

Periodic behavior of solutions to a continuous casting problem

千葉大自然科学 篠田淳一 (Junichi Shinoda)

1. Introduction

In this paper we consider a continuous casting problem

$$(P)^\nu \quad \begin{cases} \partial_t \eta + \nu \partial_z \eta - \Delta \theta = 0 & \text{in } Q_\infty :=]0, \infty[\times \Omega, \\ \eta \in \beta(\theta) & \text{in } Q_\infty, \\ \frac{\partial \theta}{\partial n} + g(t, x, \theta) = 0 & \text{on } \Sigma_\infty^N :=]0, \infty[\times \Gamma_N, \\ \theta = M & \text{on } \Sigma_\infty^0 :=]0, \infty[\times \Gamma_0, \\ \theta = -m & \text{on } \Sigma_\infty^L :=]0, \infty[\times \Gamma_L, \end{cases}$$

under periodic (in time) boundary condition

$$g(t+T, x, \theta) = g(t, x, \theta) \quad \text{on } \Sigma_\infty^N \times \mathbf{R},$$

for a given period $T > 0$. Here $\Omega =]-l, l[\times]0, L[$, $\Gamma_N = \{l, -l\} \times]0, L[$, $\Gamma_0 =]-l, l[\times \{0\}$, $\Gamma_L =]-l, l[\times \{L\}$, $L, l > 0$, $x = (y, z)$; ν, m and M are given constants with $\nu \geq 0$ and $m, M > 0$; β is a maximal monotone graph of the form

$$\beta(r) = \begin{cases} \lambda + \int_0^r b(\tau) d\tau & \text{if } r > 0, \\ [0, \lambda] & \text{if } r = 0, \\ \int_0^r b(\tau) d\tau & \text{if } r < 0, \end{cases}$$

for a given constant $\lambda > 0$ and a locally bounded measurable function b such that

$$(1.1) \quad b(r) \geq b_* > 0 \quad \text{for a.e. } r \in \mathbf{R}.$$

Furthermore $g = g(t, x, \theta)$ is a given function on $\mathbf{R}_+ \times \Gamma_N \times \mathbf{R}$ such that

(g1) $g(t, x, \cdot)$ is a nondecreasing function for a.e. $(t, x) \in \mathbf{R}_+ \times \Gamma_N$;

(g2) $g(\cdot, \cdot, \theta) \in L^2_{loc}(\mathbf{R}_+; L^2(\Gamma_N))$ for all $\theta \in \mathbf{R}$;

(g3) For any $K > 0$ there is a constant $C_g(K) > 0$ such that

$$|g(t, x, \theta_1) - g(t, x, \theta_2)| \leq C_g(K) |\theta_1 - \theta_2|$$

for all $\theta_1, \theta_2 \in [-K, K]$ and a.e. $(t, x) \in \mathbf{R}_+ \times \Gamma_N$;

(g4) There exist constants $K_1, K_2 > 0$ such that

$$g(t, x, -K_1) \leq 0, \quad g(t, x, K_2) \geq 0 \quad \text{for a.e. } (t, x) \in \mathbf{R}_+ \times \Gamma_N.$$

For details of continuous casting problems, see Rodrigues [5], Rodrigues-Yi [6], Yi [9] and the literatures in their references. We remark here that problem $(P)^0$ is a Stefan problem. For results to periodic solutions of Stefan problems we refer to Aiki *et al.* [1], Damlamian-Kenmochi [2] and Haraux-Kenmochi [3]. In the following chapters, we shall discuss problem $(P)^\nu$ due to Shinoda [7,8].

2. Main results

Throughout this paper we denote $Q_S =]0, S[\times \Omega$, $\Sigma_S^N =]0, S[\times \Gamma_N$, etc. for $S \in]0, +\infty[$.

Now let us give a notion of a weak solution on an interval of the form $[0, S]$ or $[0, +\infty[$.

