
学位申請論文

大山陽介

Self-duality and Integrable Systems

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§0. Introduction

In his lectures (1984-85) at Kyoto University, Professor M.Sato presented a program for generalizing the soliton theory ([9]; c.f. [10]). The Kadomtsev-Petviashvili (KP) equation is a typical example of the soliton theory. The KP equation is written in the form of deformation equations of a linear ordinary differential equation. The time evolutions of a solution are interpreted as dynamical motions on an infinite dimensional Grassmann manifold ([7],[9]). The Lie algebra of microdifferential operators of one variable acts on this manifold transitively. He conjectured that any integrable systems can be written in the form of deformation equations of a linear systems, and proposed to investigate a deformation of differential equations in higher dimensions. He showed a simple example of a deformation of holonomic systems in higher dimensions ([9]), and its generalization is treated in [4]. In this paper we study a deformation of $\mathcal{D}$ -modules in higher dimensions.

First we review the KP equation. We denote by $\mathcal{E}$ the ring of microdifferential operators of one variable x . We fix a microdifferential operator P , and denote by t_P a time variable with respect to P . We study the following evolution equation associated to P :

$$\frac{\partial W}{\partial t_P} + WP = (WPW^{-1})_+ W, \quad (0.1)$$

where $W = W(x, D_x) = 1 + \sum_{j < 0} w_j(x) D_x^j \in \mathcal{E}$. We denote by $\mathcal{W}$ the set of such operators W . This space $\mathcal{W}$ is a group by the composition of $\mathcal{E}$. We get the KP-hierarchy taking $P = D_x^n$ ($n = 1, 2, 3, \dots$) in (0.1). The equation (0.1) defines a dynamical motion on $\mathcal{W}$. This infinitesimal action of the Lie algebra $\mathcal{E}$ on $\mathcal{W}$ is transitive.

The purpose of this article is to give a foundation for higher dimensional generalization of the KP hierarchy. Let now $\mathcal{E}$ be the ring of microdifferential operators in several variables. Similarly to the one dimensional case, fixing an operator $P \in \mathcal{E}$, we shall study the following equation

$$\frac{\partial W}{\partial t_P} + WP = (WPW^{-1})_+ W, \quad (0.2)$$

where the operator W is a 0-th order microdifferential operator. Here we choose a decomposition $\mathcal{E} = \mathcal{D} \oplus \mathcal{E}_\phi$ and $(WPW^{-1})_+ \in \mathcal{D}$ is the component

of WPW^{-1} according to this decomposition. In general the equation (0.2) imposes some constraints on the initial value $W(t_P = 0)$, since the vector field defined by (0.2) is not tangent to the space $\mathcal{E}(0)$. There is no operator W_0 such that the equation (0.2) has a solution $W(t) \in \mathcal{E}$ with the initial value W_0 for any $P \in \mathcal{E}$. We take generators P in (0.2) only in the Lie subalgebra V of $\mathcal{E}$

$$V = \{F_0(x', D_0)x_0 + \sum_{0 \leq j < r} F_j(x', D_0)D_j + E(x', D_0); \\ F_k(x', D_0), E(x', D_0) \in \mathcal{E} \text{ for } 0 \leq k < r\},$$

where $x' = (x_1, x_2, \dots, x_{r-1})$. This Lie algebra contains the transformation groups both of the self-dual Yang-Mills equations and of the self-dual Einstein equations (see [7],[8]). In §2 we will determine the subspace $\mathcal{W}$ of $\mathcal{E}(0)$ so that the vector field defined by (0.2) for any $P \in V$ is tangent to $\mathcal{W}$. The space $\mathcal{W}$ is a subgroup in $\mathcal{E}$. The Lie algebra V acts on $\mathcal{W}$ transitively.

In the case of $r = 3$, our integrable system is nothing but a composed system of the self-dual Yang-Mills equations and the equations of self-dual metrics on Riemannian manifolds of dimension four. The Lie algebra V acts transitively on the space of self-dual connections on self-dual spaces. Thus we obtain a group-theoretical description of the twistor theory ([1],[5]).

Notations. We use the following notations: $\mathbb{Z}$ denotes the set of integers. $\mathbb{N}$ denotes the set of non-negative integers. We denote by $\mathbb{C}$ the complex number field. We denote by 1_n the unit matrix of size $n \times n$.

§1. Deformations of $\mathcal{D}$ -modules

Throughout this paper we shall work in the category of formal power series, $\mathcal{O} = \mathbb{C}[[x]] = \mathbb{C}[[x_0, x_1, \dots, x_{r-1}]]$ ($r \geq 2$). Let $\mathcal{D}$ be the ring of differential operators with coefficients in $\mathcal{O}$. Then every differential operator P of order m can be written as:

$$P = \sum_{\alpha \in \mathbb{N}^r, |\alpha| \leq m} a_\alpha(x) D_x^\alpha,$$

where $a_\alpha(x)$ are elements of $\mathcal{O}$, $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_{r-1}) \in \mathbb{N}^r$, $|\alpha| = \alpha_0 + \alpha_1 + \dots + \alpha_{r-1}$, $D_x^\alpha = D_0^{\alpha_0} D_1^{\alpha_1} \dots D_{r-1}^{\alpha_{r-1}}$ and $D_j = \frac{\partial}{\partial x_j}$ ($j = 0, 1, \dots, r-1$).

The ring $\mathcal{E}$ of formal microdifferential operators is a set of formal Laurent series in $D_0, D_1, \dots, D_{r-1}$ with only non-negative powers of $D_1, \dots, D_{r-1}$. The precise definition is as follows. We denote by $\mathcal{E}(m)$ the space of formal series :

$$P = \sum_{\alpha \in \mathbb{Z} \times \mathbb{N}^{r-1}, |\alpha| \leq m} a_\alpha(x) D_x^\alpha$$

where a_α 's are elements of $\mathcal{O}$, and the summation is taken through $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_{r-1})$, where $\alpha_0 \in \mathbb{Z}, \alpha_1 \in \mathbb{N}, \dots, \alpha_{r-1} \in \mathbb{N}$. We set

$$\mathcal{E} = \bigcup_{m \in \mathbb{Z}} \mathcal{E}(m).$$

We endow the $\mathcal{O}$ -module $\mathcal{E}$ with a structure of ring by extending the Leibniz formula. For two elements $P = \sum_\alpha a_\alpha D_x^\alpha$ and $Q = \sum_\beta b_\beta D_x^\beta$ of $\mathcal{E}$, we define the composition $P \circ Q$ by

$$P \circ Q = \sum_{\alpha, \beta \in \mathbb{Z} \times \mathbb{N}^{r-1}, \gamma \in \mathbb{N}^r} \binom{\alpha}{\gamma} a_\alpha b_\beta^{(\gamma)} D_x^{\alpha + \beta - \gamma},$$

where $b_\beta^{(\gamma)} = D_x^\gamma(b_\beta)$. The ring $\mathcal{E}$ has an increasing filtration by subspaces $\{\mathcal{E}(m)\}_{m \in \mathbb{Z}}$. We have

$$\mathcal{E}(m)\mathcal{E}(n) = \mathcal{E}(n + m).$$

For any $\mathcal{O}$ -submodule $\mathcal{L}$ of $\mathcal{E}$ we define the induced filtration $\{\mathcal{L}(m)\}_{m \in \mathbb{Z}}$ of $\mathcal{L}$ by $\mathcal{L}(m) = \mathcal{L} \cap \mathcal{E}(m)$.

Let $\mathcal{E}_\phi$ be the $\mathcal{O}$ -module consisting of the formal microdifferential operators of the form

$$\sum_{\alpha_0 < 0} a_\alpha(x) D_x^\alpha,$$

The ring $\mathcal{E}$ is the direct sum of $\mathcal{D}$ and $\mathcal{E}_\phi$. For any $P \in \mathcal{E}$, we define $P_+ \in \mathcal{D}$ and $P_- \in \mathcal{E}_\phi$ by the decomposition of $\mathcal{E}$:

$$\begin{aligned} \mathcal{E} &= \mathcal{D} \oplus \mathcal{E}_\phi \\ P &= P_+ + P_- \end{aligned}.$$

For any $\mathcal{O}$ -submodule $\mathcal{L}$ of $\mathcal{E}$ we define the $\mathcal{O}$ -module $\mathcal{L}_-$ by $\mathcal{L}_- = \mathcal{L} \cap \mathcal{E}_\phi$. Remark that $\mathcal{E}(0) = \mathcal{O} \oplus \mathcal{E}_\phi(0)$.

In the following we shall study a left $\mathcal{D}$ -submodule I of $\mathcal{E}$ which satisfies the following condition:

$$\mathcal{E}(m) = I(m) \oplus \mathcal{E}_\phi(m) \quad \text{for any } m \in \mathbb{Z}. \quad (1.1)$$

For example $I = \mathcal{D}$ satisfies (1.1). We make clear the structure of such a $\mathcal{D}$ -submodule I .

