

Game Theoretic Analysis for an Optimal Stopping Problem in Some Class of Distribution Functions

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1. Introduction

Let $X_1, X_2, \dots, X_n, \dots$ be mutually independent and identically distributed random variables with a common cdf $F(t) = P\{X \leq t\}$ such that $E[X^+] = \int_{R_+} t dF(t) < \infty$, where $R = (-\infty, \infty)$, $R_+ = [0, \infty)$. A positive observation cost $c (\in R_{++} = (0, \infty))$ is incurred to the observation of each $X_n, n \geq 1$. If the observation process is stopped after X_n is observed, a reward $X_n - nc$ is received.

The optimal stopping time N is necessarily of the form; to stop at $N = \min\{n \mid X_n \in S\}$ for some stopping set $S \subset R$, and S is stationary and of a control-limit-type $\{X \geq x\}$ or $\{X > x\}$ for some $x \in R$, where x is called a stopping level. For this, we define that a stopping level x (or $x - 0$) means a stopping set $\{X > x\}$ (or $\{X \geq x\}$) respectively.

For any stopping level x and for any cdf F , we define an expected reward $\phi(x, F) = E[X_N - cN]$ of the stopping problem by

$$(1.1) \quad \phi(x, F) = \frac{\int_{(x, \infty)} t dF(t) - c}{\bar{F}(x)} = x + \frac{\int_{(x, \infty)} (t - x) dF(t) - c}{\bar{F}(x)},$$

where $\bar{F}(x) = 1 - F(x)$. Note that $\bar{F}(x) \rightarrow 0$ and $\phi \rightarrow -\infty$ as $x \rightarrow \infty$ and that $\bar{F}(x) \rightarrow 1$ and $\phi \rightarrow \mu_F - c$ as $x \rightarrow -\infty$ where $\mu_F = E[X] = \int_R t dF(t)$.

By the assumption $E[X^+] < \infty$, define $T_F(x)$,

$$(1.2) \quad T_F(x) = \int_x^\infty (t - x) dF(t) = \int_x^\infty \bar{F}(t) dt.$$

Lemma 1. $T_F(x)$ is continuous, non-negative, convex and non-increasing function of x . It satisfies that $T_F(x) \geq (\mu_F - x)^+$ for any $x \in R$ and that $T_F(x) \rightarrow +\infty$ as $x \rightarrow -\infty$ and $T_F(x) \rightarrow 0$ as $x \rightarrow +\infty$. T_F has a derivative a.e.. Moreover, if $T_F(x)$ is positive at any point x , it is strictly decreasing at x .

Now, redefining the expected reward $\phi(x, F)$ by (1.1') for any stopping level x and for any cdf F , we will have the optimal expected reward $\phi^o(F)$ for any cdf F .

$$(1.1') \quad \phi(x, F) = x + \frac{T_F(x) - c}{\bar{F}(x)}.$$

$$(1.3) \quad \phi^o(F) \stackrel{\text{def}}{=} \sup_{x \in R} \phi(x, F) .$$

$$(1.4) \quad \frac{d\phi(x, F)}{dF(x)} = \frac{T_F(x) - c}{\bar{F}^2(x)} .$$

The right hand side of (1.4) changes the sign from $+$ to $-$ at most one time as x goes from $-\infty$ to $+\infty$. From Lemma 1, the equation $T_F(x) = c$ for any fixed $c(c > 0)$ has a unique solution $x^o(F) \stackrel{\text{def}}{=} (T_F)^{-1}(c)$, so that the set of optimal stopping levels $\mathbf{x}^o(F)$ (which must contain the point $x^o(F)$) of (1.3) is given by

$$(1.5) \quad \mathbf{x}^o(F) = \{x \mid F(x) = F(x^o(F))\} .$$

Since the cdf F is right-continuous, this set is an interval of the form $[a, b)$.

We have the optimal expected reward $\phi^o(F)$,

$$(1.3') \quad \phi^o(F) = x^o(F) = \phi(\mathbf{x}^o(F), F) ,$$

where $\phi(A, F)$ means $\phi(y, F)$ for any y in a set A .