Definition 2.1. Let S be a positive number. Then a couple $(\theta, \eta) \in L^2(0, S; H^1(\Omega)) \times L^\infty(Q_S)$ is called a weak solution of $(P)^\nu$ on $[0, S]$ when the following four conditions are satisfied:

(w1) $\eta \in C_w([0, S]; L^2(\Omega))$, that is, η is a weakly continuous function from $[0, S]$ to $L^2(\Omega)$;

(w2) $\theta = M$ a.e. on Σ_S^0 and $\theta = -m$ a.e. on Σ_S^L ;

(w3) $\eta \in \beta(\theta)$ a.e. in Q_S ;

(w4) for any $\varphi \in W_S := \{\varphi \in H^1(Q_S); \varphi(S, \cdot) = 0 \text{ a.e. in } \Omega, \varphi = 0 \text{ a.e. on } \Sigma_S^D\}$,

$$-\int_{Q_S} \eta(\partial_t \varphi + \nu \partial_z \varphi) dx dt + \int_{Q_S} \nabla \theta \nabla \varphi dx dt + \int_{\Sigma_S^N} g(\cdot, \cdot, \theta) \varphi d\Gamma dt = \int_{\Omega} \eta(0, \cdot) \varphi(0, \cdot) dx,$$

where $\Sigma_S^D =]0, S[\times \Gamma_D$, $\Gamma_D = \Gamma_0 \cup \Gamma_L$. In the case when $S = +\infty$, (θ, η) is called a weak solution of $(P)^\nu$ on $\mathbf{R}_+$, if (θ, η) is a weak solution of $(P)^\nu$ on $[0, S]$ for any finite $S > 0$.

Definition 2.2. Let $0 < S \leq +\infty$ and let (θ_0, η_0) be a pair of functions in $L^\infty(\Omega)$ satisfying $\eta_0 \in \beta(\theta_0)$ a.e. in Ω . Then we call a pair (θ, η) a weak solution for $CP(\theta_0, \eta_0)^\nu$ on $[0, S]$ ($\mathbf{R}_+$ if $S = +\infty$) if (θ, η) is a weak solution of $(P)^\nu$ on $[0, S]$ and the initial conditions $\theta(0, \cdot) = \theta_0$ and $\eta(0, \cdot) = \eta_0$ are satisfied, respectively.

Concerning the existence and the uniqueness results for $CP(\theta_0, \eta_0)^\nu$, we quote them from Rodrigues-Yi [6]. The first proposition assures the existence of a weak solution for $CP(\theta_0, \eta_0)^\nu$.

Proposition 2.1. (cf. [6;theorem 1]) Let $(\theta_0, \eta_0) \in (L^\infty(\Omega))^2$ be any pair of functions such that $\eta_0 \in \beta(\theta_0)$ a.e. in Ω . Choose two positive constants $\tilde{K}_1$ and $\tilde{K}_2$ so that $\tilde{K}_i \geq \max\{m, M, K_i\}$, $i = 1, 2$, and that

$$\beta(-\tilde{K}_1) \leq \eta_0(x) \leq \beta(\tilde{K}_2) \quad \text{for a.e. } x \in \Omega.$$

Then, there exists at least one weak solution (θ, η) for $CP(\theta_0, \eta_0)^\nu$ on $\mathbf{R}_+$ such that

$$\beta(-\tilde{K}_1) \leq \eta(t, x) \leq \beta(\tilde{K}_2) \quad \text{for a.e. } (t, x) \in Q_\infty,$$

hence

$$-\tilde{K}_1 \leq \theta(t, x) \leq \tilde{K}_2 \quad \text{for a.e. } (t, x) \in Q_\infty.$$

Remark 2.1. In view of the proof of [6;theorem 1] we may assume that the solution (θ, η) obtained in proposition 2.1 is constructed as a limit of an approximate solution $(\theta_\varepsilon, \beta_\varepsilon(\theta_\varepsilon))$ of

$$\begin{cases} \partial_t \beta_\varepsilon(\theta_\varepsilon) + \nu \partial_z \beta_\varepsilon(\theta_\varepsilon) - \Delta \theta_\varepsilon = 0 & \text{in } Q_\infty, \\ \frac{\partial \theta_\varepsilon}{\partial n} + g_\varepsilon(t, x, \theta_\varepsilon) = 0 & \text{on } \Sigma_\infty^N, \\ \theta_\varepsilon = M & \text{on } \Sigma_\infty^0, \\ \theta_\varepsilon = -m & \text{on } \Sigma_\infty^L, \\ \theta_\varepsilon(0, \cdot) = \theta_{0\varepsilon} & \text{in } \Omega, \end{cases}$$

in the sense that for some subsequence $\{\varepsilon_n\}$ of $\{\varepsilon\}$

$$(2.1) \quad \beta_{\varepsilon_n}(\theta_{\varepsilon_n}) \rightarrow \eta \quad \text{weakly}^* \text{ in } L_{loc}^\infty(\mathbf{R}_+; L^\infty(\Omega));$$