Lemma 1.1 Suppose that a $\mathcal{D}$ -submodule I of $\mathcal{E}$ satisfies the condition (1.1). Then I is generated as $\mathcal{D}$ -module by a unique operator W such that

$$W \in \mathcal{E}(0) \quad \text{and} \quad W_+ = 1. \quad (1.2)$$

Proof. The operator W is obtained by decomposing the identity operator $1 \in \mathcal{E}(0)$ into the sum of an operator in $I(0)$ and an operator in $\mathcal{E}_\phi(0)$ according to the condition (1.1):

$$\begin{aligned} \mathcal{E}(0) &= I(0) \oplus \mathcal{E}_\phi(0) \\ 1 &= W + U \end{aligned}$$

It is evident that W is contained in $\mathcal{E}(0)$ and that $W_+ = 1$. Since we have

$$\mathcal{E}(m)W = \mathcal{E}(m), \quad \mathcal{E}_\phi(m)W = \mathcal{E}_\phi(m)$$

for any $m \in \mathbb{Z}$, we obtain that

$$\mathcal{E}(m) = \mathcal{D}(m)W \oplus \mathcal{E}_\phi(m) \quad \text{for any } m \in \mathbb{Z}.$$

Thus the $\mathcal{D}$ -module $I' = \mathcal{D}W$ also satisfies (1.1). Because I contains I' and both satisfy (1.1), I coincides with I' . The uniqueness is clear. $\square$

Remark that W in Lemma 1.1 is invertible by (1.2).

We investigate nonlinear evolution equations according to the program of M. Sato ([9],[10]). For any $P \in \mathcal{E}$ and any $\mathcal{D}$ -submodule I_0 of $\mathcal{E}$ we define the time evolution I_t of I_0 by the following differential equations:

$$\frac{\partial V(t)}{\partial t} + V(t)P \in I_t \quad \text{for any } V(t) \in I_t. \quad (1.3)$$

We call $P \in \mathcal{E}$ the generator of the evolution equation (1.3).

In general we cannot find any $\mathcal{D}$ -submodule I_t which solves (1.3). In this paper we shall study the case that we can find a solution I_t of (1.3) which is a $\mathcal{D}$ -module satisfying (1.1) for any t . Then I_t is generated by an operator $W(t) \in \mathcal{E}(0)$ by Lemma 1.1 and we can rewrite the equation (1.3) in terms of the generator $W(t)$.

Lemma 1.2 We fix an operator $P \in \mathcal{E}$. We assume that the solution I_t of the evolution equation (1.3) is a $\mathcal{D}$ -submodule which satisfies (1.1) for any t . Then the equation (1.3) reduces to the following equation

$$\frac{\partial W(t)}{\partial t} + W(t)P = (W(t)PW(t)^{-1})_+ W(t), \quad (1.4)$$

where the operator $W(t)$ is the generator of I_t in Lemma 1.1.

Proof. From the equation (1.3) there exists an operator $B(t) \in \mathcal{D}$ such that

$$\frac{\partial W(t)}{\partial t} + W(t)P = B(t)W(t).$$

Thus we have

$$B(t) = W(t)PW(t)^{-1} + \frac{\partial W(t)}{\partial t}W(t)^{-1}.$$

Since the operator $W(t)$ is contained in $1 + \mathcal{E}_\phi$, the operator $\frac{\partial W(t)}{\partial t}W(t)^{-1}$ is contained in $\mathcal{E}_\phi$. Thus we obtain that $B(t) = (W(t)PW(t)^{-1})_+$. $\square$

Remark. The equation (1.4) is rewritten as

$$\begin{aligned} \frac{\partial W}{\partial t} &= (WPW^{-1})_+W - WP \\ &= -(WPW^{-1})_-W. \end{aligned} \tag{1.5}$$

The evolution equation (1.4) is associated with an infinitesimal action ρ of $\mathcal{E}$ on the space $\mathcal{E}$. For $P \in \mathcal{E}$ the vector field $\rho(P)$ is given as follows.

$$\dot{W} \longrightarrow -(WPW^{-1})_-W \in T_W\mathcal{E},$$

where the tangent space $T_W\mathcal{E}$ is identified with $\mathcal{E}$ by the structure of vector space of $\mathcal{E}$.

Theorem 1.3 For any $P, Q \in \mathcal{E}$ we have

$$\rho([P, Q]) = -[\rho(P), \rho(Q)].$$

Proof. We denote by ϵ_1 and ϵ_2 the time parameters with respect to P and Q , respectively. We set $\tilde{P} = WPW^{-1}$ and $\tilde{Q} = WQW^{-1}$. We have

$$\begin{aligned} \exp(\epsilon_1\rho(P))W &\equiv (1 - \epsilon_1(\tilde{P})_-)W \mod \epsilon_1^2, \\ \exp(\epsilon_2\rho(Q))W &\equiv (1 - \epsilon_2(\tilde{Q})_-)W \mod \epsilon_2^2. \end{aligned}$$

Hence we have, modulo $\epsilon_1^2, \epsilon_2^2$

$$\begin{aligned} \exp(\epsilon_1 \rho(P))(1 - \epsilon_2(WQW^{-1})_-) \\ \equiv 1 - \epsilon_2((1 - \epsilon_1 \tilde{P}_-) \tilde{Q}(1 + \epsilon_1 \tilde{P}_-))_- \\ \equiv 1 - \epsilon_2 \tilde{Q}_- + \epsilon_1 \epsilon_2 [\tilde{P}_-, \tilde{Q}]_-. \end{aligned}$$

Thus we obtain, modulo $\epsilon_1^2, \epsilon_2^2$

$$\begin{aligned} \exp(\epsilon_1 \rho(P)) \exp(\epsilon_2 \rho(Q)) W \\ \equiv \exp(\epsilon_1 \rho(P))(1 - \epsilon_2(WQW^{-1})_-) W \\ \equiv (1 - \epsilon_2 \tilde{Q}_- + \epsilon_1 \epsilon_2 [\tilde{P}_-, \tilde{Q}]_-)(1 - \epsilon_1 \tilde{P}_-) W \\ \equiv (1 - \epsilon_1 \tilde{P}_- - \epsilon_2 \tilde{Q}_- + \epsilon_1 \epsilon_2 ([\tilde{P}_-, \tilde{Q}]_- + \tilde{Q}_- \tilde{P}_-)) W. \end{aligned}$$

Similarly we have, modulo $\epsilon_1^2, \epsilon_2^2$

$$\begin{aligned} \exp(\epsilon_2 \rho(Q)) \exp(\epsilon_1 \rho(P)) W \\ \equiv (1 - \epsilon_2 \tilde{Q}_- - \epsilon_1 \tilde{P}_- + \epsilon_1 \epsilon_2 ([\tilde{Q}_-, \tilde{P}]_- + \tilde{P}_- \tilde{Q}_-)) W. \end{aligned}$$

By the formula

$$\begin{aligned} \exp(\epsilon_1 \rho(P)) \exp(\epsilon_2 \rho(Q)) W - \exp(\epsilon_2 \rho(Q)) \exp(\epsilon_1 \rho(P)) W \\ \equiv \epsilon_1 \epsilon_2 [\rho(P), \rho(Q)] W \quad \text{mod } \epsilon_1^2, \epsilon_2^2, \end{aligned}$$

we obtain

$$\begin{aligned} [\rho(P), \rho(Q)] W &= \{[\tilde{P}_-, \tilde{Q}]_- + \tilde{Q}_- \tilde{P}_- - [\tilde{Q}_-, \tilde{P}]_- - \tilde{P}_- \tilde{Q}_-\} W \\ &= \{[\tilde{P}_-, \tilde{Q}_+ + \tilde{Q}_-]_- - [\tilde{Q}_-, \tilde{P}_+ + \tilde{P}_-]_- + [\tilde{Q}_-, \tilde{P}_-]\} W. \end{aligned} \tag{1.6}$$

Since $[\tilde{P}_+, \tilde{Q}_+]_- = 0$, the right hand side of (1.6) is equal to

$$[\tilde{P}, \tilde{Q}]_- W = -\rho([P, Q]) W. \quad \square$$

When $P, Q \in \mathcal{E}$ commute with each other, the following equations are compatible by Theorem 1.3:

$$\begin{aligned}\frac{\partial W}{\partial s} &= (WPW^{-1})_+ W - WP, \\ \frac{\partial W}{\partial t} &= (WQW^{-1})_+ W - WQ,\end{aligned}$$

where $W = W(s, t)$.

In the case $r = 1$, for any $P \in \mathcal{E}$ and for any $I_0 = \mathcal{D}W(0)$ the solution I_t of the equation (1.3) satisfies the condition (1.1) for any t . With the choice $P = D_0^n$ ($n = 1, 2, \dots$), we obtain the KP-hierarchy ([7],[8]):

$$\frac{\partial W(t)}{\partial t} + W(t)D_0^n = (W(t)D_0^n W(t)^{-1})_+ W(t).$$

In higher dimensional case, we must choose a nice pair of the generator P and $I_0 = \mathcal{D}W(0)$ in order that I_t satisfies the condition (1.1) for any t . We shall see the evolution equation (1.4) constrains the initial value $W(0)$ in the following example.

