \geq \left\lfloor \frac{k}{d} m(f, R_i) \right\rfloor - |R_i| + 1, \qquad i = 1, \dots, s.$$

The ideal $\mathcal{J}_{P,k,d}$ is generated by the monomials satisfying the above conditions.

Example 1. *Let C be a plane curve of degree $2n+4$ such that C has only A_1 and A_3 singularities. Let P be a singular point.*

(1) *Assume that P is an A_1 singularity. Then the adjunction ideal is*

$$\mathcal{J}_{P,k,2n+4} = \left\langle u^a v^b \mid a + b \geq \left\lfloor \frac{k}{n+2} \right\rfloor - 1 \right\rangle = \mathcal{O}_P,$$

for all $k = 3, \dots, 2n+3$. Hence A_1 singularity do not contribute to computations of Alexander polynomials because $V_P(k)$ is 0 for all k.

(2) *Assume that P is an A_3 singularity. Then the adjunction ideal is*

$$\begin{aligned} \mathcal{J}_{P,k,2n+4} &= \left\langle u^a v^b \mid a + 2b \geq \left\lfloor \frac{2k}{n+2} \right\rfloor - 2 \right\rangle \\ &= \begin{cases} \mathcal{O}_P, & 3 \leq k < \left\lceil \frac{3}{2}(n+2) \right\rceil, \\ m_P, & \left\lceil \frac{3}{2}(n+2) \right\rceil \leq k \leq 2n+3 \end{cases} \end{aligned}$$

where $\lceil * \rceil = \min\{n \in \mathbb{Z} \mid * \leq n\}$ and we call it the ceil function.

2.4 The Alexander polynomials of subconfigurations of reduced curves

Let B be a reduced curve on $\mathbb{P}^2$, $B = B_1 + \cdots + B_r$ be its irreducible decomposition. Let I be a non-empty subset of the index $J := \{1, \ldots, r\}$ and $B_I := \sum_{i \in I} B_i$. We define $\mathbf{Alex}(B)$ as follows:

$$\mathbf{Alex}(B) = (\Delta_{B_I}(t))_{I \in 2^J \setminus \{\emptyset\}}.$$

Clearly, $\mathbf{Alex}(B)$ is a topological invariant of $(\mathbb{P}^2, B)$. We also define $\widetilde{\mathbf{Alex}}(B)$ as the set of the reduced Alexander polynomials of subconfigurations of B. We consider subsets of $\widetilde{\mathbf{Alex}}(B)$:

$$\widetilde{\mathbf{Alex}}(B)_s := (\tilde{\Delta}_{B_I}(t))_{\sharp I = s}, \quad 1 \le s \le r.$$

We say that $\widetilde{\mathbf{Alex}}(B)_s$ is *trivial* if any reduced Alexander polynomial in $\widetilde{\mathbf{Alex}}(B)_s$ is 1.

For our curves, $\widetilde{\mathbf{Alex}}(B)_1$ and $\widetilde{\mathbf{Alex}}(B)_2$ are trivial.

Now we consider $\widetilde{\mathbf{Alex}}(B)_s$ where $s \ge 3$. Let I be a non-empty subset of $\{1, \ldots, n+1\}$. We correspond $n+1$ to the quartic Q. If $n+1 \notin I$, then $\tilde{\Delta}_{B_I}(t) = 1$ as B_I has only A_1 singularities. Hence we consider the Alexander polynomial of B_I where I contains $n+1$.

To determine the Alexander polynomial of B_I, we consider the adjunction ideals and the map $\bar{\sigma}_k : O(k-3) \to V(k)$. The adjunction ideals for each singular point are computed in Example 1. Now we consider the multiplicity ℓ_k in the formula (1) of Loeser-Vaquié which is given as

$$\ell_k = \dim \operatorname{coker} \bar{\sigma}_k = \rho(k) + \dim \ker \bar{\sigma}_k.$$

where $\rho(k) = \sum_{P \in \mathrm{Sing}(B_I)} \dim V_k(P) - \dim O(k-3)$. For fixed k, the integer $\rho(k)$ is determined by only the adjunction ideal. Hence we should consider the dimension of $\ker \bar{\sigma}_k$.

By Lemma 1, if $k < \lceil \frac{3}{2}(n+2) \rceil$, then $V(k) = 0$. That is $\ell_k = 0$. For other cases, $V(k) = \mathbb{C}^{4n}$ and $g \in \ker \bar{\sigma}_k$ if and only if $\{g = 0\}$ passes through all A_3 singular point P. Hence we investigate the linear series $\mathcal{N}_{k-3}(\mathcal{P})$. In general, the dimension of $\mathcal{N}_{k-3}(\mathcal{P})$ is greater than or equal to $N := \frac{(k-2)(k-1)}{2} - 4n$. Note that if $\dim \mathcal{N}_{k-3}(\mathcal{P}) = N$, then $\ell_k = 0$.

Lemma 2. *If $k = 2n+3$, then* $\dim \mathcal{N}_{2n}(\mathcal{P}) = N$.