$$(2.2) \quad \theta_{\varepsilon_n} \rightarrow \theta \quad \text{weakly in } L_{loc}^2(\mathbf{R}_+; H^1(\Omega)) \cap H_{loc}^1(Q_\infty);$$

$$(2.3) \quad g_{\varepsilon_n}(\cdot, \cdot, \theta_{\varepsilon_n}) \rightarrow g(\cdot, \cdot, \theta) \quad \text{in } L_{loc}^2(\mathbf{R}_+; L^2(\Gamma_N)).$$

Here $\{\beta_\varepsilon\}$, $\{g_\varepsilon\}$ and $\{\theta_{0\varepsilon}\}$ are smooth approximations to β , g and θ_0 , respectively.

Furthermore, $\{\beta_\varepsilon\}$ satisfies (1.1) with $b_\varepsilon = \beta'_\varepsilon$, $\beta_\varepsilon(0) = 0$, $\beta'_\varepsilon \leq 1/\varepsilon$ and

$$\beta_\varepsilon(r) \rightarrow \beta(r) \quad \text{for any compact interval in } \mathbf{R} \setminus \{0\} \text{ as } \varepsilon \rightarrow 0;$$

$\{g_\varepsilon\}$ satisfies (g1)~(g4) and

$$g_\varepsilon(\cdot, \cdot, \theta) \rightarrow g(\cdot, \cdot, \theta) \quad \text{in } L_{loc}^2(\mathbf{R}_+; L^2(\Gamma_N))$$

uniformly with respect to θ on any compact set in $\mathbf{R}$ as $\varepsilon \rightarrow 0$;

$\{\theta_{0\varepsilon}\}$ satisfies the compatibility conditions

$$(2.4) \quad \theta_{0\varepsilon} = M \quad \text{on } \Gamma_0 \quad \text{and} \quad \theta_{0\varepsilon} = -m \quad \text{on } \Gamma_L$$

and

$$\beta_\varepsilon(\theta_{0\varepsilon}) \rightarrow \eta_0 \quad \text{in } L^2(\Omega) \text{ as } \varepsilon \rightarrow 0.$$

The second proposition is the continuous dependence of the weak solutions. This requires the following condition to a weak solution (θ, η) of $(P)^\nu$:

For some positive constants $\delta, \rho > 0$,

$$(2.5) \quad \theta(t, y, z) \geq \rho > 0 \quad \text{a.e. in } Q_\infty^\delta := \{(t, y, z) \in Q_\infty; 0 < z < \delta\}.$$

Proposition 2.2. (cf. [6;theorem 2]) *Fix $\nu > 0$. Let (θ_1, η_1) and (θ_2, η_2) be two weak solutions for $CP(\theta_{10}, \eta_{10})^\nu$ and $CP(\theta_{20}, \eta_{20})^\nu$, respectively. If at least one of (θ_i, η_i) satisfies (2.5), then the following is valid:*

$$(2.6) \quad \int_{Q_\infty} |\eta_1 - \eta_2| \, dx dt \leq \frac{L}{\nu} \int_\Omega |\eta_{10} - \eta_{20}| \, dx.$$

As a direct corollary we have:

Corollary 2.1. *If at least one of the weak solution (θ, η) for $CP(\theta_0, \eta_0)^\nu$ on $\mathbf{R}_+$ satisfies (2.5), then (θ, η) is the only weak solution for $CP(\theta_0, \eta_0)^\nu$ on $\mathbf{R}_+$.*

Using well-known L^1 -space technique, we have in the manner similar to that of [1]:

Proposition 2.3. *Let $\nu > 0$, and let $(\theta_1, \eta_1), (\theta_2, \eta_2)$ be two weak solutions for $CP(\theta_{10}, \eta_{10})^\nu$ and $CP(\theta_{20}, \eta_{20})^\nu$ on $\mathbf{R}_+$ satisfying (2.5), respectively. Then we have*

$$|[\eta_1(t, \cdot) - \eta_2(t, \cdot)]^+|_{L^1(\Omega)} \leq |[\eta_1(s, \cdot) - \eta_2(s, \cdot)]^+|_{L^1(\Omega)} \quad \text{for any } s, t \in \mathbf{R}_+ \text{ with } s \leq t,$$

and

$$|\eta_1(t, \cdot) - \eta_2(t, \cdot)|_{L^1(\Omega)} \leq |\eta_1(s, \cdot) - \eta_2(s, \cdot)|_{L^1(\Omega)} \quad \text{for any } s, t \in \mathbf{R}_+ \text{ with } s \leq t.$$