Example 1.1 We consider (1.4) in the case $r = 2$. We take D_0^2 as the generator of the equation (1.4). We write

$$W(t) = \sum_{i+j \leq 0, j \geq 0} w_{i,j} D_0^i D_1^j, \quad w_{0,0} \equiv 1.$$

The operator $W(t)D_0^2$ is decomposed into the sum

$$\begin{aligned}\mathcal{E} &= \mathcal{D}W \\ W(t)D_0^2 &= (W(t)D_0^2 W(t)^{-1})_+ W + U.\end{aligned}$$

Then we have

$$\begin{aligned}(W(t)D_0^2 W(t)^{-1})_+ &= (D_0^2 - 2\frac{\partial w_{-1,1}}{\partial x_0} D_1 - 2\frac{\partial w_{-1,0}}{\partial x_0}), \\ U &= \sum_{i+j \leq 1, i < 0} (-2\frac{\partial w_{i-1,j}}{\partial x_0} + 2\frac{\partial w_{-1,1}}{\partial x_0} w_{i,j-1} - \frac{\partial^2 w_{i,j}}{\partial x_0^2} \\ &\quad + 2\frac{\partial w_{-1,0}}{\partial x_0} w_{i,j} + 2\frac{\partial w_{-1,1}}{\partial x_0} \frac{\partial w_{i,j}}{\partial x_1}) D_0^i D_1^j.\end{aligned}$$

The equation (1.4) is equivalent to the following:

$$\begin{aligned} \frac{\partial w_{i,j}}{\partial t} + u_{i,j} &= 0 & \text{for } i + j \leq 0, \\ \frac{\partial w_{i-1,j}}{\partial x_0} - \frac{\partial w_{-1,1}}{\partial x_0} w_{i,j-1} &= 0, & \text{for } i + j = 1. \end{aligned}$$

The second equation constrains the initial data $W(0)$.

§2. Integrable systems in higher dimensions

In the example 1.1 we have considered the equation (1.3) for one generator $P = D_0^2$. In this section we will introduce a space V of generators, and determine the space of $\mathcal{D}$ -submodules I of $\mathcal{E}$ such that the condition (1.1) is preserved under the time evolution (1.3) for any $P \in V$.

First we review two known examples, the self-dual Yang-Mills equations and the self-dual Einstein equations. We can interpret both the equations as integrable systems of three variables (see [14]).

Example 2.1 Self-dual Yang-Mills equations (see [15],[11]).

The self-dual Yang-Mills equations are written in the following form

$$\begin{aligned} \frac{\partial A_1}{\partial x_2}(x_1, x_2, s, t) &= \frac{\partial A_2}{\partial x_1}(x_1, x_2, s, t), \\ \left[\frac{\partial}{\partial s} + A_1(x_1, x_2, s, t), \frac{\partial}{\partial t} + A_2(x_1, x_2, s, t) \right] &= 0 \end{aligned} \quad (2.1)$$

for gauge fields $A_1, A_2 \in \text{Mat}(n \times n)$ on four-dimensional manifolds.

The evolution equation (1.4) is generalized to the case that W and P have matrix coefficients. We introduce the space $\mathcal{W}_{YM}(n)$ and the Lie algebra $V_{YM}(n)$:

$$\begin{aligned} \mathcal{W}_{YM}(n) &= \{W(x_1, x_2, D_0) = \sum_{i \in \mathbb{N}} w_i(x_1, x_2) D_0^{-i}; \\ &\quad w_i \in \text{Mat}(n \times n, \mathbb{C}[[x_1, x_2]]), w_0 \equiv 1_n\}. \end{aligned}$$

$$V_{YM}(n) = \{F_1(x_1, x_2, D_0)D_1 + F_2(x_1, x_2, D_0)D_2 + E(x_1, x_2, D_0); \\ F_1, F_2 \in \text{Mat}(n \times n, \mathcal{D}), E \in \text{Mat}(n \times n, \mathcal{E})\}.$$

The evolution equation (1.4) for $P \in V_{YM}(n)$ with any initial value $W \in \mathcal{W}_{YM}(n)$ has a solution in $\mathcal{W}_{YM}(n)$. We consider the equation (1.4) for $P = D_0D_1, D_0D_2 \in V_{YM}(n)$:

$$\begin{aligned} \frac{\partial W}{\partial s} + WD_0D_1 &= (WD_0D_1W^{-1})_+W, \\ \frac{\partial W}{\partial t} + WD_0D_2 &= (WD_0D_2W^{-1})_+W. \end{aligned} \quad (2.2)$$

In terms of the coefficients w_j of W the equation (2.2) is written in the form

$$\begin{aligned} \frac{\partial w_i}{\partial s} &= \frac{\partial w_{i+1}}{\partial x_1} - \frac{\partial w_1}{\partial x_1} w_i, \\ \frac{\partial w_i}{\partial t} &= \frac{\partial w_{i+1}}{\partial x_2} - \frac{\partial w_1}{\partial x_2} w_i, \end{aligned} \quad (2.3)$$

for $i > 0$. We set $A_j = \frac{\partial w_1}{\partial x_j}$ ($j = 1, 2$). By eliminating w_2 from the equation (2.3) for $i = 1$, we obtain the equation (2.1).

Example 2.2 Self-dual Einstein equations (see [2],[12]).

The self-dual Einstein equations are written in the form (see [6])

$$\begin{aligned} \frac{\partial A_1}{\partial x_2}(x_1, x_2, s, t) &= \frac{\partial A_2}{\partial x_1}(x_1, x_2, s, t), \\ \frac{\partial B_1}{\partial x_2}(x_1, x_2, s, t) &= \frac{\partial B_2}{\partial x_1}(x_1, x_2, s, t), \\ \left[\frac{\partial}{\partial s} + A_1(x_1, x_2, s, t) \frac{\partial}{\partial x_1} + B_1(x_1, x_2, s, t) \frac{\partial}{\partial x_2}, \right. \\ &\quad \left. \frac{\partial}{\partial t} + A_2(x_1, x_2, s, t) \frac{\partial}{\partial x_1} + B_2(x_1, x_2, s, t) \frac{\partial}{\partial x_2} \right] = 0, \\ \frac{\partial B_j}{\partial x_2}(x_1, x_2, s, t) + \frac{\partial A_j}{\partial x_1}(x_1, x_2, s, t) &= 0 \quad (j = 1, 2). \end{aligned} \quad (2.4)$$

In the following we forget the last equation in (2.4) for simplicity.

We introduce the space $\mathcal{W}_E$ and the Lie algebra V_E :

$$\mathcal{W}_E = \{W = \sum_{j,k \in \mathbb{N}} \frac{1}{j!k!} G_1^j G_2^k D_1^j D_2^k; \quad G_i = \sum_{j < 0} g_{i,j}(x_1, x_2) D_0^j (i = 1, 2)\}.$$

$$V_E = \{F_1(x_1, x_2, D_0)D_1 + F_2(x_1, x_2, D_0)D_2; \quad F_1, F_2 \in \mathcal{E}\}.$$

The evolution equation (1.4) for $P \in V_E$ with any initial value $W \in \mathcal{W}_E$ has a solution in $\mathcal{W}_E$. We consider (1.4) for $P = D_0 D_1, D_0 D_2 \in V_E$:

$$\begin{aligned} \frac{\partial W}{\partial s} + W D_0 D_1 &= (W D_0 D_1 W^{-1})_+ W, \\ \frac{\partial W}{\partial t} + W D_0 D_2 &= (W D_0 D_2 W^{-1})_+ W. \end{aligned} \quad (2.5)$$

In terms of the coefficients $g_{i,j}$ the equation (2.5) is written in the form:

$$\begin{aligned} \frac{\partial g_{i,j}}{\partial s} &= \frac{\partial g_{i,j-1}}{\partial x_1} - \frac{\partial g_{1,-1}}{\partial x_1} \frac{\partial g_{i,j}}{\partial x_1} - \frac{\partial g_{2,-1}}{\partial x_1} \frac{\partial g_{i,j}}{\partial x_2}, \\ \frac{\partial g_{i,j}}{\partial t} &= \frac{\partial g_{i,j-1}}{\partial x_2} - \frac{\partial g_{1,-1}}{\partial x_2} \frac{\partial g_{i,j}}{\partial x_1} - \frac{\partial g_{2,-1}}{\partial x_2} \frac{\partial g_{i,j}}{\partial x_2} \quad \text{for } i = 1, 2, j < 0. \end{aligned} \quad (2.6)$$

We set $A_j = -\frac{\partial g_{1,-1}}{\partial x_j}, B_j = -\frac{\partial g_{2,-1}}{\partial x_j} (j = 1, 2)$. By eliminating $g_{i,-2} (i = 1, 2)$ from the equation (2.6) for $i = 1, 2, j = -1$, we obtain the equation (2.4).

We shall unify these two examples and obtain more general systems. We introduce the Lie subalgebra V of $\mathcal{E}$ which contains both the Lie algebras V_{YM} and V_E :

$$\begin{aligned} V &= \{F_0(x', D_0)x_0 + \sum_{0 < j < r} F_j(x', D_0)D_j + E(x', D_0); \\ &\quad F_k(x', D_0), E(x', D_0) \in \mathcal{E}, \quad (0 \leq k < r)\}, \end{aligned}$$

where $x' = (x_1, x_2, \dots, x_{r-1})$.

Lemma 2.1 V is isomorphic to $\mathcal{D}(1) \otimes_{\mathcal{O}} \mathbb{C}[[x]][x_0^{-1}]$ as a Lie algebra.

Proof. Set $X = \mathbb{C}^r = \{(z_0, z_1, \dots, z_{r-1}) \in \mathbb{C}^r\}$, and $Y = \{z \in X; z_0 = 0\}$. We take the transformation

$$x_0 = z_0^2 D_{z_0}, \quad D_{x_0} = z_0^{-1}, \quad x' = z', \quad D_{x'} = D_{z'}.$$

Then the transform of V is $\mathcal{D}(1) \otimes \mathbb{C}[[z]][z_0^{-1}]$. □

The Lie algebra $\mathcal{D}(1) \otimes \mathbb{C}[[x]][x_0^{-1}]$ is the direct sum of the Lie algebra $\Theta \otimes_{\mathcal{O}} \mathbb{C}[[x]][x_0^{-1}]$ of vector fields and the commutative Lie algebra $\mathbb{C}[[x]][x_0^{-1}]$, where the Lie algebra Θ is defined by

$$\Theta = \sum_{0 \leq j < r} \mathcal{O} D_j.$$

The Lie algebra $\Theta \otimes_{\mathcal{O}} \mathbb{C}[[x]][x_0^{-1}]$ corresponds to the infinitesimal coordinate transformations, and the Lie algebra $\mathbb{C}[[x]][x_0^{-1}]$ corresponds to the infinitesimal gauge transformations of a line bundle.