Proof. Assume $\dim \mathcal{N}_{2n}(\mathcal{P}) \geq N+1$. We take $4(n-r)-3$ distinct points $\mathcal{Q}_r$ on $C_{n-r} \setminus \mathcal{P}_{n-r}$ for $r = 0, \ldots, n-1$. Put $\mathcal{Q} := \mathcal{Q}_0 \cup \cdots \cup \mathcal{Q}_{n-1} \cup \{R\}$ where $R \notin C_1$. Then $\sharp\mathcal{Q} = N$ and $\dim \mathcal{N}_{2n}(\mathcal{P}, \mathcal{Q}) \geq \mathcal{N}_{2n}(\mathcal{P}) - N = 1$. Hence we can take a non-zero element $D \in \mathcal{N}_{2n}(\mathcal{P}, \mathcal{Q})$. As $\mathcal{Q}_0 \subset C_n \setminus \mathcal{P}_n$, we have $I(D, C_n) \geq 4 + 4n - 3 = 2 \cdot 2n + 1$. Hence $D \in C_n \mathcal{N}_{2n-2}(\mathcal{P}', \mathcal{Q}')$ where $\mathcal{P}' = \mathcal{P} \setminus \mathcal{P}_n$ and $\mathcal{Q}' = \mathcal{Q} \setminus \mathcal{Q}_0$. By the same argument for $r = 1, \ldots, n-2$, then D is contained in $C_n C_{n-1} \cdots C_2 \mathcal{N}_2(\mathcal{P}_1, \mathcal{Q}_{n-1}, R)$. But $\mathcal{N}_2(\mathcal{P}_1, \mathcal{Q}_{n-1}, R) = \{0\}$ because $R \notin C_1$. This is a contradiction. $\square$

Lemma 3. *If* $n \geq 3$ *and* $k = 2n+2$, *then* $\dim \mathcal{N}_{2n-1}(\mathcal{P}) = N$.

Proof. We assume $\dim \mathcal{N}_{2n-1}(\mathcal{P}) \geq N + 1$. We divide our considerations into two cases $\dim \mathcal{N}_2(\mathcal{P}_{ij}) = 0$ for all (i,j) or $\dim \mathcal{N}_2(\mathcal{P}_{ij}) \geq 1$ for some (i,j). The first case is proved by the same argument of Lemma 2.

Now we consider the second case. We may assume that $(i,j) = (1,2)$ and we take a non-zero conic $D_2 \in \mathcal{N}_2(\mathcal{P}_{12})$. We take $4n-9$ distinct points $\mathcal{Q}_0$ on $D_2 \setminus \mathcal{P}_{12}$ and $4(n-r)-9$ distinct points $\mathcal{Q}_r$ on $C_{n-r+1} \setminus \mathcal{P}_{n-r+1}$ for $r = 1, \ldots, n-3$. Put $\mathcal{Q} = \mathcal{Q}_0 \cup \cdots \cup \mathcal{Q}_{n-3} \cup \{R_1, R_2, R_3\}$ where R_1, R_2 and R_3 are not collinear. Then $\sharp\mathcal{Q} = N$ and $\dim \mathcal{N}_{2n-1}(\mathcal{P}, \mathcal{Q}) \geq \mathcal{N}_{2n-1}(\mathcal{P}) - N = 1$. Hence we can take a non-zero element $D \in \mathcal{N}_{2n-1}(\mathcal{P}, \mathcal{Q})$. By the same argument, D is in $D_2 C_n \cdots C_3 \mathcal{N}_1(R_1, R_2, R_3)$. But $\mathcal{N}_1(R_1, R_2, R_3) = \{0\}$ because R_1, R_2 and R_3 are not collinear.

$\square$

Corollary 1. $\widetilde{\mathrm{Alex}}(C)_4$ *is trivial.*

参考文献

[1] E. Artal Bartolo. Sur les couples des Zariski. *J. Algebraic Geometry*, 3:223–247, 1994.

[2] A. I. Degtyarev. Alexander polynomial of a curve of degree six. *J. Knot Theory Ramifications*, 3:439–454, 1994.

[3] H. Esnault. Fibre de Milnor d'un cône sur une courbe plane singulière. *Invent. Math.*, 68(3):477–496, 1982.

[4] M. Kawashima and M. Oka. On Alexander polynomials of certain $(2, 5)$ torus curves. *J. Math. Soc. Japan*, 62(1):213–238, 2010.

[5] A. Libgober. Alexander invariants of plane algebraic curves. In *Singularities, Part 2 (Arcata, Calif., 1981)*, volume 40 of *Proc. Sympos. Pure Math.*, pages 135–143. Amer. Math. Soc., Providence, RI, 1983.

[6] F. Loeser and M. Vaquié. Le polynôme d'Alexander d'une courbe plane projective. *Topology*, 29(2):163–173, 1990.

[7] M. Namba and H. Tsuchihashi. On the fundamental groups of Galois covering spaces of the projective plane. *Geom. Dedicata*, 104:97–117, 2004.

[8] M. Oka. *Non-degenerate complete intersection singularity.* Hermann, Paris, 1997.

[9] M. Oka. Alexander polynomial of sextics. *J. Knot Theory Ramifications*, 12(5):619–636, 2003.

[10] M. Oka. A survey on Alexander polynomials of plane curves. *Singularités Franco-Japonaise, Séminaire et congrès*, 10:209–232, 2005.