In particular, if $\eta_{10} \leq \eta_{20}$ a.e. in Ω then

$$\eta_1 \leq \eta_2 \quad \text{hence} \quad \theta_1 \leq \theta_2 \quad \text{a.e. in } Q_\infty.$$

Remark 2.2. Propositions 2.1, 2.3 and corollary 2.1 are also valid for $\nu = 0$. We can prove them by using similar techniques to those in the proofs of [6;theorem 1,4;theorem 4.2,1;lemma 2.1], respectively.

Next we state a definition of a T -periodic weak solution of $(P)^\nu$ on $\mathbf{R}_+$.

Definition 2.3. Let T be a given positive number (period). Then (θ, η) is called a T -periodic weak solution of $(P)^\nu$ on $\mathbf{R}_+$ provided that (θ, η) is a weak solution of $(P)^\nu$ on $\mathbf{R}_+$ and satisfies the periodic conditions $\theta(t + T, \cdot) = \theta(t, \cdot)$ and $\eta(t + T, \cdot) = \eta(t, \cdot)$ for all $t \in \mathbf{R}_+$.

Finally we mention the main results for the T -periodic weak solution of $(P)^\nu$ on $\mathbf{R}_+$.

Theorem 2.1. Let $\nu > 0$. Assume that the periodicity condition

$$(2.7) \quad g(t + T, x, \theta) = g(t, x, \theta) \quad \text{for all } \theta \in \mathbf{R}_+ \text{ and a.e. } (t, x) \in \mathbf{R}_+ \times \Gamma_N$$

holds. Then there exists one and only one T -periodic weak solution $(\theta_p^\nu, \eta_p^\nu)$ of $(P)^\nu$ on $\mathbf{R}_+$.

Theorem 2.2. Assume that the same conditions as in theorem 2.1 hold. Then for any weak solution (θ, η) satisfying (2.5) for some positive constants $\delta, \rho > 0$, we have

$$\eta_p^\nu(t, \cdot) - \eta(t, \cdot) \rightarrow 0 \quad \text{and} \quad \theta_p^\nu(t, \cdot) - \theta(t, \cdot) \rightarrow 0 \quad \text{in } L^q(\Omega) \text{ for all } q \geq 1 \text{ as } t \rightarrow +\infty.$$

Remark 2.3. Yi [9] treated the periodic solutions under the Dirichlet boundary condition. He proved there the existence of periodic solutions using Schauder fixed point theorem.

Remark 2.4. There exists a T -periodic weak solution (θ_p^0, η_p^0) of $(P)^0$ on $\mathbf{R}_+$ under the periodicity condition (2.7). But for the uniqueness of T -periodic weak solutions of $(P)^0$ on $\mathbf{R}_+$, we can only prove that of $g(\cdot, \cdot, \theta_p^0)$ on Σ_∞^N and moreover that of θ_p^0 in Q_∞ (see [7,8] and also [2]).

3. Lemmas

In this chapter we prepare some lemmas to prove theorems 2.1 and 2.2.

Firstly we define a function $g_* = g_*(\theta)$ by $g_*(\theta) = C_g(K_3)[\theta + K_3]^+$ for $\theta \in \mathbf{R}$, where $K_3 = \max\{M, m, K_1, K_2\}$. Then the following is valid.

Lemma 3.1. *g_* defined as above is nondecreasing and satisfies*

$$g(t, x, \theta) \leq g_*(\theta) \quad \text{for all } \theta \leq K_3 \text{ and a.e. } (t, x) \in \mathbf{R}_+ \times \Gamma_N,$$

Next we construct a smooth function $\theta_* = \theta_*(x)$ satisfying for any $\varepsilon > 0$ the following system

$$(3.1) \quad \begin{cases} \nu \partial_z \beta_\varepsilon(\theta_*) - \Delta \theta_* \leq 0 & \text{in } \Omega, \\ \frac{\partial \theta_*}{\partial n} + g_*(\theta_*) \leq 0 & \text{on } \Gamma_N, \\ \theta_* \leq M & \text{in } \overline{\Omega}, \\ \theta_* \leq -K_3 & \text{on } \Gamma_L. \end{cases}$$