Lemma 2. For any given cdf F , the following stopping sets or stopping levels (i) (ii) (iii) are optimal, and the optimal expected reward is given by (1.3'') ;

- (i) the set $\{X > a\}$ or level a where $a = \min\{x \mid x \in \mathbf{x}^o(F)\}$,
- (ii) the set $\{X \geq b\}$ or level $b - 0$ where $b = \sup\{x \mid x \in \mathbf{x}^o(F)\}$,
- (iii) the set $\{X > x\}$ ($\{X \geq x\}$) or level x ($x - 0$) where $\forall x \in (a, b)$.

First, we shall derive the maximal bound ϕ^u for $\phi(x, F)$ on $R \times \mathcal{F}$

$$(1.6) \quad \begin{aligned} \phi^u &= \sup_{x \in R} \sup_{F \in \mathcal{F}} \phi(x, F) = \sup_{F \in \mathcal{F}} \phi^o(F) \\ &= \phi(x^o(F^u), F^u) = \phi(x^u, F^u) , \end{aligned}$$

where (x^u, F^u) is a joint maximizing point of $\phi(x, F)$.

Second, we shall consider $\phi(x, F)$ as a two-person zero-sum game in which the player 1 (gambler) decides his level x in R and the player 2 (nature) chooses her cdf F in $\mathcal{F}$, before the observation of $\{X_n; n \geq 1\}$. Then the minimax value ϕ^* and the maximin value ϕ_* on $R \times \mathcal{F}$,

$$(1.7) \quad \begin{aligned} \phi^* &= \inf_{F \in \mathcal{F}} \sup_{x \in R} \phi(x, F) = \inf_{F \in \mathcal{F}} \phi^o(F) \\ &= \phi(x^o(F^*), F^*) = \phi(x^*, F^*) , \end{aligned}$$

$$(1.8) \quad \phi_* = \sup_{x \in R} \inf_{F \in \mathcal{F}} \phi(x, F) = \phi(x_*, F_*) ,$$

and the saddle value ϕ^s , the saddle point (x^s, F^s) in $R \times \mathcal{F}$,

$$(1.9) \quad \phi^s = \text{value}_{x \in R, F \in \mathcal{F}} \phi(x, F) = \phi(x^s, F^s),$$

will be derived for the following two classes $\mathcal{F}(\mu, \sigma^2)$ and $\mathcal{F}(\mu, \sigma^2, M)$ of cdf's.

The class $\mathcal{F}(\mu, \sigma^2, M)$ is the set of cdf's whose mean μ , variance σ^2 and domain $[\mu - M, \mu + M]$ are assumed to be known.

$$(1.10) \quad \mathcal{F}(\mu, \sigma^2, M) = \{F \mid \int_A dF(t) = 1, \int_A t dF(t) = \mu, \\ \int_A t^2 dF(t) = \mu^2 + \sigma^2 \text{ where } A = [\mu - M, \mu + M], M \geq \sigma\}$$

The class $\mathcal{F}(\mu, \sigma^2)$ is $\mathcal{F}(\mu, \sigma^2, M)$ where M is arbitrary in R_{++} , and $\mathcal{F}(\mu)$ is $\mathcal{F}(\mu, \sigma^2)$ where σ^2 is arbitrary in R_{++} .

Let a random variable X has a mean μ with a cdf $F_\mu(t)$, then the new random variable $X - \mu$ has the mean 0 with the cdf $F_0(t) = F_\mu(t + \mu)$. The following Lemma 3 below holds immediately from the definition (1.1) of $\phi(x, F)$.

2. Some Fundamental Lemmas

Lemma 3.

$$(2.4) \quad \phi(x, F_\mu) = \mu + \phi(x - \mu, F_0) \text{ for any } x \in R.$$

Therefore, we may assume without loss of generality that all the cdf's in $\mathcal{F}$ have the mean 0. So that, we shall analyze the stopping problem in only two classes $\mathcal{F}(0, \sigma^2)$ and $\mathcal{F}(0, \sigma^2, M)$.

Lemma 4. For cdf's F_i and non-negative numbers $\lambda_i, i = 1, 2, \dots, n$, such that $\sum_{i=1}^n \lambda_i = 1$, let $F = \sum_{i=1}^n \lambda_i F_i$. Then

$$(2.5) \quad \phi(x, F) = \sum_{j=1}^n \lambda_j \phi(x, F_j) \text{ for any } x \in R, \text{ where} \\ \lambda_j(x) = \frac{\lambda_j \bar{F}_j(x)}{\sum_{i=1}^n \lambda_i \bar{F}_i(x)}.$$

