

TOWARDS THE BILINEAR ISOMETRIES ON SPACES OF LIPSCHITZ FUNCTIONS

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1. SOME RESULTS ABOUT LINEAR ISOMETRIES ON SPACES OF CONTINUOUS FUNCTIONS

The classical Banach–Stone theorem [4, 30] states that if X and Y are compact Hausdorff topological spaces and T is a linear isometry from $C(X)$ onto $C(Y)$ (endowed with the supremum norm), then there exists a homeomorphism from Y onto X and a continuous function τ from Y into $S_{\mathbb{K}}$ such that

$$Tf = \tau \cdot (f \circ \varphi) \quad (f \in C(X)).$$

An important generalization of the Banach–Stone theorem was given by Holsztyński in [14, 1966] by considering into isometries. His theorem asserts that if T is a linear isometry from $C(X)$ into $C(Y)$, then there exists a closed subset Y_0 of Y , a continuous surjective map φ from Y_0 onto X , and a norm-one element $\tau \in C(Y)$ with $|\tau(y)| = 1$ for all $y \in Y_0$ such that

$$Tf(y) = \tau(y)f(\varphi(y)) \quad (f \in C(X), y \in Y_0).$$

These results have been generalized in many ways. We can cite the works by Jeang and Wong [16, 1996] on spaces of continuous scalar-valued functions vanishing at infinity, by Araujo and Font [3, 1997] on certain subspaces of scalar-valued continuous functions, by Hatori and Miura [13, 2013] on uniformly closed function algebras, by Koshimizu, Miura, Takagi and Takahasi [23, 2014], etc.

In the vector-valued case, on the one hand, Jerison [17, 1950] extended the Banach–Stone theorem and, on the other hand, Cambern [8, 1978] improved the Holsztyński theorem by characterizing into linear isometries between spaces of vector-valued continuous functions. Subsequently, many other studies have been published on this subject (see the monograph [9]). To mention a recent one, we cite Kawamura’s work [22, 2016] concerning surjective linear isometries between certain subspaces of vector-valued continuous functions.

This type of results can be very useful. For example, Botelho and Jamison [6, 2008] investigated the algebraic and topological reflexivity of $C(X)$ and $C(X, E)$ by using the representations of the into isometries given by Holsztyński and Cambern (extending, in this way, a theorem of Molnár and Zalar [25]).

2. LINEAR ISOMETRIES ON SPACES OF LIPSCHITZ FUNCTIONS

Let us recall that a map $f: X \rightarrow Y$ between metric spaces is said to be *Lipschitz* if

$$L(f) = \sup \left\{ \frac{d_Y(f(x), f(y))}{d_X(x, y)} : x, y \in X, x \neq y \right\} < \infty.$$

In such case, $L(f)$ is called the *Lipschitz constant* of f .

Given a metric space X and a normed space E , we denote by $\text{Lip}(X, E)$ the vector space of all bounded Lipschitz functions $f: X \rightarrow E$. If E is the field of real or complex numbers, we shall write simply $\text{Lip}(X)$.

On $\text{Lip}(X, E)$, it is usually considered the norm $\|f\| = \max\{\|f\|_\infty, L(f)\}$, where $\|f\|_\infty$ is the supremum norm of f . If E is a Banach space, then $(\text{Lip}(X, E), \|\cdot\|)$ is a Banach space too.

The spaces of Lipschitz functions appear in the works of many authors. See for example [29, 20, 32, 21, 12, 1, 7]. In particular, the study of the surjective linear isometries between Lipschitz-spaces started with Roy [28, 1968], Vasavada [31, 1969] and Novinger [27, 1975]. Later Weaver [32, 1999] improved these results by taking complete and 1-connected metric spaces (a metric space is r -connected if it cannot be decomposed into two nonempty disjoint sets whose distance is greater than or equal to r). On the other hand, Mayer-Wolf [24, 1981] provided a description of the surjective linear isometries on spaces of Hölder functions different from a weighted composition operator.

For our part, we stated a Lipschitz version of the Holsztyński theorem for into linear isometries (not necessarily surjective) on Lipschitz spaces [18], only under the assumption that the linear isometry takes the constant function 1 into a contraction. Moreover we extended our result to the vector-valued case [19], obtaining in this way a Lipschitz version of Cambern's theorem.

More recently, Botelho, Fleming and Jamison [5, 2011] gave a description of the linear surjective isometries on $\text{Lip}(X, E)$ under weaker conditions by using extreme points of the ball of the dual $\text{Lip}(X, E)^*$.