Choose a function $\chi = \chi(y) \in C^\infty([-l, l])$ such that

$$0 \leq \chi \leq \frac{M}{2} \quad \text{in }]-l, l[,$$

and

$$\pm \frac{\partial \chi}{\partial y}(\pm l) + g_*(M) \leq 0.$$

For a positive parameter μ , let us define θ_* by

$$\theta_*(y, z) = -\mu z + \chi(y) + \frac{M}{2}.$$

Then we see that θ_* satisfies for some constants $\delta, \rho > 0$

$$(3.2) \quad \theta_*(y, z) \geq \rho \quad \text{in } \Omega_\delta := \{(y, z) \in \Omega; 0 < z < \delta\}.$$

Moreover it is readily seen that (3.1) is fulfilled for sufficiently large μ dependent upon ν . Thus we have the following lemma.

Lemma 3.2. *There is a smooth function $\theta_* = \theta_*(x)$ on Ω which is independent of ε and satisfies (3.1) and (3.2) for some positive constants δ, ρ .*

Put $\eta_* = \beta(\theta_*)$. We remark that η_* is a.e. defined since the Lebesgue measure of the set $\{x \in \Omega; \theta_*(x) = 0\}$ is zero. Then we have:

Lemma 3.3. *The unique weak solution (θ, η) for $CP(\theta_*, \eta_*)^\nu$ on $\mathbf{R}_+$ satisfies (2.5) for some $\delta, \rho > 0$.*

Proof. Let $\{\theta_{0\varepsilon}\} \subset C^\infty(\overline{\Omega})$ such that $\theta_* \leq \theta_{0\varepsilon}$ in Ω , $\beta_\varepsilon(\theta_{0\varepsilon}) \rightarrow \eta_*$ in $L^2(\Omega)$ as $\varepsilon \rightarrow 0$, and that (2.4) holds. Recalling proposition 2.1 and remark 2.1, we get a weak solution (θ, η) for $CP(\theta_*, \eta_*)^\nu$ on $\mathbf{R}_+$ as a limit of an approximate solution θ_{ε_n} corresponding to initial value $\theta_{0\varepsilon_n}$ in the sense of (2.1)~(2.3) for some subsequence $\{\varepsilon_n\}$ of $\{\varepsilon\}$. We note that for any $\varepsilon \in]0, 1]$

$$(3.3) \quad \partial_t(\beta_\varepsilon(\theta_*) - \beta_\varepsilon(\theta_\varepsilon)) + \nu \partial_z(\beta_\varepsilon(\theta_*) - \beta_\varepsilon(\theta_\varepsilon)) - \Delta(\theta_* - \theta_\varepsilon) \leq 0 \quad \text{in } Q_\infty.$$

Now let us denote by $\{\sigma_m\}$ a sequence of smooth functions on $\mathbf{R}$ such that $\sigma_m(0) = 0$, and for any $r \in \mathbf{R}$, $\sigma'_m(r) \geq 0$, $-1 \leq \sigma_m(r) \leq 1$ and

$$\sigma_m(r) \rightarrow \sigma_0(r) := \begin{cases} 1 & \text{for } r > 0, \\ 0 & \text{for } r = 0, \\ -1 & \text{for } r < 0, \end{cases} \quad \text{as } m \rightarrow +\infty.$$

Multiply (3.3) by $\sigma_m([\theta_* - \theta_\varepsilon]^+)$ and integrate it over Q_t . By lemma 3.1 and 3.2,

$$\begin{aligned}
& - \int_{Q_t} \Delta(\theta_* - \theta_\varepsilon) \sigma_m([\theta_* - \theta_\varepsilon]^+) dx d\tau \\
& \geq \int_{\Sigma_t^N} (g_*(\theta_*) - g_\varepsilon(\cdot, \cdot, \theta_\varepsilon)) \sigma_m([\theta_* - \theta_\varepsilon]^+) d\Gamma d\tau \\
& \geq \int_{\Sigma_t^N} (g_*(\cdot, \cdot, \theta_\varepsilon) - g_\varepsilon(\cdot, \cdot, \theta_\varepsilon)) \sigma_m([\theta_* - \theta_\varepsilon]^+) d\Gamma d\tau \\
& \rightarrow \int_{\Sigma_t^N} (g_*(\cdot, \cdot, \theta_\varepsilon) - g_\varepsilon(\cdot, \cdot, \theta_\varepsilon)) \sigma_0([\theta_* - \theta_\varepsilon]^+) d\Gamma d\tau \quad \text{as } m \rightarrow +\infty.
\end{aligned}$$