Now we shall determine the set of $\mathcal{D}$ -submodules in $\mathcal{E}$ such that the condition (1.1) is preserved under (1.3) with respect to any $P \in V$. We introduce the subspaces $\mathcal{W}_m$ and $\mathcal{W}_{YM}$ of $\mathcal{E}$:

$$\mathcal{W}_m = \{W(x, D_x) = \sum_{\alpha \in \mathbb{N}^r} \frac{1}{\alpha!} G^\alpha x_0^{\alpha_0} D_x^\alpha \in \mathcal{E};$$

$$G^\alpha = G_0^{\alpha_0} G_1^{\alpha_1} \dots G_{r-1}^{\alpha_{r-1}} \quad \text{where } G_j = G_j(x', D_0) \in \mathcal{E}(-1)\},$$

$$\mathcal{W}_{YM} = \{W(x', D_0) = \sum_{i \in \mathbb{N}} w_i(x') D_0^{-i}; w_i \in \mathbb{C}[[x_1, \dots, x_{r-1}]], w_0 \equiv 1.\}.$$

Proposition 2.2 Let $I_0 = \mathcal{D}W_0$ be a $\mathcal{D}$ -submodule of $\mathcal{E}$. Assume that the time evolution of I_0 for any $P \in V$ also satisfies the condition (1.1). Then W_0 factorizes into the product of $W_m \in \mathcal{W}_m$ and $W_{YM} \in \mathcal{W}_{YM}$;

$$W_0 = W_m W_{YM}.$$

Remark. The operator W_m corresponds to the equation of self-dual metrics (see §3).

Proof of Proposition 2.2.

Step 1.

Lemma 2.3 Let $W(t)$ be the solution of (1.4) for $P \in V$ with $W(0) = W_0$. Then $(W(t)PW(t)^{-1})_-$ is contained in $\mathcal{E}_\phi(0)$.

Proof.

It follows from the equation (1.5) that

$$\frac{\partial W(t)}{\partial t} W(t)^{-1} = -(W(t)PW(t)^{-1})_-.$$

Since the operator $\frac{\partial W(t)}{\partial t} W(t)^{-1}$ is contained in $\mathcal{E}(0)$, we obtain Lemma 2.3. $\square$

Step 2. For any $P \in \mathcal{E}$, let $\tilde{P}$ be $W_0 P W_0^{-1}$.

Lemma 2.4 Suppose that $P, Q \in V$ commute with each other. The operator $[\tilde{P}_+, \tilde{Q}_-]_-$ is contained in $\mathcal{E}_\phi(0)$.

Proof. We consider the solution $W = W(s, t)$ of the equation (1.5) for the operators P, Q :

$$\begin{aligned} \frac{\partial W}{\partial s} &= -(W P W^{-1})_- W, \\ \frac{\partial W}{\partial t} &= -(W Q W^{-1})_- W, \end{aligned} \tag{2.7}$$

where $W = W(s, t)$. Since $[P, Q] = 0$, the system (2.7) is compatible by Theorem 1.3. It follows from Lemma 2.3 that $W P W^{-1}$ is contained in $\mathcal{D} + \mathcal{E}_\phi(0)$. We have

$$\begin{aligned} \frac{\partial}{\partial t}(W P W^{-1}) &= \frac{\partial W}{\partial t} P W^{-1} - W P W^{-1} \frac{\partial W}{\partial t} W^{-1} \\ &= -(W Q W^{-1})_- W P W^{-1} + W P W^{-1} (W Q W^{-1})_- \\ &= [W P W^{-1}, (W Q W^{-1})_-]. \end{aligned}$$

Hence $[\tilde{P}, \tilde{Q}_-]$ is contained in $\mathcal{D} + \mathcal{E}_\phi(0)$. Since $[\tilde{P}_-, \tilde{Q}_-]$ is contained in $\mathcal{E}_\phi(0)$ by Lemma 2.3, $[\tilde{P}_+, \tilde{Q}_-]$ is contained in $\mathcal{D} + \mathcal{E}_\phi(0)$. $\square$

Step 3 Recall that $\tilde{P}$ denotes $W_0 P W_0^{-1}$ for any $P \in \mathcal{E}$. We define the operators $G_i, H_i \in \mathcal{E}$ ($0 \leq i < r$) as follows:

$$\begin{aligned}\tilde{D}_0 &= D_0 - G_0(x, D_x), \\ \tilde{x}_i &= x_i + G_i(x, D_x) \quad \text{for } i = 1, 2, \dots, r-1, \\ \tilde{x}_0 &= x_0 + H_0(x, D_x), \\ \tilde{D}_i &= D_i + H_i(x, D_x) \quad \text{for } i = 1, 2, \dots, r-1.\end{aligned}\tag{2.8}$$

Lemma 2.5 Assume that W_0 satisfies the condition in Proposition 2.2. The operators G_i, H_i are written in the following form:

$$\begin{aligned}G_i &= G_i(x', D_0) \in \mathcal{E}(-1), \\ H_i &= \sum_{0 \leq j < r} K_{ij}(x', D_0) D_j + L_i(x, D_0) \quad \text{for } 0 \leq i < r, \\ \text{where } &K_{ij}(x', D_0) \in \mathcal{E}, \quad L_i(x, D_0) \in \mathcal{E}(-1), \\ &\text{and } x' = (x_1, \dots, x_{r-1}).\end{aligned}$$

Proof. We shall show the following statement $(2.9)_n$ by the induction on n :

$$\begin{aligned}G_i &\stackrel{!}{=} G'_i + G''_i, \quad H_i = H'_i + H''_i \\ \text{for some } &G'_i, H'_i, G''_i \text{ and } H''_i \text{ } (0 \leq i < r) \text{ such that} \\ &G''_0, H''_i \in \mathcal{E}(-n), \quad G''_i, H''_0 \in \mathcal{E}(-n-1) \quad \text{for } 0 < i < r, \\ &G'_i = G'_i(x', D_0) \in \mathcal{E}(-1), \\ &H'_i = \sum_{0 \leq j < r} K_{ij}(x', D_0) D_j + L_i(x, D_0) \quad \text{for } 0 \leq i < r \\ \text{with } &K_{ij}(x', D_0), L_i(x, D_0) \in \mathcal{E}(-1).\end{aligned}\tag{2.9}_n$$

It is evident that the statement $(2.9)_0$ is true. By assuming $(2.9)_n$ we shall prove $(2.9)_{n+1}$.

We expand the operators:

$$\begin{aligned} G_j'' &= \sum_{k \leq 0, \alpha' \in \mathbb{N}^{r-1}} g_{k, \alpha'}^{(j)} D_0^k D_{x'}^{\alpha'}, \\ H_j'' &= \sum_{k \leq 0, \alpha' \in \mathbb{N}^{r-1}} h_{k, \alpha'}^{(j)} D_0^k D_{x'}^{\alpha'}. \end{aligned} \quad (2.10)$$

Since G_0'' belongs to $\mathcal{E}(-n)$, we have

$$\begin{aligned} \tilde{D}_0^{n+2} &= (D_0 - G_0' - G_0'')^{n+2} \\ &\equiv (D_0 - G_0')^{n+2} - (n+2)G_0''(D_0 - G_0')^{n+1} \\ &\equiv (D_0 - G_0')^{n+2} - (n+2)G_0''D_0^{n+1} \quad \text{modulo } \mathcal{E}(0). \end{aligned}$$

The operator $(D_0 - G_0')^{n+2}$ belongs to $\mathcal{D} + \mathcal{E}(0)$, because G_0' does not contain D_j ($0 < j < r$). Hence we obtain

$$\tilde{D}_0^{n+2} \equiv -(n+2)G_0''D_0^{n+1} \quad \text{modulo } \mathcal{D} + \mathcal{E}(0).$$

Since $\tilde{D}_0^{n+2}$ is contained in $\mathcal{D} + \mathcal{E}(0)$ for any $n \in \mathbb{Z}$ by Lemma 2.3, $G_0''D_0^{n+1}$ is also contained in $\mathcal{D} + \mathcal{E}(0)$. Therefore we obtain

$$g_{-n-|\alpha'|, \alpha'}^{(0)} = 0 \quad \text{for } |\alpha'| \geq 2. \quad (2.11)$$

Similarly we have, modulo $\mathcal{D} + \mathcal{E}(0)$

$$\begin{aligned} \tilde{D}_0^{n+1} \tilde{D}_j &\equiv -(n+1)G_0''D_0^n D_j + H_j''D_0^{n+1}, \\ \tilde{x}_0 \tilde{D}_0^{n+2} &\equiv -(n+2)x_0 G_0''D_0^{n+1} + H_0''D_0^{n+2}, \\ \tilde{x}_j \tilde{D}_0^{n+1} \tilde{D}_k &\equiv G_j''D_0^{n+1} D_k + x_j (H_k''D_0^{n+1} - (n+1)G_0''D_0^n D_k). \end{aligned}$$

By Lemma 2.3, $\tilde{D}_0^{n+1}\tilde{D}_j$, $\tilde{x}_0\tilde{D}_0^{n+2}$ and $\tilde{x}_j\tilde{D}_0^{n+1}\tilde{D}_k$ are contained in $\mathcal{D}+\mathcal{E}(0)$. Thus we obtain

$$\begin{aligned} h_{-|\alpha'|-n,\alpha'}^{(j)} &= 0 \quad \text{for } |\alpha'| \geq 3, \\ h_{-2-n,\alpha'+e_j}^{(j)} &= (n+1)g_{-1-n,\alpha'}^{(0)} \quad \text{for } |\alpha'| = 1, \\ h_{-|\alpha'|-n-1,\alpha'}^{(0)} &= 0 \quad \text{for } |\alpha'| \geq 2, \\ g_{-|\alpha'|-n-1,\alpha'}^{(j)} &= 0 \quad \text{for } |\alpha'| \geq 2, \end{aligned} \tag{2.12}$$

where $j = 1, 2, \dots, r-1$ and $e_j = (0, \dots, \overset{j}{\underset{\vee}{1}}, \dots, 0) \in \mathbb{N}^{r-1}$.