Let define G_n be a discrete cdf which has n probability masses $p_i, p_i > 0$, at n points $t_i, i = 1, 2, \dots, n$, respectively ($\sum_{i=1}^n p_i = 1$), i.e., it is represented as

$$(2.6) \quad G_n(t) = (< t_1, \dots, t_n > < p_1, \dots, p_n >),$$

and $\mathcal{G}_n(\mu, \sigma^2)$ be all discrete cdf's G_n in $\mathcal{F}(\mu, \sigma^2)$. Let

$$(2.7) \quad G_2(t; q) = (< -\frac{\sigma}{q}, \sigma q > < \frac{q^2}{1+q^2}, \frac{1}{1+q^2} >).$$

for any $q, 0 < q < \infty$. Then $G_2(t; q)$ is the only two-point cdf which has the mean 0 and the variance σ^2 .

Lemma 5. The class $\mathcal{G}_2(0, \sigma^2)$ of two-point cdf's is represented with a parameter $q, 0 < q < \infty$, as follows,

$$\mathcal{G}_2(0, \sigma^2) = \{G_2(\cdot; q) \mid 0 < q < \infty\}.$$

Let us define

$$(2.8) \quad T_{\mathcal{F}}^u(x) = \sup_{F \in \mathcal{F}} T_F(x), \quad T_{\mathcal{F}}^l(x) = \inf_{F \in \mathcal{F}} T_F(x).$$

Lemma 6. Suppose $\mathcal{F} = \mathcal{F}(0)$ so that $\mu_F = 0$ for all $F \in \mathcal{F}$, then $T_{\mathcal{F}}^u(x)$ and $T_{\mathcal{F}}^l(x)$ have the same property as $T_F(x)$ in Lemma 1 with μ_F replaced by 0, except that $T_{\mathcal{F}}^l(x)$ is not always convex.

From above Lemma 6, $T_{\mathcal{F}}^u(x)$ and $T_{\mathcal{F}}^l(x)$ have inverse functions $(T_{\mathcal{F}}^u)^{-1}(c)$ and $(T_{\mathcal{F}}^l)^{-1}(c)$ for all $c, c > 0$, respectively. Thus we have shown the existence of the values of ϕ^u and ϕ^* :

$$(2.9) \quad \phi^u = \sup_{F \in \mathcal{F}} \{x \mid T_F(x) = c\} = (T_{\mathcal{F}}^u)^{-1}(c),$$

$$(2.10) \quad \phi^* = \inf_{F \in \mathcal{F}} \{x \mid T_F(x) = c\} = (T_{\mathcal{F}}^l)^{-1}(c).$$

3. The Class $\mathcal{F}(\mu, \sigma^2)$

Proposition 3. [Feller p.151] If F is an arbitrary cdf, then

$$(3.2) \quad \left(\int_A u(t)v(t)dF(t) \right)^2 \leq \left(\int_A u^2(t)dF(t) \right) \left(\int_A v^2(t)dF(t) \right)$$

for any set A and any functions u, v for which the integrals on the right exist. Furthermore, the equality sign holds if and only if

$$(3.3) \quad \int_A (au(t) + bv(t))^2 dF(t) = 0 \text{ for some } a, b \in R.$$

Note that if u and v are linearly dependent, i.e., for some $a, b \in R$, $au(t) + bv(t) = 0$, the condition (3.3) is satisfied for all $F \in \mathcal{F}$, and that if u and v are linearly independent, the condition (3.3) is satisfied only when the cdf F is degenerated at one point in a set A .

We shall calculate ϕ^u and the maximizing point (x^u, F^u) of the problem (2.9) by Proposition 3.

$$(3.4) \quad \left(\int_{(x, \infty)} (t - x) dF(t) \right)^2 \leq \left(\int_{(x, \infty)} dF(t) \right) \left(\int_{(x, \infty)} (t - x)^2 dF(t) \right),$$

$$(3.4') \quad \left(\int_{(-\infty, x]} (t-x) dF(t) \right)^2 \leq \left(\int_{(-\infty, x]} dF(t) \right) \left(\int_{(-\infty, x]} (t-x)^2 dF(t) \right).$$

Then, we obtain the maximal bound ϕ^u .

$$(3.7) \quad \phi^u = \sup_{F \in \mathcal{F}} \{x \mid T_F(x) = c\} = \frac{\sigma^2}{4c} - c.$$