By the strict monotonicity of β_ε ,

$$\begin{aligned}
& \int_{Q_t} \partial_z(\beta_\varepsilon(\theta_*) - \beta_\varepsilon(\theta_\varepsilon)) \sigma_m([\theta_* - \theta_\varepsilon]^+) dx d\tau \\
& \rightarrow \int_{Q_t} \partial_z(\beta_\varepsilon(\theta_*) - \beta_\varepsilon(\theta_\varepsilon)) \sigma_0([\theta_* - \theta_\varepsilon]^+) dx d\tau \quad \text{as } m \rightarrow +\infty \\
& = \int_{Q_t} \partial_z(\beta_\varepsilon(\theta_*) - \beta_\varepsilon(\theta_\varepsilon)) \sigma_0([\beta_\varepsilon(\theta_*) - \beta_\varepsilon(\theta_\varepsilon)]^+) dx d\tau \\
& = \int_0^t \int_{-l}^l [\beta_\varepsilon(\theta_*) - \beta_\varepsilon(\theta_\varepsilon)]^+ dx' d\tau \Big|_{z=0}^L = 0,
\end{aligned}$$

and

$$\begin{aligned}
& \int_{Q_t} \partial_t(\beta_\varepsilon(\theta_*) - \beta_\varepsilon(\theta_\varepsilon)) \sigma_m([\theta_* - \theta_\varepsilon]^+) dx d\tau \\
& \rightarrow \int_{Q_t} \partial_t(\beta_\varepsilon(\theta_*) - \beta_\varepsilon(\theta_\varepsilon)) \sigma_0([\theta_* - \theta_\varepsilon]^+) dx d\tau \quad \text{as } m \rightarrow +\infty \\
& = \int_{Q_t} \partial_t(\beta_\varepsilon(\theta_*) - \beta_\varepsilon(\theta_\varepsilon)) \sigma_0([\beta_\varepsilon(\theta_*) - \beta_\varepsilon(\theta_\varepsilon)]^+) dx d\tau \\
& = \int_{\Omega} [\beta_\varepsilon(\theta_*) - \beta_\varepsilon(\theta_\varepsilon(t, \cdot))]^+ dx
\end{aligned}$$

Therefore we have for all $t \in \mathbf{R}_+$

$$\int_{\Omega} [\beta_\varepsilon(\theta_*) - \beta_\varepsilon(\theta_\varepsilon(t, \cdot))]^+ dx + \int_{\Sigma_t^N} (g_*(\cdot, \cdot, \theta_\varepsilon) - g_\varepsilon(\cdot, \cdot, \theta_\varepsilon)) \sigma_0([\theta_* - \theta_\varepsilon]^+) d\Gamma d\tau \leq 0.$$

Taking $\varepsilon = \varepsilon_n$ and letting $n \rightarrow +\infty$ we have by lemma 3.1

$$\int_{\Omega} [\eta_* - \eta(t, \cdot)]^+ dx \leq 0 \quad \text{for all } t \in \mathbf{R}_+,$$

which implies that

$$(3.4) \quad \eta_* \leq \eta \quad \text{hence} \quad \theta_* \leq \theta \quad \text{a.e. in } Q_\infty.$$

Because of lemma 3.2, we thus have

$$\theta(t, y, z) \geq \rho \quad \text{a.e. in } Q_\infty^\delta$$

for the same constants δ and ρ as in (3.2). By corollary 2.1 we see that (θ, η) is the unique weak solution for $CP(\theta_*, \eta_*)$ on $\mathbf{R}_+$. q.e.d.

4. Proof of main theorems

Let us prove theorems 2.1 and 2.2.