We denote by $\mathcal{V}$ and $\mathcal{V}'$ the subspaces of $\mathcal{D}+\mathcal{E}(0)$:

$$\begin{aligned} \mathcal{V} &= \left\{ \sum_{0 < k < r} F_k(x, D_0)D_k + E(x, D_0); \quad F_k(x, D_0), E(x, D_0) \in \mathcal{E} \right\}, \\ \mathcal{V}' &= \left\{ \sum_{0 < k < r} F_k(x', D_0)D_k + E(x, D_0); \quad F_k(x', D_0), E(x, D_0) \in \mathcal{E} \right\}. \end{aligned}$$

For $P, Q \in \mathcal{V}'$ the commutator $[P, Q]$ is contained in $\mathcal{V}$. Since H'_j ($0 \leq j < r$) are contained in $\mathcal{V}'$, $[H'_i, H'_j]$ $[x_0, H'_j]$ ($0 \leq i, j < r$) are contained in $\mathcal{V}$.

For $j > 0$ we have

$$\begin{aligned} [\tilde{x}_0, \tilde{D}_j] &= [x_0, H_j] - \frac{\partial H_0}{\partial x_j} + [H_0, H_j] \\ &= [x_0, H_j''] - \frac{\partial H_0''}{\partial x_j} + ([H_0'', H_j''] + [H_0', H_j''] + [H_0'', H_j']) \\ &\quad + ([x_0, H_j'] - \frac{\partial H_0'}{\partial x_j} + [H_0', H_j']) \\ &\equiv [x_0, H_j''] - \frac{\partial H_0''}{\partial x_j} \quad \text{modulo } \mathcal{E}(-n-2) + \mathcal{V}. \end{aligned}$$

Since $[\tilde{x}_0, \tilde{D}_j] = 0$, $[x_0, H_j''] - \frac{\partial H_0''}{\partial x_j}$ belongs to $\mathcal{E}(-n-2) + \mathcal{V}$. By (2.10) we

have, modulo $\mathcal{E}(-n-2) + \mathcal{V}$

$$\begin{aligned}
[x_0, H_j''] - \frac{\partial H_0''}{\partial x_j} \\
\equiv \sum_{|\alpha'| \geq 2} ((n + |\alpha'|) h_{-n-|\alpha'|, \alpha'}^{(j)} - \frac{\partial h_{-n-1-|\alpha'|, \alpha'}^{(0)}}{\partial x_j}) D_0^{-1-n-|\alpha'|} D_{x'}^{\alpha'} \quad (2.13)
\end{aligned}$$

We obtain from (2.12) and (2.13) that

$$\begin{aligned}
h_{-2-n, \alpha'}^{(j)} &= 0 \quad \text{for } |\alpha'| = 2, \quad 0 < j < r, \\
g_{-1-n, \alpha'}^{(0)} &= 0 \quad \text{for } |\alpha'| = 1. \quad (2.14)
\end{aligned}$$

Similarly by the equations

$$\begin{aligned}
[\tilde{D}_0, \tilde{D}_j] &= \frac{\partial H_j}{\partial x_0} + \frac{\partial G_0}{\partial x_j} - [G_0, H_j] = 0 \quad \text{for } 0 < j < r, \\
[\tilde{D}_0, \tilde{x}_j] &= \frac{\partial G_j}{\partial x_0} + [G_j, G_0] - [G_0, x_j] = 0 \quad \text{for } 0 < j < r, \\
[\tilde{D}_0, \tilde{x}_0] &= 1 + [D_0, H_0] - [G_0, x_0] + [G_0, H_0] = 1,
\end{aligned}$$

we obtain

$$\frac{\partial h_{-1-n, e_k}^{(j)}}{\partial x_0} = 0, \quad \frac{\partial h_{-2-n, e_k}^{(0)}}{\partial x_0} = 0, \quad \frac{\partial g_{-1-n, 0}^{(j)}}{\partial x_0} = 0. \quad (2.15)$$

For $0 < i < r$ we have modulo $\mathcal{E}(-1)$

$$\begin{aligned}
\tilde{D}_0^n \tilde{D}_i &= (D_0 - G'_0 - G''_0)^n (D_i + H'_i + H''_i) \\
&\equiv ((D_0 - G'_0)^n - n G''_0 (D_0 - G'_0)^{n-1}) (D_i + H'_i + H''_i) \\
&\equiv -n G''_0 D_0^{n-1} D_i + (D_0 - G'_0)^n (D_i + H'_i) + H''_i D_0^n. \quad (2.16)
\end{aligned}$$

By (2.12) and (2.14) we have

$$\begin{aligned} H_i'' D_0^n &\equiv \sum_{0 < j < r} h_{-1-n, e_j}^{(i)} D_0^{-1} D_j \text{ modulo } \mathcal{E}(-1), \\ -n G_0'' D_0^{n-1} D_i &\equiv -n g_{-n, 0}^{(0)} D_0^{-1} D_i \text{ modulo } \mathcal{E}(-1). \end{aligned}$$

We set K_{ij} and L_i in $(2.9)_n$ as follows:

$$\begin{aligned} K_{ij}(x', D_0) &= \sum_{m < 0} k_{i, j, m}(x') D_0^m, \\ L_i(x, D_0) &= \sum_{m < 0} l_{i, m}(x') D_0^m \end{aligned}$$

By $(2.9)_n$ we have

$$\begin{aligned} (\tilde{D}_0 \tilde{D}_i)_+ &= (D_0 - G_0)(D_i + H'_i + H''_i) \\ &= D_0 D_i + \{D_0(H'_i + \sum_{1 \leq j < r} K_{ij} + L_i)\}_+ \\ &= D_0 D_i + \sum_{1 \leq j < r} (h_{-1, e_j}^{(i)} + k_{i, j, -1}) D_j + l_{i, -1} \end{aligned} \tag{2.17}$$

Sublemma. We have

$$\frac{\partial g_{-n, 0}^{(0)}}{\partial x_0} = 0. \tag{2.18}$$

Proof. The commutator $[(\tilde{D}_0 \tilde{D}_i)_+, \tilde{D}_0^n \tilde{D}_i]$ is contained in $\mathcal{D} + \mathcal{E}(0)$ by Lemma 2.4. By (2.16) and (2.17) we have, modulo $\mathcal{E}(0)$

$$\begin{aligned} &[(\tilde{D}_0 \tilde{D}_i)_+, \tilde{D}_0^n \tilde{D}_i] \\ &\equiv [(\tilde{D}_0 \tilde{D}_i)_+, -n g_{-n, 0}^{(0)} D_0^{-1} D_i \\ &\quad + \sum_{0 < j < r} h_{-1-n, e_j}^{(i)} D_0^{-1} D_j + (D_0 - G'_0)^n (D_i + H'_i)] \\ &\equiv [D_0 D_i, -n g_{-n, 0}^{(0)} D_0^{-1} D_i] \\ &\quad + [(\tilde{D}_0 \tilde{D}_i)_+, \sum_{0 < j < r} h_{-1-n, e_j}^{(i)} D_0^{-1} D_j + (D_0 - G'_0)^n (D_i + H'_i)]. \end{aligned}$$

Since $(\tilde{D}_0 \tilde{D}_i)_+$ and $\sum_{0 < j < r} h_{-1-n, e_j}^{(i)} D_0^{-1} D_j + (D_0 - G'_0)^n (D_i + H'_i)$ are contained in $\mathcal{V}$, the operator

$$\begin{aligned} [D_0 D_i, -n g_{-n,0}^{(0)} D_0^{-1} D_i] \\ = -n \frac{\partial g_{-n,0}^{(0)}}{\partial x_0} D_0^{-1} D_i^2 - n \frac{\partial g_{-n,0}^{(0)}}{\partial x_i} D_i - n \frac{\partial^2 g_{-n,0}^{(0)}}{\partial x_0 \partial x_i} \end{aligned}$$

is contained in $\mathcal{D} + \mathcal{E}(0)$. This implies (2.18). $\square$

It follows from (2.11), (2.12) and (2.14) that we have

$$\begin{aligned} G'_0 - g_{-n,0}^{(0)} D_0^{-n} &\in \mathcal{E}(-n-1), \\ G'_i - g_{-n-1,0}^{(i)} D_0^{-n-1} &\in \mathcal{E}(-n-2), \\ H'_0 - \left(\sum_{0 < k < r} h_{-n-2, e_k}^{(0)} D_0^{-n-2} D_k + h_{-n-1,0}^{(0)} D_0^{-n-1} \right) &\in \mathcal{E}(-n-2), \\ H'_i - \left(\sum_{0 < k < r} h_{-n-1, e_k}^{(i)} D_0^{-n-1} D_k + h_{-n,0}^{(i)} D_0^{-n} \right) &\in \mathcal{E}(-n-1). \end{aligned}$$

for $0 < i < r$. Thus we obtain $(2.9)_{n+1}$ from (2.15) and (2.18). $\square$

Step 4 Now we shall prove Proposition 2.2. We introduce a micro-differential operator

$$W_m = \sum_{\alpha \in \mathbb{N}^r} \frac{1}{\alpha!} G^\alpha x_0^{\alpha_0} D_{x'}^{\alpha'},$$

where $G^\alpha = G_0^{\alpha_0} G_1^{\alpha_1} \cdots G_{r-1}^{\alpha_{r-1}}$ is given in Lemma 2.5. Then we have

$$\begin{aligned} [D_0, W_m] &= \sum_{\alpha \in \mathbb{N}^r} \frac{\alpha_0}{\alpha!} G^\alpha x_0^{\alpha_0-1} D_{x'}^{\alpha'} \\ &= G_0 \sum_{\alpha \in \mathbb{N}^r} \frac{1}{\alpha!} G^\alpha x_0^{\alpha_0} D_{x'}^{\alpha'} = G_0 W_m. \end{aligned}$$