Finally Araujo and Dubarbie [2, 2011] gave a complete description of surjective linear isometries in a very general setting (only strict convexity on the normed spaces E and F is assumed). They considered *standard isometries* and *purely nonstandard isometries*. A map $T: \text{Lip}(X, E) \rightarrow \text{Lip}(Y, F)$ is a *standard isometry* if it has the form

$$T(f)(y) = Jy(f(\varphi(y))) \quad (f \in \text{Lip}(X, E), y \in Y),$$

where $Jy: E \rightarrow F$ is a surjective linear isometry for each $y \in Y$, the map J is constant on each 2-component of Y , and $\varphi: Y \rightarrow X$ satisfies that both φ and φ^{-1} preserve distances less than 2. The *purely nonstandard isometries*, however, are not weighted composition operators on a part of the metric space Y . Concretely, $S_\psi: \text{Lip}(Y, F) \rightarrow \text{Lip}(Y, F)$ is a *purely nonstandard isometry* if it can be described by

$$S_\psi(f)(y) = \begin{cases} f(y) & \text{if } y \in \mathcal{B}, \\ f(\psi(y)) - f(y) & \text{if } y \in \mathcal{U}. \end{cases}$$

where $\{\mathcal{B}, \mathcal{U}\}$ is certain partition of Y , and $\psi: \mathcal{U} \rightarrow \mathcal{B}$ is a map with certain metric properties. Araujo and Dubarbie proved that every nonstandard surjective isometry is the composition of a standard and a purely nonstandard isometry (when we are not in the case E and F complete and X or Y not complete).

3. BILINEAR ISOMETRIES ON SPACES OF CONTINUOUS FUNCTIONS

In the setting of bilinear isometries, we do not find such an extensive literature. The first result that we can cite is a bilinear version of the Holsztyński theorem obtained by Moreno and Rodríguez [26, 2005]. They proved that if X, Y, Z are compact Hausdorff spaces and $\Phi: C(X) \times C(Y) \rightarrow C(Z)$ is a bilinear mapping satisfying $\|\Phi(f, g)\| = \|f\| \|g\|$ for every $(f, g) \in C(X) \times C(Y)$, then there exists a closed subset Z_0 of Z , a continuous

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surjective mapping $\varphi: Z_0 \rightarrow X \times Y$, and a norm-one function $\tau \in C(Z)$ with $|\tau(z)| = 1$ such that

$$\Phi(f, g)(z) = \tau(z) f(\pi_X(\varphi(z))) g(\pi_Y(\varphi(z)))$$

for all $(f, g) \in C(X) \times C(Y)$ and $z \in Z_0$ (where π_X, π_Y stand for the natural coordinate projections).

Moreno and Rodríguez's theorem was extended by Font and Sanchis. Firstly, to the case of certain subspaces of scalar-valued continuous functions [10, 2010]; and, secondly, to the case of vector-valued continuous functions [11, 2012]. Moreover Hosseini, Font and Sanchis got a multilinear version of that theorem [15, 2015].

4. BILINEAR ISOMETRIES ON SPACES OF LIPSCHITZ FUNCTIONS

We follow the ideas of Moreno and Rodríguez and get a description of the bilinear isometries of $\text{Lip}(X)$ -spaces. Concretely, our main theorem is

Theorem 4.1. *Let X, Y, Z be compact metric spaces, and $\Phi: \text{Lip}(X) \times \text{Lip}(Y) \rightarrow \text{Lip}(Z)$ be a bilinear mapping taking the pair of constant functions one $(1_X, 1_Y)$ into a contraction and satisfying $\|\Phi(f, g)\| = \|f\| \|g\|$ for every $(f, g) \in \text{Lip}(X) \times \text{Lip}(Y)$. Then there exist a closed subset Z_0 of Z , a surjective mapping $\varphi: Z_0 \rightarrow X \times Y$ and a function $\tau \in \text{Lip}(Z)$ with $|\tau(z)| = 1$ for every $z \in Z_0$ such that*

$$\Phi(f, g)(z) = \tau(z) f(\varphi_1(z)) g(\varphi_2(z)) \quad ((f, g) \in \text{Lip}(X) \times \text{Lip}(Y), z \in Z_0).$$

Here, $\varphi_1: Z_0 \rightarrow X$ and $\varphi_2: Z_0 \rightarrow Y$ denote the compositions of φ with the natural coordinate projections. Moreover, if it is considered on $X \times Y$ the maximum distance d_∞ , it holds that φ is Lipschitz with $L(\varphi) \leq \max\{1, \text{diam}(X)/2, \text{diam}(Y)/2\}$, and $d_\infty(\varphi(z), \varphi(z')) \leq d(z, z')$ for every $z, z' \in Z_0$ with $d(z, z') < 2$.

By taking in this theorem the space Y reduced to a point, we can get as a consequence the description of the into linear isometries given earlier in [18, Theorem 2.4].

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