Proof of theorem 1.1. Firstly we construct a T -periodic weak solution of $(P)^\nu$ on $\mathbf{R}_+$. Let (θ, η) be as in lemma 3.3, that is, the unique weak solution for $CP(\theta_*, \eta_*)^\nu$ on $\mathbf{R}_+$. For each $m \in \mathbf{N}$ we denote by (θ_m, η_m) the weak solution for $CP(\theta(mT, \cdot), \eta(mT, \cdot))^\nu$ on $[0, T]$. By proposition 2.1 and (3.4), we have

$$\eta_* \leq \eta \leq \beta(K_3) \quad \text{a.e. in } Q_\infty.$$

In particular

$$\eta_* \leq \eta(T, \cdot) \leq \beta(K_3) \quad \text{a.e. in } \Omega.$$

Applying proposition 2.3 to η and η_1 ,

$$\eta_* \leq \eta \leq \eta_1 \leq \beta(K_3) \quad \text{a.e. in } Q_T.$$

Recursive use of this procedure derives that

$$\eta_* \leq \eta \leq \eta_1 \leq \eta_2 \leq \cdots \leq \eta_m \leq \cdots \leq \beta(K_3) \quad \text{a.e. in } Q_T,$$

hence

$$\theta_* \leq \theta \leq \theta_1 \leq \theta_2 \leq \cdots \leq \theta_m \leq \cdots \leq K_3 \quad \text{a.e. in } Q_T.$$

Then we can define $\eta_\infty(t, x) = \lim_{m \rightarrow +\infty} \eta_m(t, x)$ and $\theta_\infty(t, x) = \lim_{m \rightarrow +\infty} \theta_m(t, x)$ for a.e. $(t, x) \in Q_T$. It is easily verified that $\eta_\infty \in \beta(\theta_\infty)$ a.e. in Q_T , $\eta_\infty(0, \cdot) = \eta_\infty(T, \cdot)$ and $\theta_\infty(0, \cdot) = \theta_\infty(T, \cdot)$ a.e. in Ω . Further we have estimates

$$\eta_* \leq \eta_m \leq \beta(K_3) \quad \text{hence} \quad \theta_* \leq \theta_m \leq K_3 \quad \text{a.e. in } Q_T,$$

$$|\theta_m|_{L^2(0,T;H^1(\Omega))} \leq C_1,$$

and for any bounded subdomain A with $\bar{A} \subset Q_T$,

$$|\theta_m|_{H^1(A)} \leq C_2 := C_2(A),$$

where C_i , $i = 1, 2$ are positive constants independent of m . Then we easily see that $(\theta_\infty, \eta_\infty)$ is a weak solution of $(P)^\nu$ on $[0, T]$. Consequently, T -periodic extension $(\theta_p^\nu, \eta_p^\nu)$ of $(\theta_\infty, \eta_\infty)$ onto $\mathbf{R}_+$ is a T -periodic weak solution of $(P)^\nu$ on $\mathbf{R}_+$.

Next we prove the uniqueness of T -periodic weak solutions. To do this, we shall show that any T -periodic weak solution (θ, η) is equal to $(\theta_p^\nu, \eta_p^\nu)$ constructed as above. Since θ_p^ν satisfies (2.5), (2.6) holds for $\theta_1 = \theta_p^\nu$ and $\theta_2 = \theta$, from which it follows that

$$(4.1) \quad \int_{mT}^{(m+1)T} \int_{\Omega} |\eta_p^\nu - \eta| \, dx dt \rightarrow 0 \quad \text{as } m \rightarrow +\infty.$$

On the other hand, by T -periodicity of η_p^ν and η ,

$$\int_0^T \int_{\Omega} |\eta_p^\nu - \eta| \, dx dt = \int_{mT}^{(m+1)T} \int_{\Omega} |\eta_p^\nu - \eta| \, dx dt.$$

So we must have $\int_0^T \int_{\Omega} |\eta_p^\nu - \eta| \, dx dt = 0$. Therefore $\eta_p^\nu = \eta$ a.e. in Q_T . Again, by T -periodicity of η_p^ν and η , $\eta_p^\nu = \eta$ a.e. in Q_∞ . Hence $\theta_p^\nu = \theta$ a.e. in Q_∞ . Thus the proof has been completed. q.e.d.