Hence we get

$$W_m D_0 W_m^{-1} = D_0 - G_0. \quad (2.20)$$

Similarly we obtain that

$$W_m x_j W_m^{-1} = x_j + G_j \quad (1 \leq j < r). \quad (2.21)$$

Set $W_{YM} = W_m^{-1} W$. Then it follows from (2.8), (2.20) and (2.21) that W_{YM} commutes with $D_0, x_1, \dots, x_{r-1}$. Therefore W_{YM} is contained in $\mathcal{W}_{YM}$.

Thus we have completed the proof of Proposition 2.2. $\square$

We set

$$\mathcal{W} = \{W_m W_{YM}; \quad W_m \in \mathcal{W}_m, W_{YM} \in \mathcal{W}_{YM}\}.$$

We shall investigate the structure of $\mathcal{W}, \mathcal{W}_m$ and $\mathcal{W}_{YM}$.

Proposition 2.6 The spaces $\mathcal{W}$, $\mathcal{W}_m$ and $\mathcal{W}_{YM}$ are groups by the composition of microdifferential operators, and $\mathcal{W}_{YM}$ is a normal subgroup in $\mathcal{W}$.

Proof. It is evident that $\mathcal{W}_{YM}$ is an abelian group.

Let $W_m = \sum_{\alpha} \frac{1}{\alpha!} G^{\alpha} x_0^{\alpha_0} D_{x'}^{\alpha'}$ and $W'_m = \sum_{\beta} \frac{1}{\beta!} \tilde{F}^{\beta} x_0^{\beta_0} D_{x'}^{\beta'}$ be operators in $\mathcal{W}_m$. For any microdifferential operator $P(x', D_0)$, we set

$$\tilde{P} := W_m P W_m^{-1} = P(x_1 + G_1, \dots, x_{r-1} + G_{r-1}, D_0 - G_0).$$

The last equality follows from (2.20) and (2.21). Noting that $\tilde{P}$ commutes with G_j , the composition

$$\begin{aligned} W_m W'_m &= \sum_{\gamma} \frac{1}{\gamma!} \tilde{F}^{\gamma} W_m x_0^{\gamma_0} D_{x'}^{\gamma'} \\ &= \sum_{\alpha, \beta} \frac{1}{\beta!} \tilde{F}^{\beta} \frac{1}{\alpha!} G^{\alpha} x_0^{\alpha_0 + \beta_0} D_{x'}^{\alpha' + \beta'} \\ &= \sum_{\gamma} \frac{1}{\gamma!} (\tilde{F} + G)^{\gamma} x_0^{\gamma_0} D_{x'}^{\gamma'} \end{aligned} \quad (2.22)$$

is contained in $\mathcal{W}_m$. For $W_{YM} = \sum_i w_i(x')D_0^{-i} \in \mathcal{W}_{YM}$ the operator

$$\begin{aligned} & W_m W_{YM} W_m^{-1} \\ &= \sum_{i \geq 0} w_i(x_1 + G_1, x_2 + G_2, \dots, x_{r-1} + G_{r-1})(D_0 - G_0)^{-i} \end{aligned} \quad (2.23)$$

is contained in $\mathcal{W}_{YM}$. Since $\mathcal{W} = \mathcal{W}_m \mathcal{W}_{YM}$, it follows from (2.22) and (2.23) that $\mathcal{W}$ is a group and that $\mathcal{W}_{YM}$ is a normal subgroup of $\mathcal{W}$. $\square$

We define the Lie subalgebras V_m and V_{YM} of V

$$\begin{aligned} V_m &= \{F_0(x', D_0)x_0 + \sum_{0 < j < r} F_j(x', D_0)D_j; \\ &\quad F_k(x', D_0) \in \mathcal{E}, (0 \leq k < r)\}, \\ V_{YM} &= \{E(x', D_0); E(x', D_0) \in \mathcal{E}\}. \end{aligned}$$

We have

$$V = V_m \oplus V_{YM}.$$

Proposition 2.7 For any $P \in V$ (resp. V_m, V_{YM}) and $W \in \mathcal{W}$ (resp. $\mathcal{W}_m, \mathcal{W}_{YM}$), we have $WPW^{-1} \in V$ (resp. V_m, V_{YM}).

Proof. For $W = \sum_{\alpha} \frac{1}{\alpha!} G^{\alpha} x_0^{\alpha_0} D_{x'}^{\alpha'} \in \mathcal{W}_m$ we have

$$\begin{aligned} [D_i, W] &= \sum_{0 \leq j < r} \sum_{\alpha} \frac{\alpha_j}{\alpha!} G^{\alpha} \frac{\partial G_j}{\partial x_i} x_0^{\alpha_0} D_{x'}^{\alpha'} \\ &= \sum_{0 < j < r} \frac{\partial G_j}{\partial x_i} W D_j + \frac{\partial G_0}{\partial x_i} W x_0 \end{aligned}$$

for any $i = 1, 2, \dots, r-1$. Similarly we have

$$[x_0, W] = \sum_{0 < j < r} [x_0, G_j] W D_j + [x_0, G_0] W x_0.$$

Hence we obtain

$$\begin{aligned} WD_iW^{-1} &= D_i - \sum_{0 < j < r} \frac{\partial G_j}{\partial x_i} WD_jW^{-1} - \frac{\partial G_0}{\partial x_i} Wx_0W^{-1} \quad \text{for } 0 < i < r, \\ Wx_0W^{-1} &= x_0 - \sum_{0 < j < r} [x_0, G_j] WD_jW^{-1} - [x_0, G_0] Wx_0W^{-1}. \end{aligned} \quad (2.24)$$

We set

$$G_{ij} = \begin{cases} \frac{\partial G_j}{\partial x_i} & \text{for } 0 < i < r, \\ [x_0, G_j] & i = 0, \end{cases}$$

for $0 \leq j < r$. Then we have

$$\begin{aligned} x_0 &= (1 + G_{00})Wx_0W^{-1} + \sum_{0 < j < r} G_{0j}WD_jW^{-1}, \\ D_i &= G_{i0}Wx_0W^{-1} + \sum_{0 < j < r} (\delta_{ij} + G_{ij})WD_jW^{-1} \quad \text{for } 0 < i < r. \end{aligned} \quad (2.25)$$

Since $G_{ij} \in \mathcal{E}(-1)$, The matrix $(\delta_{ij} + G_{ij})_{0 \leq i, j < r}$ is invertible. Hence WD_iW^{-1} ($0 < i < r$) and Wx_0W^{-1} are contained in V_m , because G_{ij} is independent of $x_0, D_1, \dots, D_{r-1}$. For any operator $P = P(x', D_0) \in \mathcal{E}$, WPW^{-1} is independent of $x_0, D_1, \dots, D_{r-1}$ by (2.20) and (2.21). Hence we get Proposition 2.7 for V_m and $\mathcal{W}_m$.

The proposition is evident for V_{YM} and $\mathcal{W}_{YM}$. For $W_m \in \mathcal{W}_m$ and $E \in V_{YM}$ the operator $W_mE W_m^{-1}$ is contained in V_{YM} by (2.20) and (2.21). For $W_{YM} \in \mathcal{W}_{YM}$ we have

$$\begin{aligned} W_{YM}D_iW_{YM}^{-1} &= D_i - \frac{\partial W_{YM}}{\partial x_i} W_{YM}^{-1} \quad \text{for } 0 < i < r, \\ W_{YM}x_0W_{YM}^{-1} &= x_0 - [x_0, W_{YM}]W_{YM}^{-1}. \end{aligned}$$

Since W_{YM} commutes with $D_0, x_1, \dots, x_{r-1}$, the operator $W_{YM}PW_{YM}^{-1}$ is contained in V for any $P \in V$. Since $V = V_m \oplus V_{YM}$ and $\mathcal{W} = \mathcal{W}_m \mathcal{W}_{YM}$, we get Proposition 2.7. $\square$

Example 2.3 We shall write down the evolution equations (1.4) for $W \in \mathcal{W}$ in the case $r = 3$. We take D_0D_1, D_0D_2 as generators of (1.4).