Since the equality holds in two Schwartz inequalities (3.4) and (3.4'), from the remark of Proposition 3, the maximizing cdf F^u should be the two-point cdf. Then, we have

$$(3.8) \quad F^u(t) = G_2(t; \frac{\sigma}{2c}) = \left(< -2c, \frac{\sigma^2}{2c} > < \frac{\sigma^2}{\sigma^2 + 4c^2}, \frac{4c^2}{\sigma^2 + 4c^2} > \right),$$

$$(3.9) \quad x^u \in \mathbf{x}^u = \mathbf{x}^o(F^u) = \left[-2c, \frac{\sigma^2}{2c} \right).$$

Theorem 1. For a class $\mathcal{F}(0, \sigma^2)$ of cdf's, the maximal bound ϕ^u is $\sigma^2/4c - c$ by (3.7) and the maximizing point $(x^u, F^u) \in \mathbf{x}^u \times \{F^u\}$ is given by $F^u(t) = G_2(t; \sigma/2c)$ in (3.8) and $\mathbf{x}^u = [-2c, \sigma^2/2c]$ in (3.9).

Remark of Theorem 1. From Lemma 2, The equation (3.9) means that the player 1 may decide a stopping level x^u for some $x^u \in [-2c, \sigma^2/2c]$ or $\sigma^2/2c - 0$. If the player 1 decides any of the above stopping levels, he stops the process whenever $X_n = \sigma^2/2c$ is observed because the player 2 chooses only one cdf given by (3.8).

Second, we shall calculate the minimax value ϕ^* of (2.10) and the minimax-mizing point $(x^*, F^*) \in (\mathbf{x}^*, \mathcal{F}^*)$.

From Lemma 6, $T_F^t(x) \geq (-x)^+$ for all $x \in R$. Then it holds that $T_{F^*}(x) = (-x)^+ \leq T_F^t(x)$ for $x \in (-\infty, -c]$ if a cdf F^* , which has all the mass on $[-c, \infty)$, is contained in $\mathcal{F}$. Since $T_{F^*}(x) = (-x)^+$ is strictly decreasing on $(-\infty, -c]$, we have

$$(3.10) \quad \phi^* = \inf_{F \in \mathcal{F}} \{x \mid T_F(x) = c\} = \{x \mid T_{F^*}(x) = c\} = -c.$$

Such a class $\mathcal{F}^*$ of cdf's F^* always exists in $\mathcal{F}$ for all $c, c > 0$.

$$(3.11) \quad \mathcal{F}^* = \{F \mid \int_{[-c, \infty)} dF(t) = 1, F \in \mathcal{F}\}.$$

In particular, we can find the class $\mathcal{G}_2^* = \mathcal{G}_2^*(0, \sigma^2)$ of two-point cdf's in $\mathcal{F}^*$ from Lemma 5.

$$(3.11') \quad \mathcal{G}_2^* = \{G_2(\cdot; q) \mid q \geq \frac{\sigma}{c}\}.$$

It is easily shown that for any $F^* \in \mathcal{F}^*$ it is optimal for the player 1 to stop the process immediately. That is,

$$(3.12) \quad \mathbf{x}^* = \mathbf{x}^o(F^*) = (-\infty, -c) \text{ for all } F^* \in \mathcal{F}^*.$$

Theorem 2. For a class $\mathcal{F}(0, \sigma^2)$ of cdf's, the minimax value ϕ^* is $-c$ by (3.10) and the minimax-mizing point $(x^*, F^*) \in (\mathbf{x}^*, \mathcal{F}^*)$ is given by (3.11) and (3.12). In particular, there exists the class $\mathcal{G}_2^*$ of two-point cdf's in $\mathcal{F}^*$ by (3.11').

Now, we shall derive the saddle value ϕ^s for $\phi(x, F)$ in $\mathcal{F} = \mathcal{F}(\mu, \sigma^2)$. We have a candidate $(\mathbf{x}^s, \mathcal{F}^s)$ for a set of saddle points $(\mathbf{x}^s, \mathcal{F}^s)$.

Theorem 3. For a class $\mathcal{F}(0, \sigma^2)$ of cdf's, the saddle value ϕ^s is $-c$ and the saddle point $(x^s, F^s) \in \mathbf{x}^s \times \mathcal{F}^s$ is given by $\mathbf{x}^s = \mathbf{x}^*$, $\mathcal{F}^s = \mathcal{F}^*$ and $\mathcal{G}_2^s = \mathcal{G}_2^* \subset \mathcal{F}^s$ defined in Theorem 2.