Proof of theorem 2. Let (θ, η) be an arbitrary weak solution of $(P)^\nu$ on $\mathbf{R}_+$ satisfying (2.5). From proposition 2.3 we find that

$$d := \lim_{t \rightarrow +\infty} \|\eta_p^\nu(t, \cdot) - \eta(t, \cdot)\|_{L^1(\Omega)}$$

exists. Further as $m \rightarrow +\infty$ we have

$$\int_{mT}^{(m+1)T} \int_{\Omega} |\eta_p^\nu - \eta| dx dt \geq T \|\eta_p^\nu((m+1)T, \cdot) - \eta((m+1)T, \cdot)\|_{L^1(\Omega)} \rightarrow dT.$$

Note that (4.1) also holds for η_p^ν and η , hence we deduce $d = 0$. That is $\eta_p^\nu(t, \cdot) - \eta(t, \cdot) \rightarrow 0$ in $L^1(\Omega)$. On account of the boundedness of η_p^ν and η in Q_∞ , we obtain

$$\eta_p^\nu(t, \cdot) - \eta(t, \cdot) \rightarrow 0 \quad \text{in } L^q(\Omega) \text{ for all } q \geq 1 \text{ as } t \rightarrow +\infty.$$

From (1.1), it results that

$$b_* \|\theta_p^\nu(t, x) - \theta(t, x)\| \leq \|\eta_p^\nu(t, x) - \eta(t, x)\| \quad \text{for a.e. } (t, x) \in Q_\infty,$$

consequently

$$\theta_p^\nu(t, \cdot) - \theta(t, \cdot) \rightarrow 0 \quad \text{in } L^q(\Omega) \text{ for all } q \geq 1 \text{ as } t \rightarrow +\infty.$$

q.e.d.

In the rest of this chapter we study the convergence of the T -periodic weak solution of $(P)^\nu$ on $\mathbf{R}_+$ to that of the Stefan problem when $\nu \rightarrow 0$. The result is as follows.

Theorem 4.3. Assume that (2.7) holds. When $\nu \rightarrow 0$, $(\theta_p^\nu, \eta_p^\nu)$ converges to some periodic solution (θ_p^0, η_p^0) of the Stefan problem $(P)^0$ in the following sense:

$$\theta_p^\nu \rightarrow \theta_p^0 \quad \text{weakly in } L^2(0, T; H^1(\Omega)) \text{ and strongly in } L^q(Q_T) \text{ for all } q \geq 1,$$

$$g(\cdot, \cdot, \theta_p^\nu) \rightarrow g(\cdot, \cdot, \theta_p^0) \quad \text{in } L^2(\Sigma_T^N),$$

and there exists a subsequence $\{\nu_k\}$ of $\{\nu\}$ such that

$$\eta_p^{\nu_k} \rightarrow \eta_p^0 \quad \text{weakly in } L^\infty(Q_T).$$

We claim that the following estimates hold for $\{(\theta_p^\nu, \eta_p^\nu)\}$:

$$\beta(-K_3) \leq \eta_p^\nu(t, x) \leq \beta(K_3) \quad \text{hence} \quad -K_3 \leq \theta_p^\nu(t, x) \leq K_3 \quad \text{a.e. in } Q_T,$$

$$|\theta_p^\nu|_{L^2(0,T;H^1(\Omega))} \leq C_3,$$

and for any bounded subdomain A with $\overline{A} \subset Q_T$,

$$|\theta_p^\nu|_{H^1(A)} \leq C_4,$$

where $C_i > 0$, $i = 3, 4$, are constants independent of $\nu \in]0, 1]$. Hence there exist a subsequence $\{\nu_k\}$ of $\{\nu\}$ and $(\theta, \eta) \in L^2(0, T; H^1(\Omega)) \times L^\infty(Q_T)$ such that

$$\eta_p^{\nu_k} \rightarrow \eta \quad \text{weakly}^* \text{ in } L^\infty(Q_T),$$

$$(4.2) \quad \theta_p^{\nu_k} \rightarrow \theta \quad \text{weakly in } L^2(0, T; H^1(\Omega)) \text{ and strongly in } L^q(Q_T) \text{ for all } q \geq 1,$$

$$(4.3) \quad g(\cdot, \cdot, \theta_p^{\nu_k}) \rightarrow g(\cdot, \cdot, \theta) \quad \text{in } L^2(\Sigma_T^N).$$

We easily see that (θ, η) is a weak solution of $(P)^0$ on $[0, T]$. Moreover, since $(\theta_p^\nu, \eta_p^\nu)$ is T -periodic, (θ, η) is also T -periodic. On account of remark 2.4 we can replace $\{\nu_k\}$ with $\{\nu\}$ in (4.2) and (4.3). Therefore T -periodic extension of (θ, η) onto $\mathbf{R}_+$ is a desired one.

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