We set

$$W = \left(\sum_{\alpha \in \mathbb{N}^3} \frac{1}{\alpha!} G^\alpha x_0^{\alpha_0} D_{x'}^{\alpha'} \right) \left(\sum_i w_i(x') D_0^{-i} \right),$$

where $G_i(x_1, x_2, D_0) = \sum_{j < 0} g_{i,j}(x_1, x_2) D_0^j \quad (i = 1, 2).$

Then we have

$$\begin{aligned} (\tilde{D}_0 \tilde{D}_1)_+ &= D_0 D_1 - \frac{\partial g_{0,-1}}{\partial x_1} x_0 - \sum_{i=1,2} \frac{\partial g_{i,-1}}{\partial x_1} D_i - \frac{\partial w_1}{\partial x_1}, \\ (\tilde{D}_0 \tilde{D}_2)_+ &= D_0 D_2 - \frac{\partial g_{0,-1}}{\partial x_2} x_0 - \sum_{i=1,2} \frac{\partial g_{i,-1}}{\partial x_2} D_i - \frac{\partial w_1}{\partial x_2}. \end{aligned}$$

Taking time parameters s and t with respect to $D_0 D_1$ and $D_0 D_2$, respectively, we obtain the evolution equations

$$\begin{aligned} \frac{\partial W}{\partial s} + W D_0 D_1 &= \left(D_0 D_1 - \frac{\partial g_{0,-1}}{\partial x_1} x_0 - \sum_{i=1,2} \frac{\partial g_{i,-1}}{\partial x_1} D_i - \frac{\partial w_1}{\partial x_1} \right) W, \\ \frac{\partial W}{\partial t} + W D_0 D_2 &= \left(D_0 D_2 - \frac{\partial g_{0,-1}}{\partial x_2} x_0 - \sum_{i=1,2} \frac{\partial g_{i,-1}}{\partial x_2} D_i - \frac{\partial w_1}{\partial x_2} \right) W. \end{aligned} \quad (2.26)$$

It follows from (2.26) that we obtain the Zakharov-Shabat type equation

$$\begin{aligned} \left[\frac{\partial}{\partial s} - D_0 D_1 + \frac{\partial g_{0,-1}}{\partial x_1} x_0 + \sum_{i=1,2} \frac{\partial g_{i,-1}}{\partial x_1} D_i + \frac{\partial w_1}{\partial x_1}, \right. \\ \left. \frac{\partial}{\partial t} - D_0 D_2 + \frac{\partial g_{0,-1}}{\partial x_2} x_0 + \sum_{i=1,2} \frac{\partial g_{i,-1}}{\partial x_2} D_i + \frac{\partial w_1}{\partial x_2} \right] = 0. \end{aligned} \quad (2.27)$$

We shall investigate the infinitesimal action ρ of the Lie subalgebra V of $\mathcal{E}$. Remark that the Lie algebra of the group $\mathcal{W}$ (resp. $\mathcal{W}_m$, $\mathcal{W}_{YM}$) is

canonically isomorphic to the Lie subalgebra $V_- = V \cap \mathcal{E}_\phi$ (resp. $(V_m)_-$, $(V_{YM})_-$).

Theorem 2.8 The action ρ of V (resp. V_m, V_{YM}) preserves the space $\mathcal{W}$ (resp. $\mathcal{W}_m, \mathcal{W}_{YM}$). The action of V_- (resp. $(V_m)_-, (V_{YM})_-$) coincides with the infinitesimal right action of on the group $\mathcal{W}$ (resp. $\mathcal{W}_m, \mathcal{W}_{YM}$).

Proof. For any element $P \in V$ (resp. V_m, V_{YM}) and any operator $W \in \mathcal{W}$ (resp. $\mathcal{W}_m, \mathcal{W}_{YM}$), we have

$$\rho(P) = -(WPW^{-1})_- W \in T_W \mathcal{E}.$$

By Proposition 2.7, $-(WPW^{-1})_- W$ is contained in the tangent space of $\mathcal{W}$ (resp. $\mathcal{W}_m, \mathcal{W}_{YM}$) at W . Taking $P \in V_-$ (resp. $(V_m)_-, (V_{YM})_-$), we have $\rho(P) = -WP$. Hence the action ρ is the right action of vector fields. $\square$

By Theorem 2.8 the Lie algebra V (resp. V_m, V_{YM}) acts on $\mathcal{W}$ (resp. $\mathcal{W}_m, \mathcal{W}_{YM}$) transitively.

§3. Twistor theory and integrable systems

On oriented Riemannian manifolds of dimension four, the Weyl curvature tensor C decomposes into two components, the self-dual part C_+ and the anti-self-dual part C_- . A manifold is called self-dual (resp. anti-self-dual) when C_- (resp. C_+) vanishes. Penrose [5] showed that the vanishing of the anti-self-dual part of the Weyl tensor is precisely the integrability condition of the existence of a curved twistor space.

In this section we prove that the equation $C_+=0$ is the compatibility condition of the deformation equations of filtered $\mathcal{D}$ -submodules in $\mathcal{E}$ (See [13] in which the Frobenius integrability condition of the equations of self dual metrics is discussed). We get the equations of self dual metrics from the equation (2.26) for $W \in \mathcal{W}_m$.

Let M be a complex four-manifold and g a holomorphic metric, i.e. a non-degenerate symmetric holomorphic covariant two-tensor on M . We shall choose a holomorphic orientation on M which is necessary to define the complex Hodge $*$ -operator. Our discussion being only local, we can assume the existence of two complex vector bundles S_+ and S_- : the bundles of self-dual and anti-self-dual spinors.

Let $\{e_j\}_{j=1,2,3,4}$ denote a local coframe on M such that $g = e_1 e_2 + e_3 e_4$. We can write them in spinor language as

$$\begin{bmatrix} e_4 & e_2 \\ -e_1 & e_3 \end{bmatrix} = \begin{bmatrix} \psi_1 \phi_1 & \psi_1 \phi_2 \\ \psi_2 \phi_1 & \psi_2 \phi_2 \end{bmatrix} \quad (3.1)$$

where ψ_1, ψ_2 (resp. ϕ_1, ϕ_2) are the bases of self-dual (resp. anti-self-dual) spinor coframes.

We take $P = P(S_-)$, the projective bundle of the rank two vector bundle S_- . We parametrize S_- locally by

$$(x, \mu_1, \mu_2) \longrightarrow \mu_1 \phi_1(x) + \mu_2 \phi_2(x),$$

and $\mu = \mu_1/\mu_2$ is an affine coordinate for $\mu_2 \neq 0$.

Theorem 3.1 ([5]) The Riemannian manifold (M, g) is self-dual iff the

following Pfaffian system Ω on P is integrable:

$$\begin{aligned}\theta &:= d\mu + \omega_{21}\mu^2 - (\omega_{22} - \omega_{11})\mu - \omega_{12} = 0, \\ \Omega : \quad \sigma_1 &:= \mu e_4 + e_2 = 0, \\ \sigma_2 &:= -\mu e_1 + e_4 = 0,\end{aligned}\tag{3.2}$$

where ω_{ij} is the connection form of S_- with respect to the frame ϕ_1 and ϕ_2 .

Let $A(\Omega)$ be the sheaf of vector fields orthogonal to the Pfaffian system Ω . The sheaf $A(\Omega)$ is a Lie algebra iff Ω is integrable. In this case there exists a local basis (v_1, v_2) of $A(\Omega)$ such that $[v_1, v_2] = 0$.

Proposition 3.2 Assume (M, g) self-dual. With appropriate coordinates $(\lambda, x_1, x_2, s, t)$ of P , there exists a commuting basis (v_1, v_2) of $A(\Omega)$, in the following form

$$\begin{aligned}v_1 &= \frac{\partial}{\partial s} - \lambda \frac{\partial}{\partial x_1} - \left(\frac{\partial R}{\partial x_1} \frac{\partial}{\partial x_1} + \frac{\partial S}{\partial x_1} \frac{\partial}{\partial x_2} + \frac{\partial T}{\partial x_1} \frac{\partial}{\partial \lambda} \right), \\ v_2 &= \frac{\partial}{\partial t} - \lambda \frac{\partial}{\partial x_2} - \left(\frac{\partial R}{\partial x_2} \frac{\partial}{\partial x_1} + \frac{\partial S}{\partial x_2} \frac{\partial}{\partial x_2} + \frac{\partial T}{\partial x_2} \frac{\partial}{\partial \lambda} \right),\end{aligned}\tag{3.3}$$

where the functions R, S and T do not depend on λ .

Proof. First we notice the following lemma.