Theorem 3 says the class $\mathcal{F}(\mu, \sigma^2)$ is so rich for the player 2 that the player 1 must stop immediately. In this case, the information of the value σ^2 is useless for the player 1.

4. The Class $\mathcal{F}(\mu, \sigma^2, M)$

In this section, we shall derive the maximal bound ϕ^u and the saddle value ϕ^s in the more restrictive and interesting class $\mathcal{F} = \mathcal{F}(0, \sigma^2, M)$ (see (1.10)).

Theorem 4. For a class $\mathcal{F}(0, \sigma^2, M)$ of cdf's, $\sigma < M$, the maximal bound ϕ^u and the maximizing point $(x^u, F^u) \in \mathbf{x}^u \times \mathcal{F}^u$ are as follows:

(i) When $0 \leq c \leq \sigma^2/2M$,

$$\phi^u = M - c(1 + \frac{M^2}{\sigma^2}), \quad \mathbf{x}^u = [-\frac{\sigma^2}{M}, M),$$

$$F^u(t) = G_2(t; \frac{M}{\sigma}) = (< -\frac{\sigma^2}{M}, M > < \frac{M^2, \sigma^2}{\sigma^2 + M^2} >).$$

(ii) When $\sigma^2/2M \leq c \leq M/2$, the same result as Theorem 1 holds, i.e.,

$$\phi^u = \frac{\sigma^2}{4c} - c, \quad \mathbf{x}^u = [-2c, \frac{\sigma^2}{2c}),$$

$$F^u(t) = G_2(t; \frac{\sigma}{2c}) = (< -2c, \frac{\sigma^2}{2c} > < \frac{\sigma^2, 4c^2}{\sigma^2 + 4c^2} >).$$

(iii) When $M/2 \leq c \leq M$,

$$\phi^u = \frac{\sigma^2}{M} - c(1 + \frac{\sigma^2}{M^2}), \quad \mathbf{x}^u = [-M, \frac{\sigma^2}{M}),$$

$$F^u(t) = G_2(t; \frac{\sigma}{M}) = (< -M, \frac{\sigma^2}{M} > < \frac{\sigma^2, M^2}{\sigma^2 + M^2} >).$$

Now, we shall derive the saddle value ϕ^s .

We confine our consideration to the case:

$$(4.4) \quad 0 < c < \sigma^2/M.$$

On the other hand, it holds that

$$(4.2') \quad \inf_{F \in \mathcal{F}} \phi(x, F) \leq \phi(x, G_2(\cdot; M/\sigma)) = -c \text{ for } x \in [-M, -\sigma^2/M],$$

$$(4.3') \quad \inf_{F \in \mathcal{F}} \phi(x, F) \leq \phi(x, G_2(\cdot; \sigma/M)) = -\infty \text{ for } x \in [\sigma^2/M, M],$$

because the player 1 stops immediately in the case of (4.2') or he cannot stop in the case of (4.3'). Then, the player 1 must decide his stopping level x in the interval

$$(4.5) \quad \mathbf{x}^M \stackrel{\text{def}}{=} \left[-\frac{\sigma^2}{M}, \frac{\sigma^2}{M}\right],$$

in order not to make his reward $\inf_{F \in \mathcal{F}} \phi(x, F) \leq -c$, where $-c$ is the reward of immediately stopping or the saddle value $\phi^s = -c$ in Section 3.

Lemma 7. For any strategy (x, F) , $x \in \mathbf{x}^M$, $F \in \mathcal{F}$, if F has a probability mass p at any point y in the interval (x, M) and satisfies $\phi(x, F) \geq -c$, then there exists a cdf $F'' \in \mathcal{F}$ such that F'' has no mass in the interval (x, M) , and it satisfies $\phi(x - 0, F'') \leq \phi(x, F)$.

Lemma 8. For any strategy $x \in \mathbf{x}^M$, $F \in \mathcal{F}$, if F has probability mass p at any point y in the interval $(-M, x)$ and it satisfies $\phi(x, F) \geq -c$, then there exists a cdf $F'' \in \mathcal{F}$ such that F'' has no mass in the interval $(-M, x)$, and it satisfies $\phi(x, F'') \leq \phi(x, F)$.

Let us define for any $x \in [-\sigma^2/M, \sigma^2/M]$, a three-point cdf $G_3^M(\cdot; x) \in \mathcal{F}$ which has all the mass at three points $-M, x, M$ with the mean 0 and the variance σ^2 . This cdf is uniquely determined by

$$(4.11) \quad G_3^M(t; x) = \langle -M, x, M \rangle \langle \frac{Mx + \sigma^2}{2M(M+x)}, \frac{M^2 - \sigma^2}{M^2 - x^2}, \frac{\sigma^2 - Mx}{2M(M-x)} \rangle,$$

and let $\mathcal{G}_3^M = \{G_3^M(t; x) \mid -\sigma^2/M \leq x \leq \sigma^2/M\}$. Note that if $x = \sigma^2/M$ or $-\sigma^2/M$, $G_3^M(t; x)$ becomes the two-point cdf $G_2(t; \sigma/M)$ or $G_2(t; M/\sigma)$ respectively.

The player 1 would decide a stopping level x in the following set

$$(4.13) \quad \{x \mid \phi(x, F) \geq -c \text{ for all } F \in \mathcal{F}\} \cap \mathbf{x}^M \stackrel{\text{def}}{=} \mathbf{x}_c^M.$$

This set is not empty because $x = -c$ is contained in it.

If there exists a point $x^s \in \mathbf{x}_c^M$ such that

$$(4.15) \quad \phi(x^s, G_3^M(\cdot; x^s)) = (T_{\mathcal{G}_3^M}^t)^{-1}(c) (= \phi(x^s - 0, G_3^M(\cdot; x^s))) \geq -c,$$

the strategy $(x^s, G_3^M(\cdot; x^s))$, $x \in \mathbf{x}_c^M$, $G_3^M(\cdot; x^s) \in \mathcal{G}_3^M \subset \mathcal{F}$, is the saddle point and $\phi^s = (T_{\mathcal{G}_3^M}^t)^{-1}(c)$ is the saddle value. Because, from (4.14), Proposition 1 and (2.10), the following relation is satisfied.

$$\phi(x^s - 0, G_3^M(\cdot; x^s)) \leq \sup_{x \in \mathbf{x}_c^M} \inf_{F \in \mathcal{F}} \phi(x, F) \leq \inf_{F \in \mathcal{F}} \sup_{x \in \mathbf{x}_c^M} \phi(x, F)$$

$$\leq \inf_{F \in \mathcal{G}_3^M} \sup_{x \in \mathbf{x}_3^M} \phi(x, F) = (T_{\mathcal{G}_3^M}^\ell)^{-1}(c) = \phi(x^s, G_3^M(\cdot; x^s)).$$

Theorem 5. For a class $\mathcal{F}(0, \sigma^2, M)$ of cdf's, $\sigma \leq M$, the saddle point $(x^s, F^s) \in (\mathbf{x}^s, \mathcal{F}^s)$ is as follows:

(i) When $\sigma^2/M \leq c \leq M$, the same result as Theorem 3 holds, that is,

$$\phi^s = -c, \mathbf{x}^s = [-M, -c] \text{ and}$$

$$\mathcal{F}^s = \{F \mid \int_{[-c, M]} dF(t) = 1, F \in \mathcal{F}(0, \sigma^2, M)\}.$$

(ii) When $0 < c < \sigma^2/M$,

$$\phi^s = (\sigma^2/M - c)^+ - c, \mathbf{x}^s = \{x^s\}, x^s = (\sigma^2/M - c)^+ - c \text{ and}$$

$$\mathcal{F}^s = \{F^s\}, F^s(t) = G_3^M(t; x^s) \text{ defined by (4.11)}.$$

References

- [1] Billingsley, P., *Probability and Measure*, Wiley, New York, 1978.
- [2] Chow, Y.S., H. Robbins and D. Siegmund, *Great Expectations : The Theory of Optimal Stopping*, Houghton-Mifflin, New York, 1971.
- [3] Ekeland, I. and R. Teman, *Convex Analysis and Variational Problems*, North-Holland, 1976.
- [4] Feller, W., *An Introduction to Probability Theory and its Applications*, Vol. II, Wiley, New York, 1968.
- [5] Nakagami, J. and M. Yasuda, "Saddle Point of an Inventory Problem", *J. Information and Optimization Sciences*, 2 1981, pp. 181-191.
- [6] Pickands III, Jr., "Extreme Order Statistics with Cost of Sampling", *Adv. Appl. Prob.*, 15 1983, pp. 783-797.

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