Lemma 3.3 ([3]) Let (M, g) be a self-dual Riemannian four-manifold. Then there exist local coordinates (p_1, p_2, q_1, q_2) of M such that

$$g = \sum_{i,j=1,2} P_{ij}(p, q) dp_i dq_j.$$

It follows from Lemma 3.3 that we can take local frames $\{e_j\}_{j=1,2,3,4}$ as follows:

$$\begin{aligned}e_1 &= -dp_1, \quad e_2 = -(P_{11}dq_1 + P_{12}dq_2), \\ e_4 &= dp_2, \quad e_3 = P_{21}dq_1 + P_{22}dq_2.\end{aligned}$$

By Theorem 3.1 the ideal $\mathcal{I}$ generated by $(\theta, \sigma_1, \sigma_2)$ is closed under the exterior derivative d . Thus we have

$$\begin{aligned} d\sigma_1 \wedge \theta \wedge \sigma_1 \wedge \sigma_2 &= 0, \\ d\sigma_2 \wedge \theta \wedge \sigma_1 \wedge \sigma_2 &= 0. \end{aligned} \tag{3.4}$$

By direct calculations we have

$$\begin{aligned} d\sigma_1 \wedge \theta \wedge \sigma_1 \wedge \sigma_2 &= ((\frac{\partial P_{12}}{\partial q_1} - \frac{\partial P_{11}}{\partial q_2})\mu^2 + K\mu + L)d\mu \wedge dp_1 \wedge dp_2 \wedge dq_1 \wedge dq_2, \\ d\sigma_2 \wedge \theta \wedge \sigma_1 \wedge \sigma_2 &= ((\frac{\partial P_{22}}{\partial q_1} - \frac{\partial P_{21}}{\partial q_2})\mu^2 + M\mu + N)d\mu \wedge dp_1 \wedge dp_2 \wedge dq_1 \wedge dq_2, \end{aligned}$$

for functions K, L, M and N independent of μ . Thus we have

$$\frac{\partial P_{12}}{\partial q_1} - \frac{\partial P_{11}}{\partial q_2} = 0, \quad \frac{\partial P_{22}}{\partial p_1} - \frac{\partial P_{21}}{\partial p_2} = 0.$$

Hence we can define new coordinates (x_1, x_2, s, t, μ) by the following equations:

$$\begin{aligned} \frac{\partial x_1}{\partial q_i} &= -P_{1i}, \quad \frac{\partial x_2}{\partial q_i} = P_{2i}, \quad (i = 1, 2) \\ s &= p_2, \quad t = -p_1, \end{aligned}$$

The differential forms θ, σ_1 and σ_2 are written in these coordinates as follows:

$$\begin{aligned} \theta &= d\mu + \mu^2(E_1 e_2 + E_2 e_3) + \mu(F_1 e_2 + F_2 e_3) + \sum_j J_j e_j, \\ \sigma_1 &= \mu ds + (dx_1 + A_1 ds + A_2 dt), \\ \sigma_2 &= \mu dt + (dx_2 + B_1 ds + B_2 dt), \end{aligned}$$

For functions A_j, B_j, E_j, F_j ($j = 1, 2$), and J_j ($j = 1, 2, 3, 4$) independent of μ . It is easily verified that the following vectors v_1, v_2 belong to $A(\Omega)$:

$$\begin{aligned} v_1 &= -\mu \frac{\partial}{\partial x_1} + \frac{\partial}{\partial s} - A_1 \frac{\partial}{\partial x_1} - B_1 \frac{\partial}{\partial x_2} + (E_1 \mu^3 + F_1 \mu^2 + J_2 \mu - J_4) \frac{\partial}{\partial \mu}, \\ v_2 &= -\mu \frac{\partial}{\partial x_2} + \frac{\partial}{\partial t} - A_2 \frac{\partial}{\partial x_1} - B_2 \frac{\partial}{\partial x_2} + (E_2 \mu^3 + F_2 \mu^2 + J_3 \mu - J_1) \frac{\partial}{\partial \mu}. \end{aligned}$$

We set the vector field

$$\begin{aligned} l_1 &= \frac{\partial}{\partial s} - A_1 \frac{\partial}{\partial x_1} - B_1 \frac{\partial}{\partial x_2}, \\ l_2 &= \frac{\partial}{\partial t} - A_2 \frac{\partial}{\partial x_1} - B_2 \frac{\partial}{\partial x_2}. \end{aligned}$$

The commutator $[v_1, v_2]$ is written in the following form

$$\begin{aligned} [v_1, v_2] &= (E_2 \mu^3 + F_2 \mu^2) \frac{\partial}{\partial x_1} - (E_1 \mu^3 + F_1 \mu^2) \frac{\partial}{\partial x_2} \\ &\quad + \{ (E_2 F_1 - E_1 F_2) \mu^4 + (2E_2 J_1 - 2E_1 J_2 + l_1(E_2) - l_2(E_1)) \mu^3 \\ &\quad + (F_2 J_1 - F_1 J_2 + \frac{\partial J_2}{\partial x_2} - \frac{\partial J_3}{\partial x_1} + l_1(F_2) - l_2(F_1)) \mu^2 \} \frac{\partial}{\partial \mu} + \mu u_1 + u_0, \end{aligned} \quad (3.5)$$

where the coefficients of vectors u_1 and u_0 are independent of μ .

By the integrability condition, $[v_1, v_2]$ is a linear combination of v_1 and v_2 and since $[v_1, v_2]$ does not contain $\frac{\partial}{\partial s}$ nor $\frac{\partial}{\partial t}$, v_1 and v_2 commute with each other. It follows from (3.5) that

$$E_j = 0, \quad F_j = 0 \quad (j = 1, 2), \quad \frac{\partial J_2}{\partial x_2} = \frac{\partial J_3}{\partial x_1}.$$

Thus there exists a function $f = f(x_1, x_2, s, t)$ such that

$$\frac{\partial f}{\partial x_1} = J_2, \quad \frac{\partial f}{\partial x_2} = J_3.$$

We can take new coordinates $(\lambda = \mu + f, x_1, x_2, s, t)$. With these coordinates we have

$$\begin{aligned} v_1 &= \frac{\partial}{\partial s} - \lambda \frac{\partial}{\partial x_1} - (A_1 \frac{\partial}{\partial x_1} + B_1 \frac{\partial}{\partial x_2} + C_1 \frac{\partial}{\partial \lambda}), \\ v_2 &= \frac{\partial}{\partial t} - \lambda \frac{\partial}{\partial x_2} - (A_2 \frac{\partial}{\partial x_1} + B_2 \frac{\partial}{\partial x_2} + C_2 \frac{\partial}{\partial \lambda}), \end{aligned}$$

where $C_1 = \frac{\partial f}{\partial s} - (J_4 + J_2 A_1 + J_3 B_1)$ and $C_2 = \frac{\partial f}{\partial t} - (J_1 + J_2 A_2 + J_3 B_2)$. Again taking the coefficients of μ in $[v_1, v_2]$, we obtain that

$$\frac{\partial A_1}{\partial x_2} = \frac{\partial A_2}{\partial x_1}, \quad \frac{\partial B_1}{\partial x_2} = \frac{\partial B_2}{\partial x_1}, \quad \frac{\partial C_1}{\partial x_2} = \frac{\partial C_2}{\partial x_1}.$$

Thus there exist functions R, S and T which are independent of λ such that

$$\frac{\partial R}{\partial x_j} = A_j, \quad \frac{\partial S}{\partial x_j} = B_j, \quad \frac{\partial T}{\partial x_j} = C_j \quad \text{for } j = 1, 2.$$

This completes the proof of Proposition 3.2. □

Remark. By Theorem 3.1 and Proposition 3.2 the equations of self-dual metrics are equivalent to the compatibility condition $[v_1, v_2] = 0$:

$$\begin{aligned} &\frac{\partial^2 T}{\partial t \partial x_1} - \frac{\partial^2 T}{\partial s \partial x_2} - \frac{\partial R}{\partial x_2} \frac{\partial^2 T}{\partial x_1^2} \\ &\quad + \left(\frac{\partial R}{\partial x_1} - \frac{\partial S}{\partial x_2} \right) \frac{\partial^2 T}{\partial x_1 \partial x_2} + \frac{\partial S}{\partial x_1} \frac{\partial^2 T}{\partial x_2^2} = 0, \\ &\frac{\partial^2 R}{\partial t \partial x_1} - \frac{\partial^2 R}{\partial s \partial x_2} - \frac{\partial T}{\partial x_2} \frac{\partial^2 R}{\partial x_1^2} \\ &\quad + \left(\frac{\partial R}{\partial x_1} - \frac{\partial S}{\partial x_2} \right) \frac{\partial^2 R}{\partial x_1 \partial x_2} + \frac{\partial S}{\partial x_1} \frac{\partial^2 R}{\partial x_2^2} = 0, \\ &\frac{\partial^2 S}{\partial t \partial x_1} - \frac{\partial^2 S}{\partial s \partial x_2} + \frac{\partial T}{\partial x_1} \frac{\partial^2 S}{\partial x_2^2} \\ &\quad + \left(\frac{\partial R}{\partial x_1} - \frac{\partial S}{\partial x_2} \right) \frac{\partial^2 S}{\partial x_1 \partial x_2} + \frac{\partial S}{\partial x_1} \frac{\partial^2 S}{\partial x_2^2} = 0. \end{aligned} \tag{3.6}$$

In the equation (2.26) we take $W \in \mathcal{W}_m$. In this case w_1 vanishes. Replacing $g_{0,1}, g_{1,1}$ and $g_{2,1}$ with R, S and T , respectively. The equation (2.27) reduces to (3.6). Therefore we have

Theorem 3.4 The Lie algebra V_m acts on the space of self-dual metrics. This action is transitive.

Let $\mathcal{W}_0$ be the space of $W \in \mathcal{W}$ which commute with D_0 . In the equation (2.26) we take $W \in \mathcal{W}_0$. In this case G_0 vanishes. The equation (2.27) reduces to the composed system of the self-dual Yang-Mills equations and the self-dual Einstein equations (see Example 2.1 and 2.2). Let V_0 be the Lie subalgebra of V :

$$V_0 = \left\{ \sum_{0 \leq j < r} F_j(x', D_0) D_j + E(x', D_0); \right. \\ \left. F_k(x', D_0) \in \mathcal{E} (0 \leq k < r), E(x', D_0) \in \mathcal{E} \right\}.$$

Then V_0 acts on $\mathcal{W}_0$. Thus the self-dual Einstein equations are a specialization of our integrable system.

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